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DIFFERENTIAL SANDWICH THEOREMS FOR SOME SUBCLASSES OF P-VALENT FUNCTIONS

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Abstract. In this paper a certain multiplier operator of p-valent functions is defined. Moreover, subordination- and superordination-preserving properties for a class of multiplier operators defined on the space of normalized analytic functions in the open unit disk is obtained. Also by applying these results, sandwich theorems and generalizations of some known results are obtained.

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1. Introduction

Observations: Let $H(\Delta)$ denote the class of analytic functions in the open unit disk $\Delta = \{z : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\mathcal{H}[a, n] = \{f \in H(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

Also, let $A(n)$ be the subclass of the $\mathcal{H}[0, n]$ the form

$$f(z) = z^n + \sum_{k=n+1}^{\infty} a_k z^k, \qquad z \in \Delta.$$

Suppose that f and F are in $H(\Delta)$. The function f is said to be subordinate to F or F is said to be superordinate of f, if there exist a function $w \in H(\Delta)$, with $w(0) = 0$, and $|w(z)| < 1$ such that $f(z) = F(w(z))$ and we write $f \prec F$ or $f(z) \prec F(z)$. If function F is univalent in Δ, then we have

$$f \prec F \Longleftrightarrow f(0) = F(0) \text{ and } f(\Delta) \subset F(\Delta).$$

Let $\varphi : \mathbb{C}^2 \times \Delta \longrightarrow \mathbb{C}$ and H be analytic in Δ. If p is analytic in Δ and satisfies the (first-order) differential subordination

$$\varphi(p(z), zp'(z); z) \prec H(z), \tag{1}$$

is called a solution of the differential subordination. The univalent function q is called a dominant of the solution of the differential subordination, or dominant if $p \prec q$ for all p satisfying in (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominant of q of (1) is said to be the best dominant.

Let $\varphi : \mathbb{C}^2 \times \Delta \longrightarrow \mathbb{C}$ and H be analytic in Δ. If p and $\varphi(p(z), zp'(z); z)$ are univalent and p satisfies the (first-order) differential superordination

$$H(z) \prec \varphi(p(z), zp'(z); z), \tag{2}$$

then p is a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordinate, or more simply a subordinant if $q \prec p$ for all q satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinant of q of (2) is said to be the best subordinant.

Ali et al [1] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy $q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$, where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$.

Singh et al [14] defined the following multiplier transformation

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \qquad f \in \mathcal{H}[0,p], \lambda \geq 0, n \in \mathbb{Z}. \tag{3}$$

For this operator, one easily gets

$$z(I_p(n,\lambda)f(z))' = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z). \tag{4}$$

Also, for $-1 \leq B < A \leq 1, \delta \geq 0$, let $\Omega_p^\lambda(A,B,\delta)$ be the class of function $f \in A(p)$ such that

$$\frac{\delta}{p}\frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{p-\delta}{p}\frac{I_p(n,\lambda)f(z)}{z^p} \prec \frac{1+Az}{1+Bz}.$$

The family $\Omega_p^\lambda(A,B,\delta)$ is a comprehensive family containing various well-known as well as new classes of analytic functions.

Earlier, the operator $I_1(n,\lambda)$ was investigated by Cho and Srivastava [3] and Cho and Kim [2], whereas the operator $I_1(n,1)$ was studied by Uraleaddi and Somantha [15], $I_1(n,0)$ is well-known Salagean [12] derivative operator D^n, defined as

$$D^0 f(z) = f(z), \quad D^1 f(z) = zf'(z) \;\text{ and }\; D^n f(z) = D(D^{n-1}f(z)).$$

Making use of the principle of subordinant between analytic functions Miller et all [9] and more recently Ebadian et all [4] and Rahrovi [11] obtained some interesting subordination theorems involving certain integral operators. Also Miller and Mocanu [8] considered subordination-preserving properties of certain integral operator investigations as the dual concept of differential subordination. In the present investigation, using the technique in [4], we obtain the subordination and superordination-preserving properties of the multiplier operator $I_p(n,\lambda)$ defined by (3) with the sandwich-type theorems.

2. Definitions and Preliminaries

The following definitions and Lemmas will be required in our present investigation.

Definition 1. *[7] We denote by Q the set of function q that are analytic and injective on $\overline{\Delta}\backslash E(q)$ where*

$$E(q)=\{\xi\in\Delta:\ \lim_{z\to\xi} q(z)=\infty\},$$

and $q'(\xi)\neq 0$ for $\xi\in\partial\Delta\backslash E(q)$.

Lemma 1. *[7] Let $h(z)$ be analytic and convex univalent in Δ and $h(0)=a$. Also let $p(z)$ be analytic in Δ with $p(0)=a$. If $p(z)+\frac{zp'(z)}{\gamma}\prec h(z)$, where $\gamma\neq 0$ and $Re\gamma\geq 0$, then $p(z)\prec q(z)\prec h(z)$, where*

$$q(z)=\frac{\gamma}{z^{\gamma}}\int_0^z h(t)t^{\gamma-1}\mathrm{d}t.$$

Furthermore $q(z)$ is a convex function and is the best dominant.

Lemma 2. *[8] Let $h(z)$ be convex in Δ, $h(0)=a,\gamma\neq 0$ and $Re\gamma\geq 0$. Also $p\in\mathcal{H}[a,n]\cap Q$. If $p(z)+\frac{zp'(z)}{\gamma}$ is univalent in Δ, $h(z)\prec p(z)+\frac{zp'(z)}{\gamma}$ and $q(z)=\frac{\gamma}{z^{\gamma}}\int_0^z h(t)t^{\gamma-1}\mathrm{d}t$ then $q(z)\prec p(z)$, and $q(z)$ is a convex function and is the best subordinant.*

Lemma 3. *[13] Let $q(z)$ be a convex univalent function in Δ and $\psi,\gamma\in\mathbb{C}$ with $Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\{0,-Re\ \frac{\psi}{\gamma}\}$, $h(0)=a,\gamma\neq 0$ and $Re\gamma\geq 0$. If $p(z)$ is analytic in Δ and $\psi p(z)+\gamma zp'(z)\prec\psi q(z)+\gamma zq'(z)$ then $p(z)\prec q(z)$, and $q(z)$ is the best dominant.*

Lemma 4. *[7] Let q be analytic in Δ and let $\Theta(w)$ and $\phi(w)$ be analytic in a domain $\mathbb{D}$ containing $q(\Delta)$ with $\phi(w)\neq 0$ when $w\in q(\Delta)$. Set*

$$Q(z)=zq'(z)\phi\big(q(z)\big),\quad h(z)=\Theta\big(q(z)\big)+Q(z),$$

and suppose that

(1) $Q(z)$ *is starlike(univalent) in* Δ*;*

(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in \Delta$.

If p *is analytic in* Δ *with* $p(0) = q(0)$ *and* $p(\Delta) \subset \mathbb{D}$*, and*

$$\Theta\big(p(z)\big) + zp'(z)\phi\big(p(z)\big) \prec \Theta\big(q(z)\big) + zq'(z)\phi\big(q(z)\big) = h(z)$$

then $p(z) \prec q(z)$*, and* q *is the best dominant.*

Lemma 5. *[6] For* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *let* $h \in H(\Delta)$ *with* $h(0) = c$*. If* $Re\,(\beta h(z) + \gamma) > 0$*, then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z),$$

with $q(0) = c$ *is analytic in* Δ *and satisfies* $Re\,(\beta q(z) + \gamma) > 0$.

Lemma 6. *[5] suppose that the function* $H : \mathbb{C}^2 \times \Delta \longrightarrow \mathbb{C}$ *satisfies the following condition* $Re\ H(\rho i, \sigma) \leq 0$ *for all real* ρ *and for all*

$$\sigma \leq -n(1+\rho^2)/2 \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\}.$$

If the function $p(z) = 1 + p_n z^n + \ldots$ *is analytic in* Δ *and* $Re\{H(p(z), zp'(z); z)\} > 0$ *then*

$$Re\ p(z) > 0, \qquad z \in \Delta.$$

Definition 2. *[8] A function* $L : \Delta \times [0, \infty) \longrightarrow \mathbb{C}$ *is a subordination (or Loewner) chain if* $L(\cdot, t)$ *is analytic and univalent in* Δ *for all* $t \geq 0$*, and* $L(z, \cdot)$ *is continuously differentiable on* $[0, \infty)$ *for all* $z \in \Delta$*, and* $L(z, s) \prec L(z, t)$ *when* $0 \leq s \leq t$.

The next Lemma gives us a necessary and sufficient condition for $L(z, t)$ to be a subordination chain.

Lemma 7. *[10] The function* $L(z, t) = a_1(t)z + a_2(t)z^2 + \ldots$ *with* $a_1(t) \neq 0$ *and* $lim_{t\to\infty}|a_1(t)| = \infty$ *is a subordination chain if and only if*

$$Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) > 0.$$

Lemma 8. *[7] Let* p *be analytic in* Δ *and* q *analytic and univalent in* $\overline{\Delta}\backslash E(q)$ *with* $p(0) = q(0)$*. If* p *is not subordination to* q*, then there is a point* $z_0 \in \Delta$ *and* $\xi_0 \in \partial\Delta$ *such that* $p(|z| < |z_0|) \subset q(\Delta)$*,*$p(z_0) = q(\xi_0)$*, and* $z_0 p'(z_0) = m\xi_0 q'(\xi_0)$ *for some* m*,* $m \geq 1$.

Lemma 9. *[10] Let* $q(z)$ *be a convex univalent function in* Δ *and* $\eta \in \mathbb{C}$*, assume that* $Re\ \eta > 0$*. If* $p(z) \in \mathcal{H}[a, n] \cap Q$ *and* $q(z) + \eta zq'(z) \prec p(z) + \eta zp'(z)$ *which implies that* $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

3. Subordination for analytic functions

Throughout this paper, we will denote $\Sigma_{n,\lambda}$ by

$$\Sigma_{n,\lambda} := \{f \in A : I_p(n,\lambda)f(z) \neq 0, z \in \Delta\}.$$

Theorem 10. *For $f \in A(p)$ suppose that $f \in \Omega_p^{\lambda}(A,B,\delta)$ and $0 \leq \delta \leq p(p+\lambda)$, then $f \in \Omega_p^{\lambda}(A,B,0)$.*

Proof. Let $P(z) = \frac{I_p(n,\lambda)f(z)}{z^p}$. From the relation (3) we have

$$\frac{zP'(z)}{p+\lambda} + P(z) = \frac{I_p(n+1,\lambda)f(z)}{z^p}.$$

Since $f \in \Omega_p^{\lambda}(A,B,\delta)$, we conclude that

$$\frac{\delta}{p}\frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{p-\delta}{p}\frac{I_p(n,\lambda)f(z)}{z^p} = \frac{\delta}{p(p+\lambda)}zP'(z) + P(z) \prec \frac{1+Az}{1+Bz}.$$

Next, from Lemma 1, for $\gamma = \frac{p(p+\lambda)}{\delta}$ $\left(Re\ \frac{p(p+\lambda)}{\delta} > 0\right)$ it follows that

$$P(z) = \frac{I_p(n,\lambda)f(z)}{z^p} \prec q(z) = \frac{\gamma}{z^{\gamma}}\int_0^z h(t)t^{\gamma-1}\mathrm{d}t \prec h(z) = \frac{1+Az}{1+Bz}.$$

Thus $f \in \Omega_p^{\lambda}(A,B,0)$, furthermore $q(z)$ is the best dominant.

Letting $p = 1$ and $\lambda = 0$ in the Theorem 10, we have the following corollary.

Corollary 11. *If $f \in A$ satisfies*

$$\delta\frac{D^{n+1}f(z)}{z} + (1-\delta)\frac{D^n f(z)}{z} \prec \frac{1+Az}{1+Bz}, \qquad \delta \geq 0,$$

then $\frac{D^n f(z)}{z} \prec \frac{1+Az}{1+Bz}$.

Set $p = 1$, $\lambda = 0$ and $n = 1$ in the Theorem 10, we have the following corollary.

Corollary 12. *If $f \in A$ and*

$$f'(z)(1 + \delta\frac{zf'(z)}{f'(z)}) \prec \frac{1+Az}{1+Bz}, \qquad \delta \geq 0,$$

then $f'(z) \prec \frac{1+Az}{1+Bz}$.

Theorem 13. *Let $f \in \Omega_p^{\lambda}(A,B,\delta)$. If $0 \le \delta \le p(p+\lambda)$, then*

$$Re\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right) \ge \frac{p(p+\lambda)}{\delta} z^{-\frac{p(p+\lambda)}{\delta}} \int_0^1 u^{\frac{p(p+\lambda)}{\delta}} \frac{1-Au}{1-Bu} \mathrm{d}u.$$

Proof. Let $P(z) = \frac{I_p(n,\lambda)f(z)}{z^p}$. Then by Theorem 10, we have

$$P(z) \prec \frac{p(p+\lambda)}{\delta} z^{-\frac{p(p+\lambda)}{\delta}} \int_0^1 t^{\frac{p(p+\lambda)}{\delta}-1} \frac{1+At}{1+Bt} \mathrm{d}t \prec \frac{1+Az}{1+Bz}.$$

This is equivalent to

$$\frac{I_p(n,\lambda)f(z)}{z^p} = \frac{p(p+\lambda)}{\delta} \int_0^1 u^{\frac{p(p+\lambda)}{\delta}-1} \frac{1+Auw(z)}{1+Buw(z)} \mathrm{d}u,$$

where $w(z)$ is Schwartz function. Therefore

$$\begin{aligned} Re\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right) &\ge \frac{p(p+\lambda)}{\delta} \int_0^1 u^{\frac{p(p+\lambda)}{\delta}-1} Re\left(\frac{1+Auw(z)}{1+Buw(z)}\right) \mathrm{d}u \\ &= \frac{p(p+\lambda)}{\delta} \int_0^1 u^{\frac{p(p+\lambda)}{\delta}} \frac{1-Au}{1-Bu} \mathrm{d}u. \end{aligned}$$

Thus

$$\frac{I_p(n,\lambda)f(z)}{z^p} = \frac{p(p+\lambda)}{\delta} \int_0^1 u^{\frac{p(p+\lambda)}{\delta}-1} \frac{1+Auz}{1+Buz} \mathrm{d}u.$$

Such that for this function, we have

$$\frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{p-\delta}{p} \frac{I_p(n,\lambda)f(z)}{z^p} = \frac{1+Az}{1+Bz}.$$

Letting $z \to -1$ yields

$$\frac{I_p(n,\lambda)f(z)}{z^p} \longrightarrow \frac{p(p+\lambda)}{\delta} \int_0^1 u^{\frac{p(p+\lambda)}{\delta}} \frac{1-Au}{1-Bu} \mathrm{d}u.$$

Theorem 14. *Let $q(z)$ be univalent in the open unit disk Δ, $\delta \in \mathbb{C}$, and*

$$Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > 0, \qquad \delta > 0.$$

If $f \in A(p)$ satisfies the subordination

$$\frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p} \frac{I_p(n,\lambda)f(z)}{z^p} \prec q(z) + \frac{\delta z q'(z)}{p(p+\lambda)}, \tag{5}$$

where $I_p(n,\lambda)f(z)$ is defined by (3), then $\frac{I_p(n,\lambda)f(z)}{z^p} \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Let

$$P(z) = \frac{I_p(n,\lambda)f(z)}{z^p}. \tag{6}$$

Differentiating (6) with respect to z logarithmically, we have

$$\frac{zP'(z)}{P(z)} = \frac{z(I_p(n,\lambda)f(z))'}{I_p(n,\lambda)f(z)} - p.$$

Now, in view of (3), we obtain from (6) the following subordination

$$P(z) + \frac{\delta}{p(p+\lambda)} zP'(z) \prec q(z) + \frac{\delta}{p(p+\lambda)} zq'(z).$$

Then from the lemma 3, for $\gamma = \frac{\delta}{p(p+\lambda)}$ and $\psi = 1$, we conclude that $\frac{I_p(n,\lambda)f(z)}{z^p} \prec q(z)$ and $q(z)$ is the best dominant.

Taking $q(z) = \frac{1+Az}{1+Bz}$ in the Theorem 14 we arrive the following corollary.

Corollary 15. *Let* $-1 \le B < A \le 1$ *and* $Re\left(\frac{1+Az}{1+Bz}\right) > 0$ *If* $f \in A(p)$ *and*

$$\frac{\delta}{p}\frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p}\frac{I_p(n,\lambda)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} + \frac{\delta}{p(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2},$$

then $\frac{I_p(n,\lambda)f(z)}{z^p} \prec \frac{1+Az}{1+Bz}$ *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Putting $p = 1, \lambda = 0$ and $q(z) = \frac{1+z}{1-z}$ in the Theorem 14, we get the following corollary.

Corollary 16. *If* $f \in A$ *and*

$$\delta\frac{D^{n+1}f(z)}{z} + (1-\delta)p\frac{D^n f(z)}{z} \prec \frac{1+z}{1-z} + \frac{2\delta z}{(1-z)^2},$$

then $\frac{D^n f(z)}{z} \prec \frac{1+z}{1-z}$ *and* $\frac{1+z}{1-z}$ *is the best dominant.*

Suppose $p = 1$, $\lambda = 0$, $n = 1$ and $q(z) = \frac{1+z}{1-z}$ in the Theorem 14, we have the following corollary.

Corollary 17. *If* $f \in A$ *and*

$$f'(z)\left(1 + \delta\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z} + \frac{2\delta z}{(1-z)^2}, \qquad \delta > 0,$$

then $f'(z) \prec \frac{1+z}{1-z}$ *and* $\frac{1+z}{1-z}$ *is the best dominant.*

Theorem 18. *Let $q(z)$ be univalent in Δ, and $\gamma \neq 0$, $\mu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta \neq 0$. Let $f \in A(p)$ and suppose that $q(z)$ satisfies*

$$Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0. \tag{7}$$

If

$$1 + \gamma\mu\left[\frac{\alpha z(I_p(n+1,\lambda)f(z))' + \beta z(I_p(n,\lambda)f(z))'}{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)} - p\right] \prec 1 + \gamma\frac{zq'(z)}{q(z)},$$

then

$$\left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^{\mu} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let

$$P(z) = \left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^{\mu}, \qquad \mu \geq 0, \quad \alpha + \beta \neq 0. \tag{8}$$

Differentiating logarithmically both side of (8) and multiplying by z, we get

$$\frac{zP'(z)}{P(z)} = \mu\left[\frac{\alpha z(I_p(n+1,\lambda)f(z))' + \beta z(I_p(n,\lambda)f(z))'}{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)} - p\right].$$

By setting $\Theta(w) = 1$ and $\Phi(w) = \gamma/w$, we observe that $\Theta(w)$ is analytic in $\mathbb{C}$ and $\Phi(w) \neq 0$ is analytic in $\mathbb{C}\backslash\{0\}$. Also, we let

$$Q(z) = zq'(z)\Phi(q(z)) = \frac{\gamma zq'(z)}{q(z)},$$

$$h(z) = \Theta(q(z)) + Q(z) = 1 + \gamma\frac{zq'(z)}{q(z)}. \tag{9}$$

From (7) we see that $Q(z)$ is analytic univalent in the unit disk Δ, and from (9), we have

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0.$$

An application of Lemma 4, we conclude that

$$\left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^{\mu} \prec q(z),$$

and $q(z)$ is the best dominant.

Suppose $\alpha = 0$, $\beta = 1$, $\gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ in the Theorem 18, we arrive the following corollary.

Corollary 19. *If $f \in A(p)$ for $-1 \le B < A \le 1$, $\mu \ne 0$ and*

$$1 + \mu \left[\frac{z(I_p(n,\lambda)f(z))'}{I_p(n,\lambda)f(z)} - p \right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then $\left[\frac{I_p(n,\lambda)f(z)}{z^p}\right]^\mu \prec \frac{1+Az}{1+Bz}$ and $\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $\alpha = 0$, $\beta = 1$, $\gamma = \frac{1}{b}$, $p = 1$, $\mu = 1$, $\lambda = 0$ and $q(z) = \frac{1}{(1-z)^{2b}}$ $\big(b \in \mathbb{C}\backslash\{0\}\big)$ in the Theorem 18, we get the following corollary.

Corollary 20. *Suppose $f \in A$ and b is nonzero complex number for which*

$$1 + \frac{1}{b} \left[\frac{z(D^n f(z))'}{D^n f(z)} - 1 \right] \prec \frac{1+z}{1-z},$$

then $\frac{D^n f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$ and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By setting $\alpha = 0$, $\beta = 1$, $\gamma = \frac{1}{b}$, $p = 1$, $\mu = 1$, $\lambda = 0$, $n = 1$ and $q(z) = \frac{1}{(1-z)^{2b}}(b \in \mathbb{C}\backslash\{0\})$ in the Theorem 18, we get the following Corollary.

Corollary 21. *Suppose $f \in A$ and b is nonzero complex number and*

$$1 + \frac{1}{b} \left(\frac{zf''(z)}{f'(z)} \right) \prec \frac{1+z}{1-z},$$

then $f'(z) \prec \frac{1}{(1-z)^{2b}}$ and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

By setting $p = 1$, $\big(I_1(n,\lambda)f(z) = I_n^\lambda f(z)\big)$ we get the following Theorems and Corollaries.

Theorem 22. *Let $f, g \in \Sigma_{n,\lambda}$, with $\lambda \ge 0$, $a - 1 > 0$, $n \in \mathbb{N}$ and*

$$Re\left(1 + \frac{z\phi''}{\phi'}\right) > -\eta, \qquad z \in \Delta, \phi(z) = I_{n+1}^\lambda g(z), \tag{10}$$

where

$$\eta = \frac{1 + (\gamma-1)^2 - |1 - (\gamma-1)^2|}{4Re\,(\gamma-1)}, \qquad Re\,(\gamma-1) > 0. \tag{11}$$

Then $I_{n+1}^\lambda f(z) \prec I_{n+1}^\lambda g(z)$, implies $I_n^\lambda f(z) \prec I_n^\lambda g(z)$. Moreover, the function $I_n^\lambda g(z)$ is the best dominant.

Proof. Let us define the functions F and G by $F(z) = I_n^\lambda f(z)$, and $G(z) = I_n^\lambda g(z)$. We can assume without loss of generality that G is analytic and univalent on $\overline{\Delta}$, and $G'(\xi) \neq 0$ for $|\xi| = 1$. We first show that if the function q is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)}, \tag{12}$$

then $Re\ q(z) > 0$. From (4) and the definition of $q(z)$ and $\phi(z)$, we obtain

$$\phi(z) = \frac{zG'(z)}{1+\lambda} + \frac{\lambda}{1+\lambda} G(z). \tag{13}$$

Differentiating both of (13), we get

$$\phi'(z) = G'(z) + \frac{1}{1+\lambda} zG''(z).$$

Also, by definitions of $q(z)$, we get

$$\phi'(z) = \frac{1}{1+\lambda} q(z)G'(z) + \frac{\lambda}{1+\lambda} G'(z). \tag{14}$$

Logarithmical differentiation of (14) and through a little simplification, we obtain

$$1 + \frac{z\phi''}{\phi'} = q(z) + \frac{zq'(z)}{q(z)+\lambda} = h(z). \tag{15}$$

From (10) we have $Re\,(h(z) + \lambda) > 0$, and by using Lemma 5 we conclude that the differential equation (15) has a solution $q \in H(\Delta)$ with $q(0) = h(0) = c$ and $Re\big(q(z) + (a-1)\big) > 0$. Let

$$\psi(r, s) = r + \frac{s}{r+\lambda} + \gamma,$$

where γ is given by (11). From (10) and (15), we obtain $Re\ \psi\big(q(z), zq'(z)\big) > 0$. Now we proceed to show that $Re\ \psi(\rho i, \sigma) \leq 0)$ $\big(\rho \in \mathbb{R}, \sigma \leq -\frac{1}{2}(1+\rho^2)\big)$. For this purpose we have

$$Re\,(\psi(\rho i, \sigma)) \leq -\frac{M_\gamma(\rho)}{2|\rho i + \lambda|^2}, \tag{16}$$

where

$$M_\gamma(\rho) = \left[\lambda - 2\gamma^2\right]\rho^2 - (\lambda - 1)\left[2\gamma(\lambda-1) - 1\right].$$

For γ given by (11), we note that $M_\gamma(\rho)$ is a perfect square, therefore we see from (16) that $Re\,(\psi(\rho i, \sigma)) \leq 0$. Thus, by Lemma 6, we conclude that $Re\ q(z) > 0$.

Therefor the function G defined by (12) is convex in Δ.
Now, we prove that $F \prec G$. For this purpose we consider the function $L(z,t)$ given by

$$L(z,t) = \frac{1+t}{1+\lambda} zG'(z) + \frac{\lambda}{1+\lambda} G(z), \quad z \in \Delta, 0 \le t < \infty.$$

Since G is convex and $\lambda \ge 0$, we have

$$a_1(t) = (\frac{\partial L}{\partial z})\bigg|_{z=0} = G'(0)\left(1 + \frac{t}{1+\lambda}\right) \ne 0,$$

and

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = \lambda + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right).$$

According to G is convex and $\lambda \ge 0$, we get $Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) \ge 0$. By Lemma 7, we conclude that $L(z,t)$ is a subordination chain. From the definition of subordination chain, we have

$$\phi(z) = \frac{zG'(z)}{1+\lambda} + \frac{\lambda}{1+\lambda} G(z) = L(z,0), \quad and \quad L(z,0) \prec L(z,t), \qquad t \in [0,\infty),$$

this implies that

$$L(\xi,t) \notin L(\Delta,t) = \phi(\Delta). \tag{17}$$

for $\xi \in \partial\Delta$ and $t \in [0,\infty)$. Now suppose that F is not subordinate to G. Then, by Lemma 8, there exist points $z_0 \in \Delta$ and $\xi_0 \in \partial\Delta$ such that $F(z_0) = G(\xi_0)$ and $z_0 F'(z_0) = (1+t)\xi_0 G'(\xi_0)$. Hence, we have

$$L(\xi_0,t) = \frac{1+t}{1+\lambda}\xi_0 G'(\xi_0) + \frac{\lambda}{1+\lambda} G(\xi_0) = I_n^\lambda f(z_0) \in \phi(\Delta).$$

But this contradicts to (17), thus we have $F(z) \prec G(z)$. Considering $F = G$, we see that the function G is the best dominant. Therefore, we complete the proof of Theorem 22.

Suppose that $\lambda = 0$ and in the Theorem 22 we have the following result.

Corollary 23. *Let* $\phi(z) = D^{n+1}g(z)$ *and* $Re\left(1 + \frac{z\phi''}{\phi'}\right) > -\eta$, $z \in \Delta$. *Then* $D^{n+1}f(z) \prec D^{n+1}g(z)$ *implies that* $D^n f(z) \prec D^n g(z)$

By taking $\lambda = 0$ and $n = 1$ in the Theorem 22 we have the following result.

Corollary 24. *Let* $f, g \in \Sigma_{n,\lambda}$. *If* $\phi(z) = D^2 g(z) = zf'(z) + z^2 f''(z)$ *and*

$$Re\left(1 + \frac{z\phi''}{\phi'}\right) > -\eta \quad z \in \Delta,$$

where η *is given by* (3.11), *then* $D^2 f(z) \prec D^2 g(z)$ *implies that* $zf'(z) \prec zg'(z)$.

4. Superordination for analytic functions

Theorem 25. *Suppose $f \in A(p)$, $0 \leq \delta \leq p(p+\lambda)$ and*

$$M_1(z) = \frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p} \frac{I_p(n,\lambda)f(z)}{z^p} \in \mathcal{H}[a,1] \cap Q. \tag{18}$$

If $M_1(z)$ is univalent in Δ, $\frac{1+Az}{1+Bz} \prec M_1(z)$ and

$$q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma-1} \mathrm{d}t, \tag{19}$$

then

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^p}.$$

The function $q(z)$ is the best subordinant.

Proof. If we let

$$P(z) = \frac{I_p(n,\lambda)f(z)}{z^p}, \tag{20}$$

and $h(z) = \frac{1+Az}{1+Bz}$, then (18) becomes $h(z) \prec P(z) + \frac{\delta}{p(p+\lambda)} zP'(z)$. Since $h(z)$ is convex and $Re\gamma = Re \dfrac{\delta}{p(p+\lambda)} \geq 0$, from Lemma 2 we conclude that $q(z) \prec P(z)$, where $q(z)$ and $P(z)$ given, by from (19) and (20) restrictively.

Theorem 26. *Let $q(z)$ be convex univalent in the open unit disk Δ, $\delta \in \mathbb{C}$, and $Re\ \delta > 0$. Suppose that $\frac{I_p(n,\lambda)f(z)}{z^p} \in \mathcal{H}[q(0),1] \cap Q$. Let $\frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{p-\delta}{p} \frac{I_p(n,\lambda)f(z)}{z^p}$ be univalent in the disk Δ. If*

$$q(z) + \frac{\delta}{p(p+\lambda)} zq'(z) \prec \frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{p-\delta}{p} \frac{I_p(n,\lambda)f(z)}{z^p},$$

where $I_p(n,\lambda)f(z)$ is defined by (3), then

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{z^p}, \tag{21}$$

and $q(z)$ is the best subordinant.

Proof. Let

$$P(z) = \frac{I_p(n,\lambda)f(z)}{z^p}. \tag{22}$$

By taking logarithmic derivative in both slide of (22) we get $\frac{zP'(z)}{P(z)} = \frac{z(I_p(n,\lambda)f(z))'}{I_p(n,\lambda)f(z)} - p$, after some computation, we have

$$nP(z) + \frac{\delta}{p(p+\lambda)} zP'(z) \prec \frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p} \frac{I_p(n,\lambda)f(z)}{z^p}.$$

According to Lemma 9, we get the desired result (21).

Corollary 27. *Suppose that $\delta \in \mathbb{C}$ and satisfies Re $\delta > 0$ and $I_p(n,\lambda)f(z) \in \mathcal{H}[q(0),1] \cap Q$. Let $\frac{\delta}{p}\frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p}\frac{I_p(n,\lambda)f(z)}{z^p}$ be univalent in the unit disk Δ. If*

$$n\frac{1+Az}{1+Bz} + \frac{\delta}{p(p+\lambda)} \frac{(A-B)z}{(1+Bz)^2} \prec \frac{\delta}{p} \frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p} \frac{I_p(n,\lambda)f(z)}{z^p}.$$

Then $\frac{1+Az}{1+Bz} \prec \frac{I_p(n,\lambda)f(z)}{z^p}$ and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Corollary 28. *Let $\delta \neq 0$, $f \in A$ and $f'(z) \in \mathcal{H}[q(0),1] \cap Q$. Let $f'(z)\left(1+\delta\frac{zf''(z)}{f'(z)}\right)$ be univalent in the unit disk Δ. If*

$$\frac{2\delta z}{(1-z)^2} + \frac{1+z}{1-z} \prec f'(z)\left(1+\delta\frac{zf''(z)}{f'(z)}\right).$$

Then $\frac{1+z}{1-z} \prec f'(z)$ and $\frac{1+z}{1-z}$ is the best subordinant.

Theorem 29. *Let $q(z)$ be convex univalent in Δ, $\delta \in \mathbb{C}$, and $\gamma \neq 0, \mu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta \neq 0$. Let $f \in A(p)$ and suppose that*

$$\left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^\mu \in \mathcal{H}[q(0),1] \cap Q,$$

and

$$1 + \gamma\mu\left[\frac{\alpha z(I_p(n+1,\lambda)f(z))' + \beta z(I_p(n,\lambda)f(z))'}{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)} - p\right],$$

is univalent in Δ. If

$$1 + \gamma\frac{zq'(z)}{q(z)} \prec 1 + \gamma\mu\left[\frac{\alpha z(I_p(n+1,\lambda)f(z))' + \beta z(I_p(n,\lambda)f(z))'}{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)} - p\right],$$

then

$$q(z) \prec \left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^\mu,$$

and $q(z)$ is the best subordinant.

Theorem 30. *Let $f, g \in \Sigma_{n,\lambda}$, if $\phi(z) = I^\lambda_{n+1}g(z)$ and Re $\left(1 + \frac{z\phi''}{\phi'}\right) > -\eta$, $z \in \Delta$, where η is given by (11), also let the function $I^\lambda_{n+1}f(z)$ be univalent in Δ and $I^\lambda_n f(z) \in Q$, then the following subordination relationship $I^\lambda_{n+1}g(z) \prec I^\lambda_{n+1}f(z)$, implies $I^\lambda_n g(z) \prec I^\lambda_n f(z)$. Moreover, the function $I^\lambda_n g(z)$ is the best subordinant.*

5. Sandwich results

Combining results of differential subordinations and superordinations, we arrive at the following "sandwich results".

Theorem 31. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in the open unit disk Δ and $\delta \in \mathbb{C}$ satisfies the relation $Re\delta > 0$, and q_2 satisfies* (5)*. If*

$$\frac{I_p(n,\lambda)f(z)}{z^p} \in \mathcal{H}\left[q(0),1\right] \cap Q$$

and

$$\frac{\delta}{p}\frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p}\frac{I_p(n,\lambda)f(z)}{z^p},$$

are univalent in the disk Δ, and satisfy the following subordination relationship

$$\begin{aligned} q_1(z) + \frac{\delta}{p(p+\lambda)} z q_1'(z) &\prec \frac{\delta}{p}\frac{I_p(n+1,\lambda)f(z)}{z^p} + \frac{(p-\delta)}{p}\frac{I_p(n,\lambda)f(z)}{z^p} \\ &\prec q_2(z) + \frac{\delta}{p(p+\lambda)} z q_2'(z), \end{aligned}$$

where $I_p(n,\lambda)f(z)$ is defined by (2.1)*, then*

$$q_1(z) \prec \frac{I_p(n,\lambda)f(z)}{z^p} \prec q_2(z).$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Theorem 32. *Let $q_1(z)$ and $q_2(z)$ be convex univalent in Δ and $\delta \in \mathbb{C}$ and $\gamma \neq 0$, $\mu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ such that $\alpha + \beta \neq 0$. Suppose that $q_2(z)$ satisfies in* (7)*. Moreover suppose that*

$$\left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^{\mu} \in \mathcal{H}\left[q(0),1\right] \cap Q,$$

and

$$1 + \gamma\mu\left[\frac{\alpha z(I_p(n+1,\lambda)f(z))' + \beta z(I_p(n,\lambda)f(z))'}{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)} - p\right]$$

are univalent in Δ. If

$$\begin{aligned} 1 + \gamma\frac{z q_1'(z)}{q_1(z)} &\prec 1 + \gamma\mu\left[\frac{\alpha z(I_p(n+1,\lambda)f(z))' + \beta z(I_p(n,\lambda)f(z))'}{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)} - p\right] \\ &\prec 1 + \gamma\frac{z q_2'(z)}{q_2(z)}, \end{aligned}$$

then

$$q_1(z) \prec \left[\frac{\alpha I_p(n+1,\lambda)f(z) + \beta I_p(n,\lambda)f(z)}{(\alpha+\beta)z^p}\right]^{\mu} \prec q_2(z).$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Theorem 33. *Let* $f, g_k \in \Sigma_{n,\lambda}$ $(k = 1, 2)$ *if* $\phi(z) = I_n^{\lambda} g(z)$ *and* $Re\left(1 + \frac{z\phi''}{\phi'}\right) > -\eta$, $z \in \Delta$, *where* η *is given by* (3.11), *also let the function* $I_{n+1}^{\lambda} f(z)$ *is univalent in* Δ *and* $I_{n+1}^{\lambda} f(z) \in Q$, *then the following subordination relationship*

$$I_{n+1}^{\lambda} g_1(z) \prec I_{n+1}^{\lambda} f(z) \prec I_{n+1}^{\lambda} g_2(z),$$

implies

$$I_n^{\lambda} g_1(z) \prec I_n^{\lambda} f(z) \prec I_n^{\lambda} g_2(z).$$

Moreover, the function $I_n^{\lambda} g_1(z)$ *and* $I_n^{\lambda} g_2(z)$ *are, respectively, the best subordinant and the best dominant.*

References

[1] R. M. Ali, V. Ravichandran, M. H. Khan, K.G. Sumramanian, *Differential sandwish theorems for certain analytic functions*, Far East J. Math. Sci. 15, 1 (2010), 87-94.

[2] N. E. Cho, T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean. Math. Soc. 40 (2003), 399-410.

[3] N. E. Cho, H. M. Srivastava, *Argument estimat of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, 37 (2003), 39-49.

[4] A. Ebadian, R. Aghalari, Zhi-gang Wang, *Subordination and superorination results involving certain convolution operator*, Bull. Iranian Math. Soc. 36, 1 (2010), 137-147.

[5] S. S. Miller, P. T. Mocanu, *Differential subordintions and univalent functions*, Michigan Math. J. 28 (1981), 157-171.

[6] S. S. Miller, P. T. Mocanu, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations. 56 (1985), 297-309.

[7] S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York / Basel, 2000.

[8] S. S. Miller, P. T. Mocanu, *Subordination of solutions of differential superordinations*, Complex Var. 48, 10 (2003), 815-826.

[9] S. S. Miller, P.T. Mocanu, O. Reade, *Subordination-preserving integral operator*, Trans. Amer. Math. Soc. 283, 2 (1984), 605-615.

[10] CH. Pommerenke, *Univalent functions*, Vanderhoeck and Ruprecht, Gotlingen. (1975).

[11] S. Rahrovi, *On a certain subclass of analytic univalent function defined by using Komatu integral operator*, Stud. Univ. Babes-Bolyai Math. 61, 1 (2016), 27-36.

[12] G. S. Salagean, *Subclasses of univalent functions*, Springer-Verlag Heideberg. Leature Notes in Math. 1013 (1983), 362-372.

[13] T. N. Shanmugam, V. Ravichandran, S. Sivasubramanian, *Differential sandwich theorems for some subclass of functions*, Australian J. Math. Appl. 3, 1 (2006), 1-11.

[14] S. Shing, S. Gupta, S. Singh, *On a bounded subclass of certain analytic functions a differential inequalty*, J. Math. Inequal. 2, 1 (2008), 27-36.

[15] B. A. Uralegaddi, C. Somantha, *Certain class of univalent functions, in Current Topics in Analytic functions Theory*, World Scientific, Singapore. (1992), 371-374.

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