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SOME FAMILIES OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS GIVEN BY AN INTEGRAL OPERATOR

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Abstract. In the present study, the author define a new subclass of meromorphic functions with positive coefficient defined in the punctured unit disc $\mathbb{U} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1, by using the certain integral operator. Coefficient inequality, convex linear combinations, extreme points are obtained. We also investigated meromorphically radii of close-to-convexity, meromorphically convexity, meromorphically starlikeness and Hadamard product. Then, we prove a property using an integral operator for the functions f in this class.

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1. Introduction

Let Σ symbolized the class of analytic functions, which are whith a simple pole at the origin with residue 1 of the form in the punctured unit disc $\mathbb{U} = \{z \in \mathbb{C} : 0 < |z| < 1\}$, and are in the form of the

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n. \tag{1}$$

For functions $f \in \Sigma$ given by (1) and $g \in \Sigma$ defined by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

their Hadamard product [10] is given by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

Let $\sum_s, \sum^*(\alpha), \sum_c(\alpha)$ be the subclass of $\sum$ consisting of univalent, meromorphically starlike of order α and meromrphically convex of order $\alpha(0 \leq \alpha < 1)$ respectively.

A function given by (1) in the class $\sum^*(\alpha)$ if and only if

$$R\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

and $f \in \sum_c(\alpha)$ if and only if

$$R\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \quad (z \in \mathbb{U}) \ .$$

Recent years, many authors investigated the subclass of meromorphic functions with positive coefficient (see [1], [4], [2], [3], [6], [8], [14]). Juena and Reddy (see [12]) introduced the class of $\sum_p$ functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{2}$$

which are regular and univalent in $\mathbb{U}$. The functions in this class are said to be meromorphic functions with positive coefficient.

Analogous to the integral operator defined by Jung at all [13] on the normalized analytic functions, Lashin [15] defined the integral operator $Q^{\gamma}_{\beta} : \Sigma \to \Sigma$, for $\beta > 0, \gamma > 1$; $z \in \mathbb{U}^*$,

$$Q^{\gamma}_{\beta} = Q^{\gamma}_{\beta} f(z) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)} \frac{1}{z^{\beta+1}} \int_0^z t^{\beta} \left(1-\frac{t}{z}\right)^{\gamma-1} f\left(t\right) dt \quad ,$$

where Γ is the familiar Gamma function. Using the integral representation of the Gamma and Beta functions, it can be shown that

$$Q^{\gamma}_{\beta} f(z) = \frac{1}{z} + \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\gamma+1)} a_n z^n = \frac{1}{z} + \sum_{n=1}^{\infty} L(n,\beta,\gamma) a_n z^n$$

where

$$L(n,\beta,\gamma) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\gamma+1)}.$$

Now we introduce the following subclass of Σ_p associated with the integral operator $Q^{\gamma}_{\beta} f(z)$.

Definition 1. *A function $f \in \sum$ is said to be in the class $\sum Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$ if and only if the following condition is satisfied:*

$$R\left\{\frac{-z\,(\psi(z))'}{\psi(z)}\right\} \geq q\left|\frac{z\,(\psi(z))'}{\psi(z)}+1\right|+\zeta \tag{3}$$

where $0 \leq \zeta < 1$, $\beta > 0$, $\gamma > 1$, $0 \leq \alpha \leq \lambda < \frac{1}{2}$, $q \geq 0$ *and*

$$\psi(z) = \lambda\alpha z^2 (Q_\beta^\gamma f(z))^{\prime\prime} + (\lambda - \alpha)\, z(Q_\beta^\gamma f(z))' + (1 - \lambda + \alpha)\, Q_\beta^\gamma f(z) \tag{4}$$

It is noted that

$$\sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q) = \sum Q(\alpha, \lambda, \beta, \gamma, \zeta, q) \cap \sum_p .$$

Lemma 1. *[5] Let σ is a real number and $w = u + iv$ is a complex number. Then,*
$R(w) \geq \sigma \Longleftrightarrow |w - (1 + \sigma)| \leq |w + (1 - \sigma)|$.

Lemma 2. *[5] Let $w = u + iv$ and σ, γ are real numbers. Then*
$R(-w) > \sigma\,|-w - 1| + \gamma \Longleftrightarrow \left\{-w(1 + \sigma e^{i\Phi}) - \sigma e^{i\Phi}\right\} > \gamma, (-\pi \leq \theta \leq \pi)$.

2. Coefficient Bounds

We obtain in this section a necessary and sufficient condition for a function f to be in the class $\sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$. We employ the technique adopted by Aqlan at all [5] and Athsan and Kulkarni [7] to find the coefficient estimates for the functions f defined by the equation (1) in the class $\sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$.

Theorem 3. *A meromorphic function f defined by the equation* (1) *in the class $\sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$ if and only if*

$$\begin{aligned}&\sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)a_n\\ \leq\ &(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\end{aligned} \tag{5}$$

for some $0 \leq \zeta < 1, \beta > 0, \gamma > 1, 0 \leq \alpha \leq \lambda < \frac{1}{2}$ *and* $q \geq 0$.

Proof. Let $f \in \sum_p$ and satisfies the condition (5). Then by applying lemma 2 we have to show that

$$R\left\{\frac{-z\left(\psi(z)\right)^{'}}{\psi(z)}(1+qe^{i\theta})-qe^{i\theta}\right\}>\zeta$$
$$(-\pi\leq\theta\leq\pi, 0\leq\zeta<1, q\geq 0)$$

or equavalentily

$$R\left\{\frac{-z\left(\psi(z)\right)^{'}(1+qe^{i\theta})-qe^{i\theta}\psi(z)}{\psi(z)}\right\}>\zeta. \tag{6}$$

Let

$$\varphi(z)=-z\psi^{'}(z)\left[1+qe^{i\theta}\right]-qe^{i\theta}\psi(z).$$

Thus, the equation(6) is equivalent to

$$R\left(\frac{\varphi(z)}{\psi(z)}\right)>\zeta. \tag{7}$$

In view of Lemma 1 it is sufficient to prove that

$$|\varphi(z)+(1-\zeta)\psi(z)|-|\varphi(z)-(1+\zeta)\psi(z)|>0 \quad . \tag{8}$$

Therefore

$$\begin{aligned}
& |\varphi(z)+(1-\zeta)\psi(z)| \\
= & \left|(2-\zeta)\left(2\lambda\alpha-2\lambda+2\alpha+1\right)\left(\frac{1}{z}\right)\right. \\
& +\sum_{n=1}^{\infty}\left[(1-n-\zeta)+q(-n-1)e^{\theta}\right] \\
& \left.\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]L(n,\beta,\gamma)a_n z^n\right| \\
\geq & \quad (2-\zeta)\left(2\lambda\alpha-2\lambda+2\alpha+1\right)\left|\frac{1}{z}\right| \\
& -\sum_{n=1}^{\infty}\left[(n+\zeta-1)+q(n+1)\right] \\
& \left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]L(n,\beta,\gamma)a_n\left|z^n\right|.
\end{aligned} \tag{9}$$

Also, similarly we obtain

$$\begin{aligned} & |\varphi(z)-(1+\zeta)\psi(z)| \\ \leq\ & \zeta\left(2\lambda\alpha-2\lambda+2\alpha+1\right)\left|\frac{1}{z}\right| \\ & +\sum_{n=1}^{\infty}\left[(n+\zeta+1)+q(n+1)\right] \\ & \left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]L(n,\beta,\gamma)a_n\left|z^n\right|. \end{aligned} \tag{10}$$

Thus from (9) and (10) we get

$$\begin{aligned} & |\varphi(z)+(1-\zeta)\psi(z)|-|\varphi(z)-(1+\zeta)\psi(z)| \\ \geq\ & 2(1-\zeta)\left(2\lambda\alpha-2\lambda+2\alpha+1\right) \\ & -\left[2(n+\zeta)+2q(n+1)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]L(n,\beta,\gamma)a_n \\ \geq\ & 0. \end{aligned}$$

If we use the inequality (5) in last inequality, then we obtain the desired result. Conversely, let the function f defined by the equation (1) be in the class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. That is, the inequality (4) holds for the function f. By choosing the value of z on the positive real axis, where $0\leq z=r<1$ the inequality (4) reduces to

$$R\left\{\frac{\begin{array}{c}(1+\zeta)\left(2\lambda\alpha-2\lambda+2\alpha+1\right)-\sum_{n=1}^{\infty}\left[(n+\zeta)+q(n+1)\right]\\ \left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]L(n,\beta,\gamma)a_nr^{n+1}\end{array}}{\begin{array}{c}\left(2\lambda\alpha-2\lambda+2\alpha+1\right)+\sum_{n=1}^{\infty}\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]\\ L(n,\beta,\gamma)a_nr^{n+1}\end{array}}\right\}\geq 0,$$

where $R(-e^{i\theta})\geq-\left|e^{i\theta}\right|=-1$. Letting $r\to 1^-$ through positive values we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty}\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]L(n,\beta,\gamma)a_n \\ \leq\ & (1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right) \end{aligned}$$

and this is desired result.

Corollary 4. *Let the function f defined by the equation (2) be in the class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. Then*

$$a_n\leq\frac{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right).}{\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)+1\right]L(n,\beta,\gamma)},\quad (n\geq 1).$$

The result is sharp for the function f of the form

$$f(z)=\frac{1}{z}+\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)+1]\,L(n,\beta,\gamma)}z^{n}. \quad (n\geq 1) \tag{11}$$

3. Convex Linear Combination

Theorem 5. *The class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$ is closed under convex combination.*

Proof. Let the functions

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n \qquad \text{and} \qquad g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_n z^n$$

be in the class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. Then, by Theorem 3, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)a_n\\ \leq\ & (1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)b_n\\ \leq\ & (1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)\,.\end{aligned} \tag{13}$$

For $0\leq\tau\leq 1$, define the function h as

$$h(z)=\tau f(z)+(1-\tau)g(z).$$

Then, we get

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\tau a_n+(1-\tau)b_n\right]z^n.$$

Now, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)\,[\tau a_n+(1-\tau)b_n]\\ =\ & \tau[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)a_n\\ & +(1-\tau)[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)b_n\\ \leq\ & \tau(1-\alpha)+(1-\tau)(1-\alpha)\\ =\ & (1-\alpha).\end{aligned}$$

So, $h(z) \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$.

4. Extreme Point

Theorem 6. *Let*

$$f_0(z) = \frac{1}{z}$$

and

$$f_n(z) = \frac{1}{z} + \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)}{[(n+\zeta) + q(n+1)]\,[(n-1)(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)} z^n \quad (n = 1, 2, \ldots).$$

Then $f \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$ if and only if it can be represented in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)$$

where $\mu_n \geq 0$ and $\sum_{n=0}^{\infty} \mu_n = 1$.

Proof. Assume that $f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)$, $(\mu_n \geq 0,\ n = 0, 1, 2, \ldots;\ \sum_{n=0}^{\infty} \mu_n = 1)$. Then, we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \mu_n f_n(z) \\ &= \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \mu_n \frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)}{[(n+\zeta) + q(n+1)]\,[(n-1)(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)} z^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{n=1}^{\infty} [(n+\zeta) + q(n+1)]\,[(n-1)(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)\mu_n \\ &\frac{(1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)}{[(n+\zeta) + q(n+1)]\,[(n-1)(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)} \\ =\ & (1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1) \sum_{n=1}^{\infty} \mu_n \\ =\ & (1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1)(1 - \mu_0) \\ \leq\ & (1-\zeta)(2\alpha\lambda - 2\lambda + 2\alpha + 1). \end{aligned}$$

Hence, by Theorem3, $f \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$.
Conversely, suppose that $f \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$. Since by Corollary 4,

$$a_n \leq \frac{(1-\zeta)\left(2\alpha\lambda - 2\lambda + 2\alpha + 1\right)}{\left[(n+\zeta) + q(n+1)\right]\left[(n-1)\left(n\lambda\alpha + \lambda - \alpha\right)\right] L(n, \beta, \gamma)}, \quad (n \geq 1),$$

If we set

$$\mu_n = \frac{\left[(n+\zeta) + q(n+1)\right]\left[(n-1)\left(n\lambda\alpha + \lambda - \alpha\right)\right] L(n, \beta, \gamma)}{(1-\zeta)\left(2\alpha\lambda - 2\lambda + 2\alpha + 1\right)} a_n, \qquad (n \geq 1)$$

and $\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n$, then we obtain

$$f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z).$$

This completes the proof of the theorem.

5. Radii of Starlikeness and Convexity

We now find the radii of meromorphically close-to-convexity, starlikeness and convexity for functions f in the class $\sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$.

Theorem 7. *Let $f \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$. Then f is meromorphically close-to-convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{(1-\delta)}{n} \frac{\left[(n+\zeta) + q(n+1)\right]\left[(n-1)\left(n\lambda\alpha + \lambda - \alpha\right)\right] L(n, \beta, \gamma)}{(1-\delta)\left(2\alpha\lambda - 2\lambda + 2\alpha + 1\right)}\right]^{\frac{1}{n+1}} \tag{14}$$

$(n \geq 1)$ and the result is sharp.

Proof. Let $f \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$. It is sufficient to prove that

$$\left|z^2 f'(z) + 1\right| \leq 1 - \delta, \qquad 0 \leq \delta < 1, \;\; |z| < r_1$$

By Theorem 3, we have

$$\sum_{n=1}^{\infty} \frac{\left[(n+\zeta) + q(n+1)\right]\left[(n-1)\left(n\lambda\alpha + \lambda - \alpha\right)\right] L(n, \beta, \gamma)}{(1-\zeta)\left(2\alpha\lambda - 2\lambda + 2\alpha + 1\right)} a_n \leq 1.$$

So the inequality

$$\left|z^2 f'(z) + 1\right| = \left|\sum_{n=1}^{\infty} n a_n z^{n+1}\right| \leq \sum_{n=1}^{\infty} n a_n |z|^{n+1} \leq 1 - \delta$$

holds true if

$$\frac{n\,|z|^{n+1}}{1-\delta} \leq \frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}.$$

Then, inequality (14) holds true if

$$|z|^{n+1} \leq \frac{(1-\delta)}{n}\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)},\ \ (n\geq 1)$$

which yields the close-to-convexity of the family and completes the proof and the result is sharp for the function given by (11).

Theorem 8. *Let $f\in\sum_p$. Then f is meromorphically starlike of order δ $(0\leq\delta<1)$ in the disk $|z|<r_2$, where*

$$r_2=\inf_n\left[\frac{(1-\delta)}{(n+2-\delta)}\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}\right]^{\frac{1}{n+1}}\quad (n\geq 1)\,.$$

The result is sharp.

Proof. Let $f\in\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. It is sufficient to prove that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\leq 1-\delta,\qquad 0\leq\delta<1,|z|<r_2 \tag{15}$$

Then we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}+1\right| &= \left|\frac{\sum_{n=1}^{\infty}(n+1)a_nz^{n+1}}{\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n}\right|\\ &\leq \frac{\sum_{n=1}^{\infty}(n+1)a_n\,|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n\,|z|^{n+1}}\\ &\leq 1-\delta.\end{aligned}$$

By Theorem 3 we have

$$\sum_{n=1}^{\infty}\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}a_n\leq 1.$$

Then inequality (15) holds true if

$$\frac{(n+2-\delta)}{(1-\delta)}\,|z|^{n+1}\leq\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}$$

which is equivalent to

$$|z|^{n+1} \leq \frac{(1-\delta)\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\beta,\gamma)}{(n+2-\delta)\left(1-\zeta\right)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}$$

or

$$|z| \leq \left[\frac{(1-\delta)}{(n+2-\delta)}\frac{\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\beta,\gamma)}{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}\right]^{\frac{1}{n+1}}$$

which yields the starlikeness of the family and completes the proof. Also, the result is sharp for the function given by the equation (11).

Theorem 9. *Let $f \in \sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. Then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in the disk $|z| < r_3$, where*

$$r_3 = \inf_n \left[\frac{(1-\delta)}{n\,(n+2-\delta)}\frac{\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\beta,\gamma)}{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}\right]^{\frac{1}{n+1}}, \quad (n \geq 1)\,. \tag{16}$$

The result is sharp for the extremal function f given by

$$f_n(z) = \frac{1}{z} + \frac{n\,(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}{\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\beta,\gamma)}z^n, \qquad (n \geq 1)\,. \tag{17}$$

Proof. By using the technique employed in the proof of Theorem 8 and 9, we can show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| < 1-\mu,$$

for $|z| < r_3$, and prove that the assertion of the theorem is true.

Theorem 10. *For functions $f,g \in \Sigma_p$ defined by (1) let $f,g \in \sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. Then the Hadamard product $f * g \in \sum_p Q(\alpha,\lambda,\beta,\gamma,\rho,q)$, where*

$$\rho \leq \frac{\left[\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]\right]^2 L(n,\beta,\gamma)-n(1-\alpha)^2}{\left[\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]\right]^2 L(n,\beta,\gamma)+(1-\alpha)^2\left[1-\delta(n+1)\right]}.$$

Proof. From Theorem 3, we have

$$\sum_{n=1}^{\infty}\frac{\left[(n+\zeta)+q(n+1)\right]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\beta,\gamma)}{(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}a_n \leq 1, \tag{18}$$

$$\sum_{n=1}^{\infty}\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}b_n\leq 1. \tag{19}$$

From (18) and (19) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{n=1}^{\infty}\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}\sqrt{a_nb_n}\leq 1. \tag{20}$$

We need to find the largest ρ such that

$$\sum_{n=1}^{\infty}\frac{[(n+\rho)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\rho)\,(2\alpha\lambda-2\lambda+2\alpha+1)}a_nb_n\leq 1.$$

Thus it is enough to show that

$$\begin{aligned}&\frac{[(n+\rho)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\rho)\,(2\alpha\lambda-2\lambda+2\alpha+1)}a_nb_n\\ \leq\quad&\frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}\sqrt{a_nb_n},\end{aligned}$$

that is,

$$\sqrt{a_nb_n}\leq\frac{(1-\rho)\,[(n+\zeta)+q(n+1)]}{(1-\zeta)\,[(n+\rho)+q(n+1)]}. \tag{21}$$

On the other hand, from 20 we have

$$\sqrt{a_nb_n}\leq\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}. \tag{22}$$

Therefore in view of 21 and 22 it is enough to show that

$$\begin{aligned}&\frac{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)}{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}\\ \leq\quad&\frac{(1-\rho)\,[(n+\zeta)+q(n+1)]}{(1-\zeta)\,[(n+\rho)+q(n+1)]}\end{aligned}$$

which simplifies to

$$\begin{aligned}&\rho\leq 1\\ &-\frac{2(1-\zeta)^2(2\alpha\lambda-2\lambda+2\alpha+1)L(n,\beta,\gamma)}{[(n+\zeta)+q(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]L(n,\beta,\gamma)+2(1-\zeta)^2(2\alpha\lambda-2\lambda+2\alpha+1)}.\end{aligned}$$

Let

$$\begin{aligned}&\psi(n)\\ &=\frac{2(1-\zeta)^2(2\alpha\lambda-2\lambda+2\alpha+1)L(n,\beta,\gamma)}{[(n+\zeta)+q(n+1)]^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]L(n,\beta,\gamma)+2(1-\zeta)^2(2\alpha\lambda-2\lambda+2\alpha+1)}\end{aligned}$$

Clearly $\psi(n)$ is an increasing function of $n(n\geq 1)$. Letting $n=1$, we prove the assertion.

Theorem 11. *For functions $f, g \in \Sigma_p$ defined by the equation 1 let $f, g \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$. Then the function $k(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)z^n$ is in the class $\sum_p Q(\alpha, \lambda, \beta, \gamma, \rho, q)$ and*

$$\rho \leq 1 - \frac{4(1-\zeta)^2\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)\,L(n,\beta,\gamma)}{\{[(n+\zeta) + q(n+1)]\}^2\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma) - 2(1-\zeta)^2[n + q(n+1)]\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}$$

Proof. Since $f, g \in \sum_p Q(\alpha, \lambda, \beta, \gamma, \zeta, q)$ we have

$$\sum_{n=1}^{\infty}\left\{\frac{[(n+\zeta) + q(n+1)]\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}a_n\right\}^2 \leq 1 \tag{23}$$

$$\sum_{n=1}^{\infty}\left\{\frac{[(n+\zeta) + q(n+1)]\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}b_n\right\}^2 \leq 1 \tag{24}$$

combining the inequalities (23) and (24), we get

$$\sum_{n=1}^{\infty}\frac{1}{2}\left\{\frac{[(n+\zeta) + q(n+1)]\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}\right\}^2 (a_n^2 + b_n^2) \leq 1.$$

But, we need to find the largest ρ such that

$$\sum_{n=1}^{\infty}\frac{[(n+\rho) + q(n+1)]\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)}{(1-\rho)\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}(a_n^2 + b_n^2) \leq 1 \tag{25}$$

The inequality (25) would hold if

$$\begin{aligned}&\frac{[(n+\rho) + q(n+1)]\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)}{(1-\rho)\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}\\ \leq\;& \frac{1}{2}\left\{\frac{[(n+\zeta) + q(n+1)]\,[(n-1)\,(n\lambda\alpha + \lambda - \alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda - 2\lambda + 2\alpha + 1)}\right\}^2.\end{aligned}$$

Then we have

$$\rho \leq 1 - \frac{4(1-\zeta)^2(2\alpha\lambda - 2\lambda + 2\alpha + 1)L(n,\beta,\gamma)}{\{[(n+\zeta) + q(n+1)]\}^2[(n-1)(n\lambda\alpha + \lambda - \alpha)]L(n,\beta,\gamma) - 2(1-\zeta)^2[n + q(n+1)](2\alpha\lambda - 2\lambda + 2\alpha + 1)}$$

Let

$$\phi(n) = \frac{4(1-\zeta)^2(2\alpha\lambda-2\lambda+2\alpha+1)L(n,\beta,\gamma)}{\{[(n+\zeta)+q(n+1)]\}^2[(n-1)(n\lambda\alpha+\lambda-\alpha)]L(n,\beta,\gamma)-2(1-\zeta)^2[n+q(n+1)](2\alpha\lambda-2\lambda+2\alpha+1)}$$

A simple computation shows that $\phi(n+1)-\phi(n)>0$ for all n. This means that $\phi(n)$ is increasing and $\phi(n)\geq\phi(1)$. Letting $n=1$, we prove the assertion.

6. Integral Operators

In this section, we consider integral transforms of functions in the class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$ of the type considered by Goel and Sohi [11].

Theorem 12. . *Let the function $f\in\Sigma_p$ given by (1) is in the class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. Then the integral operator*

$$F(z)=c\int_0^z u^c f(uz)du, \qquad 0<u\leq 1,\ 0<c<\infty \tag{26}$$

is in $\sum_p Q(\alpha,\lambda,\beta,\gamma,\rho,q)$, where

$$\rho\leq 1-\frac{2c(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}{c(1-\zeta)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)+(c+2)(2k+\zeta+1)}$$

and the result is sharp.

Proof. Let the function $f\in\Sigma_p$ given by (1) is in the class $\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$. Then by a simple computation we have

$$\begin{aligned} F(z) &= c\int_0^1 u^c f\left(uz\right)du \\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n z^n \end{aligned} \tag{27}$$

We have to show that

$$\sum_{n=1}^{\infty}\frac{c[(n+\rho)+q(n+1)]\left[(n-1)\left(n\lambda\alpha+\lambda-\alpha\right)\right]L(n,\beta,\gamma)}{(c+n+1)\left(1-\rho\right)\left(2\alpha\lambda-2\lambda+2\alpha+1\right)}a_n\leq 1. \tag{28}$$

Since $f\in\sum_p Q(\alpha,\lambda,\beta,\gamma,\zeta,q)$, we have

$$\sum_{n=1}^{\infty} \frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)} a_n \leq 1.$$

We note that the inequality (28) is satisfied if

$$\begin{aligned} & \frac{c[(n+\rho)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(c+n+1)\,(1-\rho)\,(2\alpha\lambda-2\lambda+2\alpha+1)} \\ \leq \; & \frac{[(n+\zeta)+q(n+1)]\,[(n-1)\,(n\lambda\alpha+\lambda-\alpha)]\,L(n,\beta,\gamma)}{(1-\zeta)\,(2\alpha\lambda-2\lambda+2\alpha+1)} \end{aligned}$$

Then we get

$$\rho \leq 1 - \frac{2c\,(1-\zeta)\,[n+k(n+1)]}{[n+\xi-\xi\beta(n+1)](n+c+1)+(1-\xi)[1-\beta(n+1)]}$$

By a simple computation, we can show that the function

$$\phi(n) = 1 - \frac{(1-\xi)[1+\beta(n+1)]+cn}{(c+n+1)\,[n+\beta+k(n+1)]-c(1-\beta)[n+k(n+1)]}$$

is an increasing function of $n(n \geq 1)$ and $\phi(n) \geq \phi(1)$. Using this, we obtain the desired result.

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