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UNIVALENCE CRITERIA FOR INTEGRAL OPERATORS ON THE BESSEL AND STRUVE CLASS OF FUNCTIONS

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Abstract. In this paper we consider the class of Bessel and Struve functions. First we obtain an univalence criteria for the integral operator

$$F(z) = \int_0^z \left(\frac{f_v(t)}{t}\right)^{\alpha} \cdot \left(\frac{g_v(t)}{t}\right)^{\beta} dt$$

then for the integral operator

$$G(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(\frac{f_v(t)}{t}\right)^{\alpha} \cdot \left(\frac{g_v(t)}{t}\right)^{\beta} dt\right]^{\frac{1}{\gamma}}.$$

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1. Introduction and Preliminaries

Let

$$U(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

be the disc with center z_0 and and of radius r, the particular case $U(0,1)$ will be denote by U. Let $H(U)$ be the set of functions which are regular in the unit disc U. Consider $A = \{f \in H(U) : f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$

Theorem 1.1. [1] If the function f is regular in unit disc U, $f(z) = z + a_2z^2 + ...$ and

$$(1 - |z|^2) \cdot \left|\frac{zf''(z)}{f'(z)}\right| \le 1 \qquad (1)$$

for all $z \in U$, then the function is univalent in U.

Theorem 1.2. [5] Let α be a complex number, $\operatorname{Re}\alpha > 0$, and $f(z) = z + a_2 z^2 + ...$ be a regular function in U. If

$$\frac{1-|z|^{\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2}$$

for all $z \in U$, then for any complex number $\beta, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\int_0^z t^{\beta-1} f'(t)dt\right]^{\frac{1}{\beta}} \tag{3}$$

is in the class S.

Theorem 1.3. [3] If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$and $z \in U$ the following inequalities hold

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)} \cdot g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z} \cdot \xi}\right| \tag{4}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - z^2} \tag{5}$$

the equalities hold in case $g(z) = \varepsilon \dfrac{z + u}{1 + \overline{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 1.1. [2] For $z = 0$ from inequality (4) we obtain for every $\xi \in U$

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi| \tag{6}$$

and hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|} \tag{7}$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \tag{8}$$

for all $z \in U$.

Let us consider the second-order inhomogeneous differential equation(([7]), p.341)

$$z^2w''(z)+zw'(z)+(z^2-v^2)w(z)=\frac{4(\frac{z}{2})^{v+1}}{\sqrt{\pi}\Gamma(v+\frac{1}{2})} \tag{9}$$

whose homogeneous part is Bessel's equation, where v is an unrestricted real(or complex) number. The function H_v, which is called the Struve function of order v, is defined as a particular solution of (9). This function has the form

$$H_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\frac{3}{2})\cdot\Gamma(v+n+\frac{3}{2})}\cdot\left(\frac{z}{2}\right)^{2n+v+1}\quad\text{for all } z\in\mathbb{C} \tag{10}$$

We consider the transformation

$$g_v(z)=2^v\sqrt{\pi}\Gamma(v+\frac{3}{2})\cdot z^{\frac{-v-1}{2}}H_v(\sqrt{z}) \tag{11}$$

After some calculus we obtain

$$g_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\frac{3}{2})\Gamma(v+\frac{3}{2})}{4^n\cdot\Gamma(n+\frac{3}{2})\Gamma(v+n+\frac{3}{2})}\cdot z^n \tag{12}$$

Using Theorem 2.1 ([4]) for our case with $b=c=1, \kappa=v+\frac{3}{2}$ we obtain that:

Theorem 1.4. [4] If $v>\frac{\sqrt{3}-7}{8}$ then the function g_v is univalent in U.

The Bessel function of the first kind is defined by

$$J_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+v+1)}\left(\frac{z}{2}\right)^{2n+v}. \tag{13}$$

We consider the transformation

$$f_v(z)=2^v\Gamma(1+v)z^{-\frac{v}{2}}J_v(\sqrt{z}) \tag{14}$$

After some calculus we obtain

$$f_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(1+v)}{n!\Gamma(n+v+1)\cdot 4^n}\cdot z^n \tag{15}$$

Theorem 1.5. [6] If $v>-2$ then $Ref_v'(z)<0$ for $z\in U_1(0,4(v+2))$ and f_v is univalent in $U_1(0,4(v+2))$.

2. Main Result

Theorem 2.1. Let $\alpha, \beta \in C$, f_v a Bessel function and g_v a Struve function. If $z \in U_1 \cap U$, $v \in (-2, -1)$ and

$$\left| \frac{zf_v'(z) - f_v(z)}{zf_v(z)} \right| \leq 1, \quad (\forall)\ z \in U_1 \cup U \tag{16}$$

$$\left| \frac{zg_v'(z) - g_v(z)}{zg_v(z)} \right| \leq 1, \quad (\forall)\ z \in U_1 \cup U \tag{17}$$

$$\frac{|\alpha| + |\beta|}{|\alpha \cdot \beta|} < 1 \tag{18}$$

$$|\alpha \cdot \beta| \leq \frac{1}{\max\limits_{x \in [0,1]} \left[(1 - |z|^2) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c||z|} \right]} \tag{19}$$

where $|c| = \left| \frac{\alpha}{32(2+v)(1+v)} + \frac{\beta}{15(2v+3)(2v+5)} \right| \cdot \frac{1}{|\alpha \cdot \beta|}$

then $F(z) = \int_0^z \left(\frac{f_v(t)}{t} \right)^\alpha \cdot \left(\frac{g_v(t)}{t} \right)^\beta dt$ is univalent.

Proof.
We have $f_v \in S, g_v \in S$ and $\frac{f_v(z)}{z} \neq 0, \frac{g_v(z)}{z} \neq 0$.
For $z = 0$ we have $\left(\frac{f_v(z)}{z} \right)^\alpha \cdot \left(\frac{g_v(z)}{z} \right)^\beta = 1$.
Consider the function

$$h(z) = \frac{1}{\alpha \cdot \beta} \cdot \frac{F''(z)}{F'(z)}$$

The function h has the form:

$$h(z) = \frac{1}{\alpha \cdot \beta} \cdot \alpha \cdot \frac{zf_v'(z) - f_v(z)}{zf_v(z)} + \frac{1}{\alpha \cdot \beta} \cdot \beta \cdot \frac{zg_v'(z) - g_v(z)}{zg_v(z)}$$

By using the relations (16), (18) and (17) we obtain

$$|h(z)| < 1$$

and

$$|h(0)| = |\frac{|\alpha \cdot a_2 + \beta \cdot b_2|}{|\alpha \cdot \beta|} = |c|$$

where
$a_2 = \dfrac{(-1)^2 \cdot \Gamma(1+v)}{2! \cdot \Gamma(2+v+1)4^2} = \dfrac{\Gamma(1+v)}{32 \cdot \Gamma(3+v)} = \dfrac{v \cdot \Gamma(v)}{32 \cdot (2+v)(1+v)v\Gamma(v)} = \dfrac{1}{32(2+v)(1+v)}$
and

$$b_2 = \frac{(-1)^2 \cdot \Gamma(\frac{3}{2}) \cdot \Gamma(v+\frac{3}{2})}{4^2 \cdot \Gamma(2+\frac{3}{2}) \cdot \Gamma(v+\frac{3}{2}+2)} = \frac{\Gamma(\frac{3}{2}) \cdot \Gamma(v+\frac{3}{2})}{16 \cdot \Gamma(\frac{7}{2})\Gamma(v+\frac{7}{2})}$$

We calculate
$\dfrac{\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} = \dfrac{\frac{1}{2}\sqrt{\pi}}{\frac{15}{8}\sqrt{\pi}} = \dfrac{4}{15}$

Using the formula $\Gamma(n+\frac{1}{2}) = \dfrac{(2n-1)!\sqrt{\pi}}{2^{2n-1}(n-1)!}$ we calculate

$\dfrac{\Gamma(v+\frac{3}{2})}{\Gamma(v+\frac{7}{2})} = \dfrac{4}{(2v+3)(2v+5)}.$

Then $|c| = \left| \dfrac{\alpha}{32(2+v)(1+v)} + \dfrac{\beta}{15(2v+3)(2v+5)} \right| \cdot \dfrac{1}{|\alpha \cdot \beta|}$

Applying Remark 1.1 for the function h we obtain

$$\frac{1}{|\alpha \cdot \beta|} \left| \frac{F''(z)}{F'(z)} \right| \leq \frac{|z| + |c|}{1 + x|c| \cdot |z|}$$

$$\Longleftrightarrow \quad \left| (1-|z|^2) \cdot z \cdot \frac{F''(z)}{F'(z)} \right| \leq |\alpha \cdot \beta| \cdot (1-|z|^2) \cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|},$$

for all $z \in U$.
Let's consider the function $H : [0,1] \to \mathbb{R}$

$$H(x) = (1-x^2)x\frac{x+|c|}{1+|c|}; x = |z|$$

$H(\frac{1}{2}) = \dfrac{3}{8} \cdot \dfrac{1+|c|}{2+|c|} > 0$ then $\max\limits_{x \in [0,1]} H(x) > 0.$
We obtain

$$\left| (1-|z|^2) \cdot z \cdot \frac{F''(z)}{F'(z)} \right| \leq |\alpha \cdot \beta| \cdot \max_{x \in [0,1]} \left[(1-|z|^2) \cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|} \right]. \qquad (20)$$

Applying the condition (19) we obtain:

$$(1-|z|^2)\left|\frac{zF''(z)}{F'(z)}\right| \leq 1,$$

for all $z \in U$ and from Theorem 1.1 F is univalent.
Considering in Theorem 2.1 $\alpha = 1$ and $\beta = 1$ we obtain the following corollary:

Corollary 2.1. Let f_v a Bessel function and g_v a Struve function. If $z \in U_1 \cap U$, $v \in (-2,-1)$ and

$$\left|\frac{zf_v'(z) - f_v(z)}{zf_v(z)}\right| < \frac{1}{2}, \quad (\forall)\ z \in U_1 \cup U \tag{21}$$

$$\left|\frac{zg_v'(z) - g_v(z)}{zg_v(z)}\right| < \frac{1}{2}, \quad (\forall)\ z \in U_1 \cup U \tag{22}$$

$$\max_{x\in[0,1]}\left[(1-|z|^2)\cdot|z|\cdot\frac{|z|+|c|}{1+|c||z|}\right] \leq 1 \tag{23}$$

where $|c| = \left|\frac{1}{32(2+v)(1+v)} + \frac{1}{15(2v+3)(2v+5)}\right|$

then $F(z) = \int_0^z \frac{f_v(t)}{t}\cdot\frac{g_v(t)}{t}dt$ is univalent.

Theorem 2.2. Let $\alpha, \beta, \gamma, \delta \in C$, f_v a Bessel function, g_v a Struve function. If $z \in U_1 \cap U$, $v \in (-2,-1)$ and

$$\left|\frac{zf_v'(z) - f_v(z)}{zf_v(z)}\right| \leq 1, \quad (\forall)\ z \in U_1 \cup U \tag{24}$$

$$\left|\frac{zg_v'(z) - g_v(z)}{zg_v(z)}\right| \leq 1, \quad (\forall)\ z \in U_1 \cup U \tag{25}$$

$$\frac{|\alpha|+|\beta|}{|\alpha\cdot\beta|} < 1 \tag{26}$$

$$\operatorname{Re}\gamma \geq \operatorname{Re}\delta > 0 \tag{27}$$

$$|\alpha\cdot\beta| \leq \frac{1}{\max_{x\in[0,1]}\left[(1-|z|^2)\cdot|z|\cdot\frac{|z|+|c|}{1+|c||z|}\right]} \tag{28}$$

where $|c| = \left|\dfrac{\alpha}{32(2+v)(1+v)} + \dfrac{\beta}{15(2v+3)(2v+5)}\right| \cdot \dfrac{1}{|\alpha\cdot\beta|}$

then $G(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(\dfrac{f_v(t)}{t}\right)^{\alpha} \cdot \left(\dfrac{g_v(t)}{t}\right)^{\beta} dt\right]^{\dfrac{1}{\gamma}}$ is univalent.

Proof.
We consider the function

$$h(z) = \int_0^z \left(\frac{f_v(t)}{t}\right)^{\alpha} \cdot \left(\frac{v_v(t)}{t}\right)^{\beta} dt$$

$$p(z) = \frac{1}{|\alpha\cdot\beta|} \cdot \frac{h''(z)}{h'(z)}$$

$p(z) = \dfrac{1}{|\alpha\cdot\beta|} \cdot \alpha \dfrac{zf_v'(z) - f_v(z)}{zf_v(z)} + \dfrac{1}{|\alpha\cdot\beta|} \cdot \beta \dfrac{zg_v'(z) - g_v(z)}{zg_v(z)}$
By using the relations (24), (25) and (26) we obtain

$$|p(z)| < 1$$

and $p(0) = \dfrac{|\alpha\cdot a_2 + \beta\cdot b_2|}{|\alpha\cdot\beta|} = |c|$
Applying Remark 1.1 for the function p we obtain

$$\frac{1}{|\alpha\cdot\beta|} \cdot \left|\frac{h''(z)}{h'(z)}\right| \le \frac{|z|+|c|}{1+|c||z|}$$

$\iff \left|\dfrac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot z \cdot \dfrac{h''(z)}{h'(z)}\right| \le |\alpha\cdot\beta| \cdot \dfrac{1-|z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \dfrac{|z|+|c|}{1+|z||c|}$,
for all $z \in U$.

Let's consider the function $Q : [0,1] \to \mathbb{R}$

$$Q(x) = \frac{1-x^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} x \frac{x+|a|}{1+|a|x}; x = |z|$$

$$Q\left(\frac{1}{2}\right) > 0 \Rightarrow \max_{x\in[0,1]} Q(x) > 0.$$

Using this result we have:

$$\frac{1-|z|^{2Re\delta}}{Re\delta} \left|\frac{zh''(z)}{h'(z)}\right| \le$$

$$\leq |\alpha \cdot \beta| \cdot \max_{|z|<1} \left[\frac{1-|z|^{2Re\delta}}{Re\delta} \cdot |z| \cdot \frac{|z|+|c|}{1+|z||c|} \right], \ (\forall) z \in U.$$

Applying the condition (28) we obtain:

$$(1-|z|^2) \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \ (\forall) z \in U,$$

and from Theorem 1.2, $G \in S$.

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