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ON THE UNIVALENCE OF GENERAL INTEGRAL OPERATOR

B.A. Frasin, T. Al-Hawary, F. Yousef

Abstract. The object of the present paper is to obtain new univalence conditions for the general integral operator $I_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{j=1}^n \left(\frac{f_j(t)}{g_j(t)}\right)^{\alpha_j} \left(\frac{f_j'(t)}{g_j'(t)}\right)^{\beta_j} dt\right]^{\frac{1}{\delta}}$ defined in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ where the functions f_j, g_j are analytic in $\mathcal{U}$, α_j, β_j and δ are complex numbers.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\gamma)$ if it satisfies

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 - \gamma, \tag{1}$$

for some $\gamma (0 \leq \gamma < 1)$ and for all $z \in \mathcal{U}$. The class $\mathcal{B}(\gamma)$ was introduced and studied by Frasin and Darus [13].

Ponnusamy and Sing [20] studied the subclass $\mathcal{S}_\mu$ of analytic functions defined as follows

$$\mathcal{S}_\mu = \left\{ f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \mu\,|z|\,,\ 0 < \mu \leq 1,\ z \in \mathcal{U} \right\}. \tag{2}$$

The problem of finding sufficient conditions for univalence of various integral operators has been investigated in many recent works (see, for example, [2, 4, 5, 6, 8, 10, 16, 17, 18]).

In [19] (see also [1]), Pescar obtained new univalence criteria for the integral operator defined by

$$I_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{g_j(t)} \right)^{\alpha_j} \left(\frac{f_j'(t)}{g_j'(t)} \right)^{\beta_j} dt \right]^{\frac{1}{\delta}}, \tag{3}$$

where the functions $f_j, g_j \in \mathcal{A}, \alpha_j, \beta_j \in \mathbb{C}$; $j = 1, \ldots, n$, $n \in \mathbb{N}$ and $\delta \in \mathbb{C} \backslash \{0\}$. Here and throughout in the sequel every many-valued function is taken with the principal branch.

We observe that the above integral operator $I_n(z)$ generalizes some integral operators introduced by several researchers, for example.

1. If $g_j(z) = z$; $j = 1, \ldots, n$, then the integral operator $I_n(z)$ reduces to the integral operator

$$F_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{t} \right)^{\alpha_j} \left(f_j'(t) \right)^{\beta_j} dt \right]^{\frac{1}{\delta}}, \tag{4}$$

introduced by Frasin [12] (see also [11]).

2. If $\beta_j = 0$; $j = 1, \ldots, n$, then the integral operator $I_n(z)$ reduces to the integral operator

$$G_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{g_j(t)} \right)^{\alpha_j} dt \right]^{\frac{1}{\delta}}, \tag{5}$$

introduced by Moldoveanu et al. [14].

3. If $g_j(z) = z$ and $\beta_j = 0$; $j = 1, \ldots, n$, then the integral operator $I_n(z)$ reduces to the integral operator

$$D_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{t} \right)^{\alpha_j} dt \right]^{\frac{1}{\delta}}, \tag{6}$$

introduced by Breaz and Breaz [3].

In our present investigation, we study some univalence conditions for the integral operator $I_n(z)$ if the analytic functions $f_j(z)$ and $g_j(z)$ are in the classes $\mathcal{B}(\gamma_j)$ and $\mathcal{S}_{\mu_j}$; $j = 1, \ldots, n$.

In order to derive our main results, we need the following lemmas.

Lemma 1. *([7]) If $f(z) \in \mathcal{B}(\gamma)$, then*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{(1-\gamma)(1+|z|)}{1-|z|} \qquad (0 \le \gamma < 1,\ z \in \mathcal{U}). \tag{7}$$

Lemma 2. *([9]) If $f(z) \in \mathcal{B}(\gamma)$, then*

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{(1-\gamma)(2+|z|)}{1-|z|} \qquad (0 \le \gamma < 1,\ z \in \mathcal{U}). \tag{8}$$

Lemma 3. *([15]) Let $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$. If $k \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2Re(\lambda)}}{Re(\lambda)} \left| \frac{zk''(z)}{k'(z)} \right| \le 1,$$

for all $z \in \mathcal{U}$, then, for any complex number ζ, with $Re(\zeta) \ge Re(\lambda)$, the integral operator

$$F_\zeta(z) = \left\{ \zeta \int_0^z t^{\zeta-1} k'(t)\,dt \right\}^{\frac{1}{\zeta}}, \tag{9}$$

is in the class $\mathcal{S}$.

2. Univalence Conditions for the Integral Operator I_n

First, we give univalence conditions for the integral operator $I_n(z)$ where the analytic functions $f_j(z)$ and $g_j(z)$ are in the class $\mathcal{B}(\gamma_j)$; $j = 1, \ldots, n$.

Theorem 4. *Let the analytic functions $f_j(z)$ and $g_j(z)$ be in the class $\mathcal{B}(\gamma_j)$; $0 \le \gamma_j < 1$; $j = 1, \ldots, n$ and satisfy the inequality*

$$\sum_{j=1}^{n} (1-\gamma_j)(4|\alpha_j| + 6|\beta_j|) \le \begin{cases} a, & \text{if } 0 < a < \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} < a < \infty \end{cases} \tag{10}$$

where $\lambda \in \mathbb{C}$ with $Re(\lambda) = a > 0$, then the integral operator $I_n(z)$ defined by (3) is analytic and univalent in $\mathcal{U}$.

Proof. Define the regular function $H(z)$ by

$$H(z)=\int_0^z\left(\frac{f_1(t)}{g_1(t)}\right)^{\alpha_1}\cdots\left(\frac{f_n(t)}{g_n(t)}\right)^{\alpha_n}\left(\frac{f_1'(t)}{g_1'(t)}\right)^{\beta_1}\cdots\left(\frac{f_n'(t)}{g_n'(t)}\right)^{\beta_n}dt.$$

Clearly $H\in\mathcal{A}$, i.e. $H(0)=H'(0)-1=0$. On the other hand, it is easy to see that

$$\frac{zH''(z)}{H'(z)}=\sum_{j=1}^{n}\alpha_j\left[\left(\frac{zf_j'(t)}{f_j(t)}-\frac{zg_j'(t)}{g_j(t)}\right)+\beta_j\left(\frac{zf_j''(t)}{f_j'(t)}-\frac{zg_j''(t)}{g_j'(t)}\right)\right], \tag{11}$$

or, equivalently,

$$\frac{zH''(z)}{H'(z)}=\sum_{j=1}^{n}\left\{\alpha_j\left[\left(\frac{zf_j'(t)}{f_j(t)}-1\right)-\left(\frac{zg_j'(t)}{g_j(t)}-1\right)\right]+\beta_j\left(\frac{zf_j''(t)}{f_j'(t)}-\frac{zg_j''(t)}{g_j'(t)}\right)\right\}. \tag{12}$$

Since the analytic functions $f_j(z)$ and $g_j(z)$ are in the class $\mathcal{B}(\gamma_j)$; $j=1,\ldots,n$, from (7), (8) and (12), we obtain

$$\begin{aligned}\left|\frac{zH''(z)}{H'(z)}\right| &\le \sum_{j=1}^{n}\left\{|\alpha_j|\left[\left|\frac{zf_j'(t)}{f_j(t)}-1\right|+\left|\frac{zg_j'(t)}{g_j(t)}-1\right|\right]\right.\\ &\quad\left.+\left|\beta_j\right|\left[\left|\frac{zf_j''(t)}{f_j'(t)}\right|+\left|\frac{zg_j''(t)}{g_j'(t)}\right|\right]\right\}\\ &\le \sum_{j=1}^{n}\left\{2(1-\gamma_j)\left|\alpha_j\right|\left(\frac{1+|z|}{1-|z|}\right)+2(1-\gamma_j)\left|\beta_j\right|\left(\frac{2+|z|}{1-|z|}\right)\right\}\\ &\le \frac{1}{1-|z|}\sum_{j=1}^{n}(1-\gamma_j)(4\left|\alpha_j\right|+6\left|\beta_j\right|),\end{aligned} \tag{13}$$

for all $z\in\mathcal{U}$. Multiply both sides of (13) by $\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}$, we get

$$\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{\mathrm{Re}(\lambda)}\left|\frac{zH''(z)}{H'(z)}\right|\le\frac{1-|z|^{2\mathrm{Re}(\lambda)}}{(\mathrm{Re}(\lambda))(1-|z|)}\sum_{j=1}^{n}(1-\gamma_j)(4\left|\alpha_j\right|+6\left|\beta_j\right|), \tag{14}$$

for all $z\in\mathcal{U}$.

Define the function $\Phi(x)=\frac{1-x^{2a}}{1-x}$, where $|z|=x$, $x\in[0,1)$, and $\mathrm{Re}(\lambda)=a>0$. It is easy to prove that

$$\Phi(x)\le\begin{cases}1, & \text{if } 0<a<\frac{1}{2}\\ 2a, & \text{if } \frac{1}{2}<a<\infty.\end{cases} \tag{15}$$

From (14), (15) and the hypothesis (10), we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zH''(z)}{H'(z)}\right| \leq \begin{cases} \frac{1}{a}\sum\limits_{j=1}^{n}(1-\gamma_j)(4|\alpha_j|+6|\beta_j|), & \text{if } 0<a<\frac{1}{2} \\ 2\sum\limits_{j=1}^{n}(1-\gamma_j)(4|\alpha_j|+6|\beta_j|), & \text{if } \frac{1}{2}<a<\infty \end{cases}$$
$$\leq 1,$$

for all $z \in \mathcal{U}$. Applying Lemma 3 for the function $H(z)$, we prove that $I_n(z) \in \mathcal{S}$. Thus, the proof is complete ∎

Our next result gives univalence conditions for the integral operator $I_n(z)$ where the analytic functions $f_j(z)$ and $g_j(z)$ are in the class $\mathcal{S}_{\mu_j}$; $j = 1, \ldots, n$.

Theorem 5. *Let the analytic functions $f_j(z)$ and $g_j(z)$ be in the class $\mathcal{S}_{\mu_j}$; $0 < \mu_j \leq 1$; $j = 1, \ldots, n$ and satisfy the inequality*

$$\left|\frac{zf_j''(z)}{f_j'(z)} - \frac{zg_j''(z)}{g_j'(z)}\right| < |z| \qquad (j = 1, \ldots, n,\ z \in \mathcal{U}). \tag{16}$$

If $\lambda \in \mathbb{C}$ with $Re(\lambda) = a > 0$ and

$$\sum_{j=1}^{n}(2\mu_j|\alpha_j| + \beta_j) \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}, \tag{17}$$

then the integral operator $I_n(z)$ defined by (3) is analytic and univalent in $\mathcal{U}$.

Proof. Let the analytic functions $f_j(z)$ and $g_j(z)$ be in the class $\mathcal{S}_{\mu_j}$; $0 < \mu_j \leq 1$; $j = 1, \ldots, n$. Then from (2) and (12) we get

$$\left|\frac{zH''(z)}{H'(z)}\right| \leq \sum_{j=1}^{n}\left\{|\alpha_j|\left[\left|\frac{zf_j'(t)}{f_j(t)} - 1\right| + \left|\frac{zg_j'(t)}{g_j(t)} - 1\right|\right] + |\beta_j|\left|\frac{zf_j''(t)}{f_j'(t)} - \frac{zg_j''(t)}{g_j'(t)}\right|\right\}$$
$$\leq \sum_{j=1}^{n}(2\mu_j|\alpha_j| + \beta_j)|z|.$$

Thus, we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zH''(z)}{H'(z)}\right| \leq \frac{1}{a}\sum_{j=1}^{n}(2\mu_j|\alpha_j| + \beta_j)|z|(1-|z|^{2a}) \qquad (z \in \mathcal{U}).$$

Let us denote $|z| = x$, $x \in [0,1]$, $\mathrm{Re}(\lambda) = a > 0$ and $\Psi(x) = x(1-x^{2a})$. It is easy to prove that the maximum is attained at the point $x = 1/(2a+1)^{1/2a}$ and therefore we have

$$\Psi(x) \leq \frac{2a}{(2a+1)^{\frac{2a+1}{2a}}}.$$

In view of this inequality and (17), we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zH''(z)}{H'(z)}\right| \leq 1 \qquad (z \in \mathcal{U}).$$

Applying Lemma 3 for the function $H(z)$, we prove that $I_n(z) \in \mathcal{S}$. ∎

We conclude our present investigation by mentioning that, by suitably specializing the parameters involved, our main results would yield the following new univalence conditions for some integral operators introduced in Section 1.

Corollary 6. *Let the analytic functions* $f_j(z)$ *in the class* $\mathcal{B}(\gamma_j)$*;* $0 \leq \gamma_j < 1$*;* $j = 1, \ldots, n$ *and satisfy the inequality*

$$\sum_{j=1}^{n}(1-\gamma_j)(4|\alpha_j| + 6|\beta_j|) \leq \begin{cases} a, & \text{if } 0 < a < \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} < a < \infty \end{cases} \tag{18}$$

where $\lambda \in \mathbb{C}$ *with* $Re(\lambda) = a > 0$, *then the integral operator* $F_n(z)$ *defined by (4) is analytic and univalent in* $\mathcal{U}$*.*

Corollary 7. *Let the analytic functions* $f_j(z)$ *in the class* $\mathcal{B}(\gamma_j)$*;* $0 \leq \gamma_j < 1$*;* $j = 1, \ldots, n$ *and satisfy the inequality*

$$\sum_{j=1}^{n}(1-\gamma_j)|\alpha_j| \leq \begin{cases} \frac{a}{4}, & \text{if } 0 < a < \frac{1}{2} \\ \frac{1}{8}, & \text{if } \frac{1}{2} < a < \infty \end{cases} \tag{19}$$

where $\lambda \in \mathbb{C}$ *with* $Re(\lambda) = a > 0$, *then the integral operator* $G_n(z)$ *defined by (5) is analytic and univalent in* $\mathcal{U}$*.*

Corollary 8. *Let the analytic functions* $f_j(z)$ *be in the class* $\mathcal{S}_{\mu_j}$*;* $0 < \mu_j \leq 1$*;* $j = 1, \ldots, n$ *and satisfy the inequality*

$$\left|\frac{zf_j''(z)}{f_j'(z)}\right| < |z| \qquad (j = 1, \ldots, n,\ z \in \mathcal{U}). \tag{20}$$

If $\lambda \in \mathbb{C}$ *with* $Re(\lambda) = a > 0$ *and*

$$\sum_{j=1}^{n} (2\mu_j |\alpha_j| + \beta_j) \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}, \tag{21}$$

then the integral operator $F_n(z)$ *defined by (4) is analytic and univalent in* $\mathcal{U}$*.*

Corollary 9. *Let the analytic functions* $f_j(z)$ *and* $g_j(z)$ *be in the class* $\mathcal{S}_{\mu_j}$*;* $0 < \mu_j \leq 1$*;* $j = 1, \ldots, n$ *and satisfy the inequality*

$$\left| \frac{z f_j''(z)}{f_j'(z)} - \frac{z g_j''(z)}{g_j'(z)} \right| < |z| \qquad (j = 1, \ldots, n,\ z \in \mathcal{U}). \tag{22}$$

If $\lambda \in \mathbb{C}$ *with* $Re(\lambda) = a > 0$ *and*

$$\sum_{j=1}^{n} \mu_j |\alpha_j| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{4}, \tag{23}$$

then the integral operator $D_n(z)$ *defined by (6) is analytic and univalent in* $\mathcal{U}$*.*

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B.A. Frasin
Department of Mathematics, Faculty of Science,
Al al-Bayt University,
Mafraq, Jordan
email: *bafrasin@yahoo.com*

Tariq Al-Hawary
Department of Applied Science, Ajloun College,
Al-Balqa Applied University,
Ajloun 26816, Jordan
email: *tariq_amh@yahoo.com*

Feras Yousef
Department of Mathematics, Faculty of Science,
University of Jordan,
Amman 11942, Jordan
email: *fyousef@ju.edu.jo*