

## ON SPECIAL FIXED POINTS OF A CONTINUOUS MULTIFUNCTION RELATED TO CONTINUOUS SELECTIONS

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**ABSTRACT.** In this paper, we study fixed point theory for a continuous multifunction with compact and convex values. The basic aim is to show some special characteristics of the fixed points in a finite dimensional Euclidean space. We also discover these special fixed points using continuous selections. We give more information about the location of the continuous selections and the fixed points, respectively.

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### 1. INTRODUCTION

Let  $\|\cdot\|$  be the Euclidean norm on  $R^n$  for  $n \geq 1$ ,  $d(x, y) = \|x - y\|$  be the Euclidean distance between  $x, y \in R^n$  and  $\tau$  be the Euclidean topology induced by  $d$ . Members of  $\tau$  are called open sets, their complements are called closed sets. It is not hard to see that the Euclidean norm is strictly convex.

The existences of fixed points for single- and multi-valued mappings are obtained from Luitzen Brouwer in 1912 and Shizuo Kakutani in 1941, respectively. Brouwer's and Kakutani's Fixed Point Theorems are two celebrated results in mathematics.

Brouwer's Fixed Point Theorem [5] is well known in the following more general version: "Let  $h : S \rightarrow S$  be a continuous function from a nonempty compact and convex set  $S \subset R^n$  into itself, then  $h$  has a fixed point, i.e. there exists a point  $x \in S$  such that  $x = h(x)$ ".

We also know Kakutani's Fixed Point Theorem [9] in its classical variant: "Let  $S \subset R^n$  be a nonempty compact and convex set, and  $\varphi : S \rightrightarrows S$  be an upper semi-continuous multifunction with compact and convex values, then  $\varphi$  has a fixed point, i.e. there exists a point  $x \in S$  such that  $x \in \varphi(x)$ ". Here we note that every value of a multifunction is a nonempty set, i.e.  $\varphi(x)$  is nonempty for all  $x \in S$ .

These two theorems show the existence of fixed points, but they give no information about the uniqueness, determination or location of the solutions.

Additionally, recall some definitions regarding the fixed point property.

**Definition 1.** The set  $S$  is said to have the fixed point property if and only if every continuous function  $h : S \rightarrow S$  from this set into itself has a fixed point, i.e. there is a point  $x \in S$  such that  $x = h(x)$ .

**Definition 2.** The set  $S$  is said to have the Kakutani fixed point property if and only if every upper semi-continuous multifunction with compact and convex values  $\varphi : S \rightrightarrows S$  has a fixed point, i.e. there is a point  $x \in S$  such that  $x \in \varphi(x)$ .

The study of fixed points for a multifunction in a finite dimensional Euclidean space was initiated by Kakutani in 1941. In [3 - Theorem 8], it is proven that if  $S \subset R^n$  is a nonempty and compact set, and  $\varphi : S \rightrightarrows S$  is an upper semi-continuous multifunction with compact values, then  $\varphi$  has a fixed compact set, i.e. there exists a nonempty and compact subset  $K \subset S$  such that  $K = \varphi(K)$ . In [10] and [11], the authors discuss the fixed point properties related to multi-valued mappings on a paracompact convex subset of a locally convex linear topological space. In [10], the authors consider the existence of interior fixed points in Euclidean maps.

Fixed point theory has become one of the most useful and powerful tools in optimization theory, see [6] for more details. Many questions in optimization can be reduced to the analysis of a fixed point problem. We can see some applications of fixed point theory in multi-criteria optimization in [15] and [16].

Now, we will focus our attention on a nonempty compact and convex set  $S \subset R^n$ , and a continuous multifunction  $\varphi : S \rightrightarrows S$  with compact and convex values. According to Kakutani's theorem  $\varphi$  has a fixed point. But, we will add an extra condition, that is,  $\varphi$  is lower semi-continuous on  $S$ . It follows that this condition may give some special characteristics of the fixed points and we will discuss this problem.

The aim of this paper is to show some special characteristics of the fixed points in a finite dimensional Euclidean space. We wish to discover these special fixed points using continuous selections. We will also give more information and some new facts on the location of the continuous selections and the special fixed points, respectively.

The paper is organized as follows. Section 2 describes some definitions, concepts and notions which we use in our study. Section 3 presents our three new theorems for continuous selections and fixed points.

## 2. DEFINITIONS, CONCEPTS AND NOTIONS

When we talk about a multifunction in this paper it can be upper semi-continuous, lower semi-continuous or continuous. So, we recall some topological definitions and their equivalent statements for a multifunction.

**Remark 1.** Consider a multifunction  $\varphi : S \rightrightarrows R^n$ .

(a)  $\varphi$  is called upper semi-continuous (briefly usc) at a point  $x \in S$  if and only if for each open set  $V \subset R^n$  such that  $\varphi(x) \subset V$ , there exists a set  $U$  of  $\tau$  containing  $x$  such that  $y \in U \cap S$  implies  $\varphi(y) \subset V$ .  $\varphi$  is usc on  $S$  if and only if  $\varphi$  is usc at each  $x \in S$ . This is equivalent to "The multifunction  $\varphi : S \rightrightarrows R^n$  is upper semi-continuous at a point  $x \in S$  if and only if  $\{x_k\}_{k=1}^\infty \subset S$  and  $\{y_k\}_{k=1}^\infty \subset \varphi(S)$  are a pair of sequences such that  $\lim_{k \rightarrow \infty} x_k = x$  and  $y_k \in \varphi(x_k)$  for all  $k \in N$ , then there exists a convergent subsequence of  $\{y_k\}_{k=1}^\infty$  whose limit belongs to  $\varphi(x)$ ".

(b)  $\varphi$  is called lower semi-continuous (briefly lsc) at a point  $x \in S$  if and only if for each open set  $V \subset R^n$  such that  $\varphi(x) \cap V \neq \emptyset$ , there exists a set  $U$  of  $\tau$  containing  $x$  such that  $y \in U \cap S$  implies  $\varphi(y) \cap V \neq \emptyset$ .  $\varphi$  is lsc on  $S$  if and only if  $\varphi$  is lsc at each  $x \in S$ . This is equivalent to "The multifunction  $\varphi : S \rightrightarrows R^n$  is lower semi-continuous at a point  $x \in S$  if and only if  $\{x_k\}_{k=1}^\infty \subset S$  is a sequence convergent to  $x$  and  $y \in \varphi(x)$ , then there exists a sequence  $\{y_k\}_{k=1}^\infty \subset \varphi(S)$  such that  $y_k \in \varphi(x_k)$  for all  $k \in N$  and  $\lim_{k \rightarrow \infty} y_k = y$ ".

(c)  $\varphi$  is called continuous at a point  $x \in S$  if and only if  $\varphi$  is both usc and lsc at  $x \in S$ .  $\varphi$  is continuous on  $S$  if and only if  $\varphi$  is continuous at each  $x \in S$ .

As was mentioned before, every nonempty compact and convex set has the fixed point property and the Kakutani fixed point property. The Fixed point property and the Kakutani fixed point property are topological properties. These properties of sets are preserved under homeomorphism and retraction [8] [16].

From a mathematical point of view, a natural extension of the continuity of a function is the concept to the upper semi-continuity of a multifunction. So, we understand that a natural generalization of Brouwer's Fixed Point Theorem is Kakutani's Fixed Point Theorem. We can also see that Kakutani's theorem is a multifunction analog of Brouwer's theorem.

**Remark 2.** The Kakutani fixed point property is very closely related to the fixed point property. The key idea is very simple. Observe that every point-to-point multifunction is a function. If  $S \subset R^n$  has the Kakutani fixed point property, then since any continuous function from  $S$  into itself can be viewed as an upper semi-continuous multifunction with compact and convex values it follows that set  $S$  will also have the fixed point property.

**Remark 3.** Let  $S \subset R^n$  be compact. It is proven that set  $S$  having the Kakutani fixed point property is equivalent to  $S$  having the fixed point property [4] [16].

Let  $S \subset R^n$  be nonempty. A point  $x \in R^n$  is called a boundary point of  $S$  if and only if every neighborhood  $N(x) \in \tau$  contains a point in  $S$  and a point in  $R^n \setminus S$ , i.e.  $N(x) \cap S \neq \emptyset$  and  $N(x) \cap (R^n \setminus S) \neq \emptyset$ . The set of all boundary points of  $S$  is denoted by  $bdS$ . The set  $clS = S \cup bdS$  is called the topological closure of  $S$  and  $clS$  is closed. In other words,  $clS$  is the intersection of all closed sets contains  $S$ . A point  $x \in S$  is called an interior point of  $S$  if and only if there is a neighborhood  $N(x) \in \tau$  such that  $N(x) \subset S$ . The set of all interior points of  $S$  is denoted by  $intS$ , i.e.  $intS$  is the union of all sets  $U \in \tau$  such that  $U \subset S$  and  $intS$  is open. It should be noted that  $bdS = clS \setminus intS$  and  $intS = clS \setminus bdS$ .

The convex hull of  $S \subset R^n$  is the convex set which is the intersection of all convex sets which contain  $S$  and it is denoted by  $convS$ .

**Remark 4.** Let  $S \subset R^n$  be a nonempty and bounded set. A point  $z(z_1, z_2, \dots, z_n) \in R^n$  is called a peak point for  $S$  if and only if for each  $i \in \{1, 2, \dots, n\}$  either  $z_i = \sup\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$  or  $z_i = \inf\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$ . Point  $\bar{z}(z_1, z_2, \dots, z_n) \in R^n$  is called the first peak point for  $S$  if and only if for each  $i \in \{1, 2, \dots, n\}$ ,  $\bar{z}_i = \sup\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$ . Here the interest is the set of all peak points and its convex hull. Denote the set of all peak points for  $S$  by  $P(S)$ ,  $convP(S)$  is called a proper hypercuboid which contains  $S$ . It is easy to show that  $\bar{z} \in P(S)$  and  $N = |P(S)| = 2^n$ . Clearly, it is possible to have coincidence of two or more peak points.

Note that if  $S \subset R^n$  is closed, then  $P(S) \cap S \neq \emptyset$  is possible, but if  $S$  is open, then  $P(S) \cap S = \emptyset$ .

**Remark 5.** Let  $S \subset R^n$  be a nonempty and bounded set. A point  $c \in R^n$  is called a central point for  $S$  if and only if  $d(c, z_1) = d(c, z_2)$  for all  $z_1, z_2 \in P(S)$ . It is easy to conclude from  $convP(S) = \prod_{i=1}^n [a_i, b_i]$  where  $a_i = \inf\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$ ,  $b_i = \sup\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$  and  $[a_i, b_i] = \{x \in R^n | x = (1-t).a_i + t.b_i, t \in [0, 1]\}$  is a closed linear segment with ends  $a_i$  and  $b_i$  that the proper hypercuboid  $convP(S)$  is the smallest hypercuboid which contains  $S$  and the set  $S$  has a unique central point  $c \in convP(S)$ . In this work every hypercuboid has edges parallel to the axes and its adjacent faces are orthogonal.

**Remark 6.** It is important to note the relationship between the fixed point theory and selection theory, for details see [12], [13] and [14]. The research in selection theory was started by Ernest Michael in 1956 with the proof of several continuous selection theorems [14]. Consider a multifunction  $\varphi : S \rightrightarrows R^m$ ,  $S \subset R^n$  and  $m \geq 1$ .

(1) By a continuous selection  $f$  for  $\varphi$ , we mean a continuous function  $f : S \rightarrow \varphi(S)$  such that  $f(x) \in \varphi(x)$  for all  $x \in S$ .

(2) Let  $\varphi(x)$  have a central point for all  $x \in S$ . By a central selection  $f$  for  $\varphi$ , we

mean a function  $f : S \rightarrow \varphi(S)$  such that  $f(x) \in \varphi(x)$  and  $f(x)$  is the central point for  $\varphi(x)$  for all  $x \in S$ .

(3) Let  $\varphi$  have compact values. By a boundary selection  $f$  for  $\varphi$ , we mean a function  $f : S \rightarrow \varphi(S)$  such that  $f(x) \in bd\varphi(x)$  for all  $x \in S$ .

(4) Let  $int\varphi(x)$  be nonempty for all  $x \in S$ . By an interior selection  $f$  for  $\varphi$ , we mean a function  $f : S \rightarrow \varphi(S)$  such that  $f(x) \in int\varphi(x)$  for all  $x \in S$ .

Observe that if  $\bigcap_{x \in S} \varphi(x)$  is nonempty and  $y \in \bigcap_{x \in S} \varphi(x)$ , then  $f(x) = y$  for all  $x \in S$  is a continuous selection for  $\varphi$ .

**Example 1.** Consider a multifunction  $\varphi : [-1, +1] \rightrightarrows [-1, +1]$  such that

$$\varphi(x) = \begin{cases} \{-1\}, & x \in [-1, 0) \\ [-1, +1], & x = 0 \\ \{+1\}, & x \in (0, +1] \end{cases} .$$

It is easy to show that  $\varphi$  is an upper semi-continuous multifunction on  $[-1, +1]$  with compact and convex values, but it is not lower semi-continuous on  $[-1, +1]$ . This example shows that  $\varphi$  has a Kakutani fixed point and no continuous selection.

**Example 2.** Consider a multifunction  $\varphi : [0, +1] \rightrightarrows [0, +1]$  such that

$$\varphi(x) = \begin{cases} \{0\}, & x = 0 \\ [0, +1], & x \in (0, +1] \end{cases} .$$

In this case  $\varphi$  is a lower semi-continuous multifunction on  $[0, +1]$  with compact and convex values, but it is not upper semi-continuous on  $[0, +1]$ . One can easily see that  $\varphi$  has a continuous selection and a fixed point, for more information see the so-called Michael's theorem in [2].

It is known that there exists a continuous multifunction from a unit ball in  $R^2$  into its compact subset that has neither a continuous selection nor fixed points [1].

### 3. MAIN RESULTS

In this section, we establish our three theorems which show the existences of three different types of continuous selections and three different types of fixed points, respectively.

**Theorem 1.** Let  $S \subset R^n$  be a nonempty compact and convex set, and  $\varphi : S \rightrightarrows S$  be a continuous multifunction with compact and convex values. The following statements are true.

- (a) There exists a central continuous selection  $f_c$  for  $\varphi$  and it is unique.
- (b)  $\varphi$  has a central fixed point, i.e. there exists a point  $x \in S$  such that  $x \in \varphi(x)$  and  $x$  is the central point for  $\varphi(x)$ .

**Remark 7.** We often use the Berge's Maximum Theorem as a mathematical tool in optimization and nonlinear analysis. In order to give the proof of Theorem 1 we

will use this theorem in its classical variant: "Let  $S \subset R^n$  and  $\Theta \subset R^m$ ,  $g : S \times \Theta \rightarrow R$  be a continuous function, and  $D : \Theta \rightrightarrows S$  be a compact-valued and continuous multifunction. Then, the function  $g^* : \Theta \rightarrow R$  defined by  $g^*(\theta) = \max\{g(x, \theta) | x \in D(\theta)\}$  is continuous on  $\Theta$ , and the multifunction  $D^* : \Theta \rightrightarrows S$  defined by  $D^*(\theta) = \{x \in D(\theta) | g(x, \theta) = g^*(\theta)\}$  is compact-valued and upper semi-continuous on  $\Theta$ " [3] [17]. Let us assume that  $S = \Theta$ . In this case we obtain the following statement: "If  $S \subset R^n$ ,  $g : S \times S \rightarrow R$  is continuous and  $D : S \rightrightarrows S$  is continuous and compact-valued, then  $g^* : S \rightarrow R$  defined by  $g^*(x) = \max\{g(y, x) | y \in D(x)\}$  is continuous, and  $D^* : S \rightrightarrows S$  defined by  $D^*(x) = \{y \in D(x) | g(y, x) = g^*(x)\}$  is upper semi-continuous and compact-valued".

**Remark 8.** Let  $S \subset R^n$  be a nonempty closed and convex set, and  $x \in R^n$ . A point  $y \in S$  is called a metric projection of  $x$  onto  $S$  if and only if  $d(x, y) = d(x, S) = \inf\{d(x, s) | s \in S\}$ . The projection of  $x$  onto  $S$  is denoted by  $y = \pi_S(x)$ . It is known that the function  $\pi_S : R^n \rightarrow S$  is continuous and nonexpansive, that is,  $\|\pi_S(x_1) - \pi_S(x_2)\| \leq \|x_1 - x_2\|$  for all  $x_1, x_2 \in R^n$  [7].

**Remark 9.** Let  $S \subseteq R^n$  be a nonempty and bounded set,  $\varphi : S \rightrightarrows S$  be a continuous multifunction with compact and convex values,  $Q = \varphi(S)$  and  $P(Q) = \{z^1, z^2, \dots, z^N\}$ . So, for each  $z^k \in P(Q)$  and for each  $x \in S$  there is a peak point  $z_x^k \in P(\varphi(x))$  such that  $z^k$  corresponds to  $z_x^k$ , i.e.  $d(z^k, z_x^k) = d(z^k, P(\varphi(x)))$ . It is easy to show that for every  $i \in \{1, 2, \dots, n\}$ ,  $(z^k)_i$  is a result of supremum is equivalent to  $(z_x^k)_i$  is a result of maximum, and  $(z^k)_i$  is a result of infimum is equivalent to  $(z_x^k)_i$  is a result of minimum. Thus, for each  $z^k \in P(Q)$  we get a collection  $\{z_x^k\}_{x \in \varphi(x)}$  such that  $z^k$  corresponds to  $z_x^k$  for all  $x \in S$ . If  $\bar{z} \in P(Q)$  is the first peak point for  $S$ , then for each  $x \in S$ ,  $\bar{z}_x \in P(\varphi(x))$  is the first peak point for  $\varphi(x)$ . When we say  $k$ -peak point or  $z^k$ -peak point, then we will understand that we have  $z^k$  or  $z_x^k$  for some  $x \in S$ , or the whole lot.

**Remark 10.** A hyperplane  $H$  in  $R^n$  is an  $(n - 1)$ -dimensional affine subset of  $R^n$  for  $n \geq 2$ , that is,  $H = \{x \in R^n | L(x) = \alpha\}$  is the level set of a nontrivial linear function  $L : R^n \rightarrow R$ . If  $L$  is given by  $L(x) = \langle \lambda, x \rangle$  for some  $\lambda \in R^n$ , then  $H(\lambda, \alpha) = \{x \in R^n | \langle \lambda, x \rangle = \alpha\}$ . We also consider two open half-spaces  $H^+(\lambda, \alpha) = \{x \in R^n | \langle \lambda, x \rangle > \alpha\}$  and  $H^-(\lambda, \alpha) = \{x \in R^n | \langle \lambda, x \rangle < \alpha\}$ , and two closed half-spaces  $H^{+0}(\lambda, \alpha) = \{x \in R^n | \langle \lambda, x \rangle \geq \alpha\}$  and  $H^{-0}(\lambda, \alpha) = \{x \in R^n | \langle \lambda, x \rangle \leq \alpha\}$  [7].

**Remark 11.** Let  $S \subset R^n$  be a nonempty and compact set,  $c \in \text{conv}P(S)$  be the central point for  $S$ , and  $H$  be a hyperplane in  $R^n$  such that either  $S \cap H$  is empty or  $\emptyset = S \cap H \subset H$ . One can easily see that:  $S \subset H^{+0}$  is equivalent to  $c \in H^{+0}$ , and  $S \subset H^{-0}$  is equivalent to  $c \in H^{-0}$ , i.e.  $S$  and  $c$  lie in the same closed half-space of  $H$ , otherwise we get that  $\text{conv}P(S)$  is not the smallest hypercuboid which constants

$S$ .

The above remarks allow us to prove the first theorem.

**Proof of Theorem 1.** (a) Without loss of generality let us choose the first peak point  $\bar{z} \in P(Q)$  as one peak point for  $Q = \varphi(S)$ . So, for each  $i \in \{1, 2, \dots, n\}$  we define a function  $m_{\bar{z},i} : S \rightarrow R$  by  $m_{\bar{z},i}(x) = \max\{y_i | y(y_1, y_2, \dots, y_n) \in \varphi(x)\}$  for all  $x \in S$ . It is easy to see that these functions  $\{m_{\bar{z},i}\}_{i=1}^n$  are continuous on  $S$ , see Berge's Maximum Theorem in Remark 7. As a result we get a continuous function  $m_{\bar{z}} : S \rightarrow \text{conv}P(Q)$  such that  $m_{\bar{z}}(x) = (m_{\bar{z},1}, m_{\bar{z},2}, \dots, m_{\bar{z},n})$  for all  $x \in S$ . Observe that for each  $x \in S$ ,  $m_{\bar{z}}(x)$  is a first peak point for  $\varphi(x)$ .

More generally, for each  $z \in P(Q)$  we construct a continuous function  $m_z : S \rightarrow \text{conv}P(Q)$  such that if  $z_i = \sup\{x_i | x(x_1, x_2, \dots, x_n) \in P(Q)\}$ , then  $m_{z,i}(x) = \max\{y(y_1, y_2, \dots, y_n) \in \varphi(x)\}$  and if  $z_i = \inf\{x_i | x(x_1, x_2, \dots, x_n) \in P(Q)\}$ , then  $m_{z,i}(x) = \min\{y(y_1, y_2, \dots, y_n) \in \varphi(x)\}$ . On the one hand, it is easy to show that  $m_z(x)$  is continuous on  $S$ . On the other hand, for each  $x \in S$ ,  $m_z(x)$  is a peak point for  $\phi(x)$  which corresponds to  $z$ .

In summary, we get a collection of peak continuous functions  $\{m_k\}_{k=1}^N$  such that  $m_k : S \rightarrow \text{conv}P(Q)$  and for each  $x \in S$ ,  $m_k(x)$  is a peak point for  $\varphi(x)$ .

Now let us define a function  $f_c : S \rightarrow \text{conv}P(Q)$  such that  $f_c(x) = \frac{1}{N} \sum_{k=1}^N m_k(x)$  for all  $x \in S$ , see also Remark 9. In view of the above results, we have that  $f_c$  is continuous on  $S$  and for each  $x \in S$ ,  $f_c(x)$  is the unique central point for  $\varphi(x)$ .

We will prove that  $f_c(x) \in \varphi(x)$  for all  $x \in S$ .

There are two cases.

Case 1. Let  $n = 1$ .

In this case we have that  $S = [a_0, b_0]$ ,  $Q = [a, b]$ ,  $\varphi(x) = [a_x, b_x]$ ,  $a_0 \leq a \leq a_x \leq b_x \leq b \leq b_0$  and  $f_c(x) = \frac{1}{2}(a_x + b_x)$  for all  $x \in S$ . As a result we derive that  $f_c(x) \in \varphi(x)$  for all  $x \in S$ .

Case 2. Let  $n \geq 2$ .

Let us fix  $x \in S$  and assume that  $f_c(x) \notin \varphi(x)$ . In this case, let  $p(x) = \pi_{\varphi(x)}(f_c(x)) \in \varphi(x)$ , i.e.  $p(x)$  be a projection to  $f_c(x)$  onto  $\varphi(x)$ . It is easy to show that  $p(x) \neq f_c(x)$ . Now consider a separating hyperplane

$$H = \{y \in R^n | \langle p(x) - f_c(x), \frac{1}{2}(p(x) + f_c(x)) - y \rangle = 0\}.$$

An easy computation shows that  $\varphi(x) \cap H$  is empty, and  $\varphi(x)$  and  $f_c(x)$  are not in the same closed half-space of  $H$ . This leads to a contradiction, see Remarks 10 and 11. As a result we also derive that  $f_c(x) \in \varphi(x)$ .

Finally,  $f_c(x)$  is the central point for  $\varphi(x)$  and  $f_c(x) \in \varphi(x)$ , i.e.  $f_c$  is a central continuous selection for  $\varphi$  and it is unique.

(b) Let  $f_c$  be the central continuous selection for  $\varphi$ , see item (a). According to Brouwer's Fixed Point Theorem it follows that there exists a fixed point  $x \in S$  such that  $x = f_c(x)$ . Thus, we obtain  $x = f_c(x) \in \varphi(x)$ , i.e. there exists a point  $x \in S$

such that  $x \in \varphi(x)$  and  $x$  is the central point for  $\varphi(x)$ .

The theorem is proven.

**Remark 12.** From Remark 9 and Theorem 1 it follows that when we think about  $k$ -peak point in general, we can understand that we also think about the  $k$ -peak continuous function  $m_k : S \rightarrow \text{conv}P(Q)$ ,  $k \in \{1, 2, \dots, N\}$ , i.e. they are equivalent.

**Remark 13.** Let  $S \subset R^n$  be a nonempty and bounded set. A point  $z' \in P(S)$  is called an antipodal point of  $z \in P(S)$  if and only if for each  $i \in \{1, 2, \dots, n\}$ ,  $z'_i = \sup\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$  is equivalent to  $z_i = \inf\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$  and  $z'_i = \inf\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$  is equivalent to  $z_i = \sup\{x_i | x(x_1, x_2, \dots, x_n) \in S\}$ . It is easy to show that  $z'' = z$  and for the central point  $c$  for  $S$  we have  $c = \frac{1}{2}(z + z')$  for all  $z \in P(S)$ . We note first that, for each  $x \in S$  we obtain  $c(x) = \frac{1}{2}(m_z(x) + m_{z'}(x))$  and second, if  $z'_x = z_x$ , then  $|\varphi(x)| = 1$ .

Continuing with our analysis we introduce the following theorem.

**Theorem 2.** Let  $S \subset R^n$  be a nonempty compact and convex set, and  $\varphi : S \Rightarrow S$  be a continuous multifunction with compact and convex values. The following statements are true.

- (a) For each  $z \in P(S)$  there exists a boundary continuous selection  $f_z$  for  $\varphi$ .
- (b)  $\varphi$  has a boundary fixed point, i.e. there exists a point  $x \in S$  such that  $x \in \text{bd}\varphi(x)$ .

**Proof.** (a) In the proof of Theorem 1 for each  $z \in P(Q)$  we construct a continuous function  $m_z : S \rightarrow \text{conv}P(Q)$  such that  $m_z(x)$  is a peak point for  $\varphi(x)$  which corresponds to  $z$  for all  $x \in S$ .

Let us fix  $z \in P(Q)$  and define a function  $g : [0, 1] \times S \rightarrow R$  by  $g(t, x) = m_{z'} + t.(m_z(x) - m_{z'}(x))$  for all  $t \in [0, 1]$  and for all  $x \in S$ . Observe that  $g$  is continuous on  $[0, 1] \times S$ .

According to Theorem 1 there exists the unique central continuous selection  $f_c$  and  $f_c(x) \in \varphi(x)$  for all  $x \in S$ . Thus, we find that  $f_c(x) \in g([0, 1], x) \cap \varphi(x)$  for all  $x \in S$ , see Remark 12.

It is easy to conclude from  $\varphi(x)$  is nonempty compact and convex for all  $x \in S$  that there exists a point  $t_s \in [0, 1]$  such that  $g(t_s, x) = \max\{g(t, x) | t \in [0, 1]\}$ . By using Berge's Maximum Theorem, we obtain that the function  $g^*(t)$  is continuous on  $[0, 1]$  and the multifunction  $\varphi^*(x) = \{y \in \varphi(x) | g(t, x) = g^*(t)\}$  is compact-valued and upper semi-continuous on  $S$ . But, the construction of function  $g$  implies that  $\varphi^*(x) \in \text{bd}\varphi(x)$  and  $|\varphi^*(x)| = 1$  for all  $x \in S$ . As a result we deduce that  $f_z = \varphi^*$  is a boundary continuous selection for  $\varphi$ .

(b) Let  $z \in P(S)$  and  $f_z$  be a boundary continuous selection for  $\varphi$ , see item (a). According to Brouwer's Fixed Point Theorem we get that there exists a fixed point



$x \in S$  such that  $x = f_z(x)$ . Thus, we obtain  $x = f_z(x) \in bd\varphi(x)$ , i.e. there exists a point  $x \in S$  such that  $x \in bd\varphi(x)$ .

The theorem is proven.

**Example 3.** Consider a multifunction  $\varphi : [0, 2] \rightrightarrows [0, 2]$  such that

$$\varphi(x) = \begin{cases} [0, 1], & x \in [0, 2] \setminus \{1\} \\ [0, 2], & x = 1 \end{cases}.$$

It is easy to show that  $\varphi$  is an upper semi-continuous multifunction on  $[0, 2]$  with compact and convex values, but it is not lower semi-continuous. This example also shows that  $\varphi$  has a Kakutani fixed point and a central fixed point, but it has no central continuous selection. If we choose point 2 as a peak point for  $[0, 2]$ , then  $\varphi$  has no boundary fixed point. Hence, the fact that  $\varphi$  is lower semi-continuous on  $S$  is very important for existence of central and boundary continuous selections and special fixed points.

**Remark 14.** Let  $S \subset R^n$  be a nonempty and bounded set. Clearly, if  $convP(S)$  has no interior point, then there exists a hyperplane  $H$  such that  $convP(S) \subset H$  and if  $convP(S)$  has a interior point, then there is no hyperplane  $H$  such that  $convP(S) \subset H$ .

In the end we will prove the third theorem in this paper. It has an extra condition for interior point.

**Theorem 3.** Let  $S \subset R^n$  be a nonempty compact and convex set,  $\varphi : S \rightrightarrows S$  be a continuous multifunction with compact and convex values, and  $int\varphi(x)$  be nonempty for all  $x \in S$ . The following statements are true.

- (a) There exists an interior continuous selection  $f$  for  $\varphi$ .
- (b)  $\varphi$  has an interior fixed point, i.e. there exists a point  $x \in S$  such that  $x \in int\varphi(x)$  and  $x \in intS$ .

**Proof.** (a) In Theorem 2 we proved that for each  $z \in P(Q)$  there exists a boundary continuous selection  $f_z$  for  $\varphi$ . Now, in this new theorem, we conclude that the peak points for  $Q$  are different and from the construction of their boundary selections it follows that they are different too. As a result we get a collection  $\{f_i\}_{i=1}^N$  of the different boundary continuous selections for  $\varphi$ .

Define a function  $f : S \rightarrow Q$  such that  $f(x) = \frac{1}{N} \sum_{i=1}^N f_i(x)$  for all  $x \in S$ . From  $\varphi(x)$  is convex it follows that  $f(x) \in \varphi(x)$  for all  $x \in S$ .

In summary, we obtain  $f$  is a continuous selection for  $\varphi$ .

We will prove that  $f$  is an interior continuous selection for  $\varphi$ , i.e.  $f(x) \in int\varphi(x)$  for all  $x \in S$ .

There are two cases.

Case 1. Let  $n = 1$ .

In this case we have that  $S = [a_0, b_0]$ ,  $Q = [a, b]$ ,  $\varphi(x) = [a_x, b_x]$ ,  $a_0 \leq a \leq a_x < b_x \leq b \leq b_0$  and  $f(x) = \frac{1}{2}(a_x + b_x)$  for all  $x \in S$ . As a result we obtain  $f(x) \in \text{int}\varphi(x)$  for all  $x \in S$ .

Case 2. Let  $n \geq 2$ .

Let us fix  $x \in S$  and assume that  $f(x) \in \text{int}\varphi(x)$ . In this case, since  $f(x) \in \varphi(x)$ ,  $\text{int}\varphi(x)$  is nonempty and  $\varphi(x)$  is a compact and convex set, we conclude that  $f(x) \in \text{bd}\varphi(x)$  and there exists a supporting hyperplane  $H(\lambda, \alpha) = \{x \in R^n \mid \langle \lambda, x \rangle = \alpha\}$  of  $\varphi(x)$  at  $f(x)$  such that  $f(x) \in H \cap \text{bd}\varphi(x)$  and either  $\varphi(x) \subset H^{+0}$  or  $\varphi(x) \subset H^{-0}$ .

Without loss of generality let us assume that  $\varphi(x) \subset H^{+0}$ . So, we have that  $f_k(x) \in H^{+0} \cap \text{bd}\varphi(x)$  for all  $z^k \in P(Q)$ ; therefore,  $\langle \lambda, f_k(x) \rangle \geq \alpha$  for all  $z^k \in P(Q)$ . But, we know that  $\langle \lambda, f(x) \rangle = \alpha$ ; therefore, we obtain  $\langle \lambda, f_k(x) \rangle = \alpha$  for all  $z^k \in P(Q)$ ; i.e.  $f_k(x) \in H$ .

In Theorem 1 we proved that  $f_c(x) \in \varphi(x)$ ; therefore,  $f_c(x) \in H^{+0}$ . Observe  $f_c(x) \in [f_z(x), f_{z'}(x)] \subset [z, z']$  along the linear segment  $[z, z'] = \{y \in R^n \mid y = (1-t) \cdot z + t \cdot z', t \in [0, 1]\}$ . As a result we obtain  $f_c(x) \in H$ ; therefore,  $P(\varphi(x)) \subset H$ . This means that  $\text{conv}P(\varphi(x)) \subset H$ ; therefore, we have a contradiction, see also Remark 14.

Finally, we get  $f(x) \in \text{int}\varphi(x)$  for all  $x \in S$ , i.e.  $f$  is an interior continuous selection for  $\varphi$ .

(b) Let  $f$  be an interior continuous selection for  $\varphi$ , see item (a). According to Brouwer's Fixed Point Theorem we have that there exists a fixed point  $x \in S$  such that  $x = f(x)$ . Thus, we obtain  $x = f(x) \in \varphi(x)$ , i.e. there exists an interior fixed point  $x \in S$ .

From  $\varphi(x) \subset S$  for all  $x \in S$  and for fixed point  $x \in \text{int}\varphi(x)$  it follows that  $x \in \text{int}S$ .

The theorem is proven.

#### REFERENCES

- [1] J. Aubin, A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer, 1984.
- [2] J. Aubin, H. Frankowska, *Set-Valued Analysis*, Birkhauser Boston, 1990.
- [3] C. Berge, *Topological Spaces, Including Treatment of Multi-Valued Function, Vector Spaces, and Convexity*, Oliver and Boyd, Edinburgh, 1963.
- [4] J. Borwein, A. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*. Springer, 2000.
- [5] L. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Mathematische Annalen (71) 1912, 97-115.

- [6] H. Corley, *Some Hybrid Fixed Point Theorems Related to Optimization*, Journal of Mathematical Analysis and Applications (120) 1986, 528-532.
- [7] ] O. Guler, *Foundations of Optimization*, Springer, 2010.
- [8] A. Hatcher, *Algebraic Topology*. Cambridge University Press, 2002.
- [9] S. Kakutani, *A Generalization of Brouwer's Fixed Point Theorem*, Duke Mathematical Journal (8) 1941, 457-459.
- [10] N. Khamsemanan, R. Brown, C. Lee, S. Dhompangsa, *Interior Fixed Points of Unit-Sphere-Preserving Euclidean Maps*, Fixed Point Theory and Applications 2012, 1-8. doi:10.1186/1687-1812-2012-183
- [11] H. Komiya, *Fixed Point Properties Related to Multi-valued Mappings*, Fixed Point Theory and Applications 2010, 1-4. doi:10.1155/2010/ 581728
- [12] S. Park, *Applications of Michael's Selection theorems to Fixed Point Theory*, Topology and Applications 155 (2008), 861-870. doi:10.1016/j.topol. 2007.02.017
- [13] A. Petrusel, *Fixed Points and Selections for Multi-valued Operators*, Seminar on Fixed Point Theory Cluj-Napoca (2) 2001, 3-22.
- [14] D. Repovš, P. Semenov, *Continuous Selections of Multivalued Mapping*, Springer, 1998.
- [15] Z. Slavov, *The Fixed Point Property in Convex Multi-Objective Optimization Problem*, ACTA UNIVERSITATIS APULENSIS (15) 2008, 405-414.
- [16] Z. Slavov, C. Evans, *Compactness, Contractibility and Fixed Point Properties of the Pareto Sets in Multi-Objective Programming*, Applied Mathematics (2) 2011, 556-561. doi:10.4236/am.2011.25073
- [17] Z. Slavov, C. Evans, *An Application of the Maximum Theorem in Multi-Criteria Optimization*, Properties of Pareto-Retract Mapping, and the Structure of Pareto Sets, Applied Mathematics (3) 2012, 1415-1422. doi:10.4236/am.2012.330199

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