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About differential sandwich theorems using multiplier transformation and Ruscheweyh derivative

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Abstract: In this paper we obtain some subordination and superordination results for the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems. The operator $IR_{\lambda,l}^{m,n}$ is defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.

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1 Introduction

Consider $\mathcal{H}(U)$ the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}(a,n)$ the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\}$ with $\mathcal{A} = \mathcal{A}_1$.

Next we remind the definition of differential subordination and superordination.

Let the functions f and g be analytic in U. The function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2 p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \qquad z \in U, \tag{2}$$

then p is a solution of the differential superordination (2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all

p satisfying (2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

Miller and Mocanu [6] obtained conditions h, q and ψ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

We need the following differential operators.

Definition 1 *[5] For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m, \lambda, l) f(z)$ is defined by the following infinite series*

$$I(m, \lambda, l) f(z) := z + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda (j-1) + l}{1 + l} \right)^m a_j z^j.$$

Remark 1 *We have*

$$(l+1) I(m+1, \lambda, l) f(z) = (l + 1 - \lambda) I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))', \quad z \in U.$$

Remark 2 *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ for $\lambda = 1$.*

Definition 2 *(Ruscheweyh [8]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the Ruscheweyh derivative R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\ldots \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

Definition 3 *([2]) Let $\lambda, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m, \lambda, l)$ and the Ruscheweyh derivative R^n,*

$$IR_{\lambda,l}^{m,n} f(z) = (I(m, \lambda, l) * R^n) f(z),$$

for any $z \in U$ and each nonnegative integers m, n.

Remark 4 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

By simple computation we obtain the relation

Proposition 1 *[1]For $m, n \in \mathbb{N}$ and $\lambda, l \geq 0$ we have*

$$(n+1) IR_{\lambda,l}^{m,n+1} f(z) - nIR_{\lambda,l}^{m,n} f(z) = z \left(IR_{\lambda,l}^{m,n} f(z) \right)'. \tag{3}$$

We need the following

Definition 4 *[7] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 2 *[7] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z) \phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

1. Q is starlike univalent in U and

2. $Re \left(\frac{zh'(z)}{Q(z)} \right) > 0$ for $z \in U$.

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z) \phi(p(z)) \prec \theta(q(z)) + zq'(z) \phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 3 *[4] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

1. $Re \left(\frac{\nu'(q(z))}{\phi(q(z))} \right) > 0$ for $z \in U$ and

2. $\psi(z) = zq'(z) \phi(q(z))$ is starlike univalent in U.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z) \phi(p(z))$ is univalent in U and

$$\nu(q(z)) + zq'(z) \phi(q(z)) \prec \nu(p(z)) + zp'(z) \phi(p(z)),$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2 Main results

We intend to find sufficient conditions for certain normalized analytic functions f such that $q_1(z) \prec \frac{z^\delta IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, $0 < \delta \leq 1$, where q_1 and q_2 are given univalent functions.

Theorem 4 *Let $\frac{z^\delta IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in U such that $q(z) \neq 0$, for all $z \in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$Re \left(\frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z\frac{q''(z)}{q(z)} - z\frac{q'(z)}{q(z)} \right) > 0, \tag{4}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ and

$$\psi_{\lambda,l}^{m,n}(\alpha, \xi, \mu, \beta; z) := \alpha + \beta\delta(n+1) + \beta(n+1) \frac{IR_{\lambda,l}^{m,n+2} f(z)}{IR_{\lambda,l}^{m,n+1} f(z)} - \tag{5}$$

$$\beta\left(1+\delta\right)\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f\left(z\right)}{\left(IR_{\lambda,l}^{m,n}f\left(z\right)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m,n+1}f\left(z\right)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f\left(z\right)\right)^{2+2\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)\prec\alpha+\xi q\left(z\right)+\mu\left(q\left(z\right)\right)^{2}+\beta\frac{zq'\left(z\right)}{q\left(z\right)},\tag{6}$$

for $\alpha,\xi,\beta,\mu\in\mathbb{C}$, $\beta\neq0$, then

$$\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f\left(z\right)}{\left(IR_{\lambda,l}^{m,n}f\left(z\right)\right)^{1+\delta}}\prec q\left(z\right),\tag{7}$$

and q is the best dominant.

Proof. Consider $p\left(z\right):=\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\neq0$, $f\in\mathcal{A}$. We have $p'\left(z\right)=\delta\left(1+n\right)\frac{z^{\delta-1}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\left(n+1\right)\frac{z^{\delta-1}IR_{\lambda,l}^{m,n+2}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\left(1+\delta\right)\left(n+1\right)\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m,n+1}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.

By using the identity (3), we obtain

$$\frac{zp'\left(z\right)}{p\left(z\right)}=\delta\left(1+n\right)+\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+2}f\left(z\right)}{IR_{\lambda,l}^{m,n+1}f\left(z\right)}-\left(1+\delta\right)\left(n+1\right)\frac{z^{\delta-1}IR_{\lambda,l}^{m,n+1}f\left(z\right)}{IR_{\lambda,l}^{m,n}f\left(z\right)}.\tag{8}$$

By setting $\theta\left(w\right):=\alpha+\xi w+\mu w^{2}$ and $\phi\left(w\right):=\frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi\left(w\right)\neq0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q\left(z\right)=zq'\left(z\right)\phi\left(q\left(z\right)\right)=\beta\frac{zq'(z)}{q(z)}$ and $h\left(z\right)=\theta\left(q\left(z\right)\right)+Q\left(z\right)=\alpha+\xi q\left(z\right)+\mu\left(q\left(z\right)\right)^{2}+\beta\frac{zq'(z)}{q(z)}$, we find that $Q\left(z\right)$ is starlike univalent in U.

We get $h'\left(z\right)=\xi q'\left(z\right)+2\mu q\left(z\right)q'\left(z\right)+\beta\frac{q'(z)}{q(z)}+\beta z\frac{q''(z)}{q(z)}-\beta z\left(\frac{q'(z)}{q(z)}\right)^{2}$ and $\frac{zh'(z)}{Q(z)}=\frac{\xi}{\beta}q\left(z\right)+\frac{2\mu}{\beta}q^{2}\left(z\right)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\xi}{\beta}q\left(z\right)+\frac{2\mu}{\beta}q^{2}\left(z\right)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}\right)>0$.

By using (8), we obtain

$\alpha+\xi p\left(z\right)+\mu\left(p\left(z\right)\right)^{2}+\beta\frac{zp'(z)}{p(z)}=\alpha+\beta\delta\left(n+1\right)+\beta\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+2}f(z)}{IR_{\lambda,l}^{m,n+1}f(z)}-$

$\beta\left(1+\delta\right)\left(n+1\right)\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m,n+1}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}$.

By using (6), we have $\alpha+\xi p\left(z\right)+\mu\left(p\left(z\right)\right)^{2}+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\xi q\left(z\right)+\mu\left(q\left(z\right)\right)^{2}+\beta\frac{zq'(z)}{q(z)}$.

By an application of Lemma 2, we have $p\left(z\right)\prec q\left(z\right)$, $z\in U$, i.e. $\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q\left(z\right)$, $z\in U$ and q is the best dominant. ■

Corollary 5 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (4) holds. If $f\in\mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\frac{\beta\left(A-B\right)z}{\left(1+Az\right)\left(1+Bz\right)},$$

for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (5), then*

$$\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 4 we get the corollary. ■

Corollary 6 *Let* $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (4) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2},$$

for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (5), then*

$$\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 4 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ■

Theorem 7 *Let* q *be analytic and univalent in* U *such that* $q(z) \neq 0$ *and* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* U. *Assume that*

$$Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0, \text{ for } \xi, \beta, \mu \in \mathbb{C},\ \beta \neq 0. \tag{9}$$

If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ *is univalent in* U, *where* $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ *is as defined in (5), then*

$$\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \tag{10}$$

implies

$$q(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}, \quad z \in U, \tag{11}$$

and q *is the best subordinant.*

Proof. Consider $p(z) := \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C} \backslash \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \backslash \{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z) q(z) [\xi + 2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$.

By using (8) and (10) we obtain

$$\alpha+\xi q(z)+\mu(q(z))^{2}+\frac{\beta zq'(z)}{q(z)}\prec\alpha+\xi p(z)+\mu(p(z))^{2}+\frac{\beta zp'(z)}{p(z)}.$$

Applying Lemma 3, we get

$$q(z)\prec p(z)=\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}},\quad z\in U,$$

and q is the best subordinant. ■

Corollary 8 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (9) holds. If $f\in\mathcal{A}$, $\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$ and*

$$\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z),$$

for $\alpha,\beta,\xi,\mu\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then

$$\frac{1+Az}{1+Bz}\prec\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}},$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$ in Theorem 7 we get the corollary. ■

Corollary 9 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that (9) holds. If $f\in\mathcal{A}$, $\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$ and*

$$\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^{2}}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z),$$

for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, $0<\gamma\leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}},$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.

Proof. For $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$ in Theorem 7 we get the corollary. ■
Combining Theorem 4 and Theorem 7, we state the following sandwich theorem.

Theorem 10 *Let q_1 and q_2 be analytic and univalent in U such that $q_1(z)\neq 0$ and $q_2(z)\neq 0$, for all $z\in U$, with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (4) and q_2*

satisfies (9). If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q\left(0\right),1\right] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)$ *is as defined in (5) univalent in* U*, then*

$$\alpha + \xi q_1\left(z\right) + \mu\left(q_1\left(z\right)\right)^2 + \frac{\beta z q_1'\left(z\right)}{q_1\left(z\right)} \prec \psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right) \prec \alpha + \xi q_2\left(z\right) + \mu\left(q_2\left(z\right)\right)^2 + \frac{\beta z q_2'\left(z\right)}{q_2\left(z\right)},$$

for $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, *implies*

$$q_1\left(z\right) \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f\left(z\right)}{\left(IR_{\lambda,l}^{m,n} f\left(z\right)\right)^{1+\delta}} \prec q_2\left(z\right),$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1\left(z\right) = \frac{1+A_1 z}{1+B_1 z}$, $q_2\left(z\right) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 11 *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$*. Assume that (4) and (9) hold. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q\left(0\right),1\right] \cap Q$ *and*

$$\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu\left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \frac{\beta\left(A_1 - B_1\right) z}{\left(1+A_1 z\right)\left(1+B_1 z\right)} \prec \psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)$$

$$\prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu\left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{\beta\left(A_2 - B_2\right) z}{\left(1+A_2 z\right)\left(1+B_2 z\right)},$$

for $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (5), then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f\left(z\right)}{\left(IR_{\lambda,l}^{m,n} f\left(z\right)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z},$$

hence $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1\left(z\right) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2\left(z\right) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 12 *Let* $m,n \in \mathbb{N}$, $\lambda,l \geq 0$*. Assume that (4) and (9) hold. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q\left(0\right),1\right] \cap Q$ *and*

$$\alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma_1} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}\left(\alpha,\beta,\mu;z\right)$$

$$\prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^{\gamma_2} + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2},$$

for $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (5), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f\left(z\right)}{\left(IR_{\lambda,l}^{m,n} f\left(z\right)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2},$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ *are the best subordinant and the best dominant, respectively.*

Changing the functions θ and ϕ we obtain the following results.

Theorem 13 *Let* $\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}(U), f\in\mathcal{A}$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0)=1$, $z\in U$. *Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0, \tag{12}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, *and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z):=(\alpha+\beta\delta(n+1))\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+ \tag{13}$$

$$\beta(n+1)\frac{z^{\delta}IR_{\lambda,l}^{m,n+2}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\beta(1+\delta)(n+1)\frac{z^{\delta}\left(IR_{\lambda,l}^{m,n+1}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}.$$

If q *satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha q(z)+\beta zq'(z), \tag{14}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, *then*

$$\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z),\quad z\in U, \tag{15}$$

and q *is the best dominant.*

Proof. Consider $p(z):=\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$

We have $p'(z)=\delta(1+n)\frac{z^{\delta-1}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+(n+1)\frac{z^{\delta-1}IR_{\lambda,l}^{m,n+2}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-(1+\delta)(n+1)\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m,n+1}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.

By using the identity (3), we obtain

$$zp'(z)=\delta(1+n)\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+(n+1)\frac{z^{\delta}IR_{\lambda,l}^{m,n+2}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-(1+\delta)(n+1)\frac{z^{\delta}\left(IR_{\lambda,l}^{m,n+1}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}. \tag{16}$$

By setting $\theta(w):=\alpha w$ and $\phi(w):=\beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta zq'(z)$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta zq'(z)$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0$.

By using (16), we obtain $\alpha p(z)+\beta zp'(z)=(\alpha+\beta\delta(n+1))\frac{z^{\delta}IR_{\lambda,l}^{m,n+1}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+$

$\beta(n+1)\frac{z^{\delta}IR_{\lambda,l}^{m,n+2}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\beta(1+\delta)(n+1)\frac{z^{\delta}\left(IR_{\lambda,l}^{m,n+1}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.

By using (14), we have $\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z)$.

Applying Lemma 2, we get $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q(z)$, $z \in U$, and q is the best dominant. ■

Corollary 14 *Let $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \le B < A \le 1$, $m,n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (12) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha \frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \le B < A \le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (13), then

$$\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 13 we get the corollary. ■

Corollary 15 *Let $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n \in \mathbb{N}$, $\lambda, l \ge 0$. Assume that (12) holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma},$$

for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\beta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (13), then

$$\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.

Proof. Corollary follows by using Theorem 13 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \le 1$. ■

Theorem 16 *Let q be convex and univalent in U such that $q(0) = 1$. Assume that*

$$Re\left(\frac{\alpha}{\beta} q'(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C},\ \beta \neq 0. \tag{17}$$

If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is univalent in U, where $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (13), then

$$\alpha q(z) + \beta z q'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \tag{18}$$

implies

$$q(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}, \quad \delta \in \mathbb{C},\ \delta \neq 0,\ z \in U, \tag{19}$$

and q is the best subordinant.

Proof. Consider $p(z) := \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta} q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\beta} q'(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (18) we obtain

$$\alpha q(z) + \beta z q'(z) \prec \alpha p(z) + \beta z p'(z), \quad z \in U.$$

Applying Lemma 3, we get

$$q(z) \prec p(z) = \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}, \quad z \in U,$$

and q is the best subordinant. ■

Corollary 17 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (17) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q$, *and*

$$\alpha \frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (13), then*

$$\frac{1+Az}{1+Bz} \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}},$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 16 we get the corollary. ■

Corollary 18 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that (17) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha \left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (13), then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}},$$

and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 16 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ■

Combining Theorem 13 and Theorem 16, we state the following sandwich theorem.

Theorem 19 *Let q_1 and q_2 be convex and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that q_1 satisfies (12) and q_2 satisfies (17). If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$, and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (13) univalent in U, then*

$$\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha q_2(z) + \beta z q_2'(z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z), \quad z \in U,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 20 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (12) and (17) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\alpha \frac{1+A_1 z}{1+B_1 z} + \frac{\beta (A_1 - B_1) z}{(1+B_1 z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$$

$$\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \frac{\beta (A_2 - B_2) z}{(1+B_2 z)^2}, \quad z \in U,$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}, \quad z \in U,$$

hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 21 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (12) and (17) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\alpha \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$$

$$\prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma_2}, \quad z \in U,$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), then

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z^{\delta} IR_{\lambda,l}^{m,n+1} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}, \quad z \in U,$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.

Competing interests

The author declares that she has no competing interests.

Author's contributions

The author drafted the manuscript, read and approved the final manuscript.

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