Generalized Measures and Integrals

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Abstract: This is a brief survey of the present state of the generalized theory of measure and integral. Monotone measures (possibly non additive) and new integrals (possibly nonlinear) are introduced. Also, some numerical devices for the computation of Choquet integrals are introduced.

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1 Introduction

Classical measure and integration theory is based upon additive (even more, σ-additive) measures.

Recently, it was seen that more varied tools, other than additive measures, are necessary in order to describe a multitude of phenomena. These tools are the generalized measures, which are monotone and possibly non additive. Here is a rough discussion pertaining to this type of measures: superadditivity indicates a cooperative action or synergy between the measured items (sets); subadditivity indicates lack of cooperation, inhibitory effects or incompatibility between the measured items (sets); additivity can express non interaction or indifference. In this spirit, it is perhaps worth remembering the fact that already the economist Shackle proposed non additive measures in 1949 (Review of Economic Studies).

Concerning "the way of speaking", we feel obliged to point out that the general "trend" was to call both non additive measures and nonlinear integrals "fuzzy". Nowadays, more varied expressions are in use, e.g. "generalized" measures and integrals a.s.o.

It is generally accepted that the most important generalized measures are **the λ-measures**, introduced by M. Sugeno in his doctoral thesis [14] which actually is the starting point of generalized measure and integration theory. An important result, due to Z. Wang (see [17]) states that any λ-measure can be obtained from a classical measure via composition with a special increasing function (see also [18] and [19]). We called the generalized measures which can be obtained in such a way **representable measures** (see [1]). In the same paper it was pointed out that λ-measures appear naturally also within the framework of functional equations. In the recent papers [15] and [16] M. Sugeno calls (in a more particular setting) the representable measures **distorted measures**. The discrete countable λ-measures with preassigned values were completely characterized in [3]. Interesting considerations concerning the theory of generalized measures can be found in [10].

In a natural way, integrals with respect to generalized measures appeared. Because of the possible non additivity of the measures involved, these integrals are possibly nonlinear. The most popular such integrals are the Sugeno and the Choquet integral. The Sugeno integral

(introduced and studied in [14]) is very far away from standard integrals (see also [2], [5], [7], [11], [12], [18], [19]). The Choquet integral is a direct generalization of the abstract Lebesgue integral. The very roots of this integral are in the classical work [4] of G. Choquet. It seems that the name "Choquet integral" was given by D. Schmeidler in [13]. Concerning the Choquet integral, the reader can consult [11], [18], [19] and also [2], [5], [7]. A very interesting point of view concerning the computation of Choquet integrals on intervals appears in [15] and [16].

Concerning the applications of the generalized measure and integration theory, we begin by underlining the importance of the Choquet integral as an aggregation tool (see [6] and [19]) i.e. as an instrument of compressing a multitude of numerical data into a single numerical date. It is very important to underline the fact that the use of non additive measures in computing the integrals allows us to take into consideration the interaction between the generators of data (as already pointed out). Nonlinear integrals can be used in other directions, e.g. multiregression, classification, social welfare, decision (qualitative, multiattribute, multicriteria, cost), model evaluation, image processing and recognition a.s.o. (see [7] and [19]). To be synthetic, we can use nonlinear integrals in **data mining**.

A short presentation of the content follows. Namely, the remainder of the paper is divided in three parts: the first part is dedicated to generalized (monotone) measures, the second part is dedicated to Choquet and Sugeno integrals and the third part is dedicated to some computing devices for Choquet integrals.

2   Monotone Measures

We introduce the generalized (monotone) measures which encompass the usual additive (or σ-additive) measures. The reader will recognize many properties of usual measures.

We shall write $N = \{1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$, $\overline{\mathbb{R}_+} = [0, \infty] = \mathbb{R}_+ \cup \{\infty\}$. All sequences will be indexed either with $\mathbb{N}$ or with $\{i, i+1, \ldots\}$ for some $i \in \mathbb{N}$.

If T is a non empty set, $\mathcal{P}(T)$ is the boolean of T (i.e. the set of all subsets of T). For any $A \in \mathcal{P}(T)$, $\varphi_A : T \to \mathbb{R}_+$ will be the characteristic (indicator function of A).

Throughout this paragraph, T will be a non empty set (*the total set*).

The main definition of this paragraph follows.

Definition 1 *Let $\mathcal{T} \subset \mathcal{P}(T)$ such that $\emptyset \in \mathcal{T}$. A monotone measure is a non null function $\mu : \mathcal{T} \to \overline{\mathbb{R}_+}$ having the properties that $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$, whenever A, B in $\mathcal{T}$ are such that $A \subset B$ (of course, any additive measure on a ring is a monotone measure).*

Monotone measures can have special properties, as in the following definition.

Definition 2 *Let $\mu : T \to \overline{\mathbb{R}_+}$ be a monotone measure.*

*1. We say that μ is **finite** in case $\mu(A) < \infty$ for any $A \in \mathcal{T}$ (in case $T \in \mathcal{T}$, this means that $\mu(T) < \infty$).*

*2. We say that μ is **continuous** if it has the following two properties:*

*a) μ is **continuous from below**, i.e. for any increasing sequence $(A_n)_n$ in $\mathcal{T}$ such that $\bigcup_n A_n \in \mathcal{T}$, one has $\mu\left(\lim_n A_n\right) = \lim_n \mu(A_n)$, which means $\mu\left(\bigcup_n A_n\right) = \sup_n \mu(A_n)$;*

b) μ **is continuous from above**, *i.e., for any decreasing sequence* $(A_n)_n$ *in* $\mathcal{T}$ *such that* $\bigcap_n A_n \in \mathcal{T}$ *and there exists* $n_0 \in \mathbb{N}$ *with* $\mu(A_{n_0}) < \infty$, *one has* $\mu\left(\lim_n A_n\right) = \lim_n \mu(A_n)$, *which means* $\mu\left(\bigcap_n A_n\right) = \inf_n \mu(A_n)$.

The reader can see that $\sigma-$additive measures on rings are continuous.

We introduced the general definitions which are in use. In the present exposure, we shall simplify facts, working within the following *particular framework*:

T will be a non empty set, $\mathcal{T} \subset \mathcal{P}(\mathcal{T})$ will be a $\sigma-$algebra (hence $(T, \mathcal{T})$ is a *measurable space*) and $\mu : \mathcal{T} \to \mathbb{R}_+$ is a finite monotone measure (we say that $(T, \mathcal{T}, \mu)$ is a *monotone measure space*). In case μ is also continuous, we say that $(T, \mathcal{T}, \mu)$ is a *continuous monotone measure space*.

As we said, the most important monotone measures are the λ-measures which we introduce now.

Definition 3 *Let* $(T, \mathcal{T}, \mu)$ *be a monotone measure space. Let* $\lambda \in \left(-\dfrac{1}{\mu(T)}, \infty\right)$ *(we say that* λ *is* $\mu-admissible$*).*

1. We say that μ **satisfies the** λ-**rule** *(or* μ *is* λ-**additive***) if*

$$\mu(E \cup F) = \mu(E) + \mu(F) + \lambda\mu(E)\mu(F)$$

whenever E, F *are in* $\mathcal{T}$ *and* $E \cap F = \emptyset$.

2. One can see that μ *satisfies the* λ-*rule if and only if* μ **satisfies the finite** λ-**rule**, *i.e.*

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \frac{1}{\lambda}\left(\prod_{i=1}^n (1 + \lambda\mu(E_i)) - 1\right), \ \text{if } \lambda \neq 0$$

or

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i), \ \text{if } \lambda = 0$$

whenever $E_1, E_2, \ldots, E_n$ *are mutually disjoint sets* $E_i \in \mathcal{T}$.

3. We say that μ **satisfies the** $\sigma - \lambda-$**rule** *if*

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \frac{1}{\lambda}\left(\prod_{i=1}^n (1 + \lambda\mu(E_i)) - 1\right), \ \text{if } \lambda \neq 0$$

or

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i), \ \text{if } \lambda = 0$$

whenever $E_i \in \mathcal{T}$ *are mutually disjoint sets.*

Remarks.

1. In case $\lambda = 0$, we have additivity (for 1.), finite additivity (for 2.) or $\sigma-$additivity (for 3.).

2. For 3. : we have $1 + \lambda\mu(A) > 0$ for any $A \in \mathcal{T}$, hence the infinite product in the definition is convergent.

3. Another way of speaking: in case there exists a $\mu-$admissible number δ such that μ satisfies the $\delta-$rule (respectively the finite $\delta-$rule, respectively the $\delta - \sigma-$rule) we say that μ **satisfies some** $\lambda-$**rule** (respectively **some finite** $\lambda-$**rule**, respectively **some** $\sigma - \lambda-$**rule**).

Definition 4 *Let* $(T, \mathcal{T}, \mu)$ *be a monotone measure space.*

1. If λ *is a* $\mu-$*admissible number and* μ *satisfies the* $\sigma - \lambda -$*rule, we say that* μ ***is a*** $\lambda-$***measure***. *In case there exists a* $\mu-$ *admissible number* δ *such that* μ *is a* $\delta-$*measure, we say that* μ ***is a*** ***some*** $\lambda-$***measure***.

2. If μ *is a* $\lambda-$*measure and* $\mu(T) = 1$, *we say that* μ ***is a*** $\lambda-$***Sugeno measure***. *Clearly, the* $0-$*Sugeno measure are the probabilities on* $\mathcal{T}$. *In case* μ *is a some* $\lambda-$*measure with* $\mu(T) = 1$ *we say that* μ ***is a Sugeno measure***.

An important result, essentially due to Z. Wang (see [17], [18] and [19]) asserts that $\lambda-$additive measures are representable (see [1] for the terminology). More precisely, we have

Theorem 1 *Let* $(T, \mathcal{T}, \mu)$ *be a monotone measure space with* $\mu(T) = A > 0$ *and assume that* μ *satisfies the* $\lambda-$*rule with* $\lambda \neq 0$ *(hence* $-\dfrac{1}{A} < \lambda < \infty$*).*

Then, for any $0 < \alpha < \infty$, *there exists an additive measure* $m : \mathcal{T} \to \mathbb{R}_+$ *such that* $m(T) = \alpha$ *and* $\mu = h_\lambda \circ m$ *where* $h_\lambda : [0, a] \to [0, A]$ *acts via*

$$h_\lambda(x) = \frac{1 + \lambda A)^{\frac{x}{\alpha}} - 1}{\lambda}.$$

Namely, the measure m is obtained from μ via the formula

$$m = h_\lambda^{-1} \circ \mu$$

where $h_\lambda^{-1} : [0, A] \to [0, \alpha]$ acts via

$$h_\lambda^{-1}(y) = \frac{\alpha \, ln(1 + \lambda y)}{ln(1 - \lambda A)}.$$

In case $\alpha = A = 1$ (the most popular case), the formulae from above $(0 \neq \lambda > -1)$ become, for $h_\lambda, h_\lambda^{-1} : [0, 1] \to [0, 1]$:

$$h_\lambda(x) = \frac{(1 + \lambda)^x - 1}{\lambda}$$

$$h_\lambda(x)^{-1}(y) = \frac{ln(1 + \lambda y)}{ln(1 + \lambda)}.$$

This result can be interpreted as follows.

Theorem 1.5.' Assume that $(T, \mathcal{T})$ is a measurable space.

Let $0 < a < \infty$ and $0 < A < \infty$. Let also $0 \neq \lambda \in \left(-\dfrac{1}{A}, \infty \right)$. Denote by $\mathcal{P}$ the set of all $\sigma-$additive measures $m : \mathcal{T} \to \mathbb{R}_+$ such that $m(T) = a$. Also, denote by $\mathcal{P}(\lambda)$ the set of all $\lambda-$measures $\mu : \mathcal{T} \to \mathbb{R}_+$ such that $\mu(T) = A$.

Then (see notations of Theorem 1) there exists a bijection $\Omega : \mathcal{P} \to \mathcal{P}(\lambda)$, acting via $\Omega(m) = h_\lambda \circ m$. The inverse $\Omega^{-1} : \mathcal{P}(\lambda) \to \mathcal{P}$ acts via $\Omega^{-1}(\mu) = h_\lambda^{-1} \circ \mu$.

Disregarding a and A, we shall denote $h_\lambda \circ m \overset{def}{=} u(\lambda, m)$.

An interesting problem is whether there exist some $\lambda-$measures having preassigned values. For instance, this problem can be raised in the case of the discrete measures, where $\mathcal{T} = \mathcal{P}(T)$, T being either a finite set or $\mathbb{N}$. In this case, one is asked to find a some $\lambda-$measure $\mu : \mathcal{T} \to \mathbb{R}_+$ with $\mu(\{a\})$ given, for any $a \in T$.

Under certain conditions this is possible in case T is finite (see [18] and [19]) . In case $T = \mathbb{N}$, the problem was positively solved in [3].

3 Choquet and Sugeno Integrals

In this paragraph we shall integrate positive measurable functions.

We start with a measurable space $(T, \mathcal{T})$ and let us consider the set $M(\mathcal{T})$ of all $\mathcal{T}-$measurable functions $f : T \to \overline{\mathbb{R}_+}$. For any such function f and any $\alpha \in [0, \infty)$, one can consider the sets

$$F_\alpha(f) \overset{def}{=} F_\alpha = \{t \in T \mid f(t) \geq a\} \in \mathcal{T}$$

$$F_{\alpha+}(f) \overset{def}{=} F_{\alpha+} = \{t \in T \mid f(t) > \alpha\} \in \mathcal{T}.$$

Considering also a monotone measure $\mu : \mathcal{T} \to \mathbb{R}_+$ (i.e. the monotone measure space $(T, \mathcal{T}, \mu)$) we have, for any $A \in \mathcal{T}$, the decreasing functions

$$\varphi : [0, \infty) \to [0, \infty), \ \varphi(\alpha) = \mu(F_\alpha \cap A)$$

$$\varphi_+ : [0, \infty) \to [0, \infty), \ \ \varphi_+(\alpha) = \mu(F_{\alpha+} \cap A).$$

A. The Sugeno Integral

Let $(T, \mathcal{T}, \mu)$ be a monotone measure space.

We consider $f \in M(\mathcal{T})$ and $A \in \mathcal{T}$.

Definition 5 *The Sugeno integral of f with respect to μ on A is*

$$(S) \qquad \int_A f \mathrm{d}\mu \overset{def}{=} \sup_{\alpha \in \mathbb{R}_+} \left(\alpha \wedge \mu\left(F_\alpha \cap A\right)\right) \leq \mu(T) < \infty.$$

In case $A = T$, we write only

$$(S) \qquad \int f \mathrm{d}\mu = \sup_{\alpha \in \mathbb{R}_+} \left(\alpha \wedge \mu\left(F_\alpha\right)\right) \leq (\mu(T) < \infty$$

(this is the Sugeno integral of f with respect to μ).

Here, as usual, $x \wedge y \overset{def}{=} \min(x, y)$, if x, y are real numbers.

Remarks.

1. One has the formula

$$(S) \int_A f \mathrm{d}\mu = (S) \int f \varphi_A \mathrm{d}\mu \leq (S) \int f \mathrm{d}\mu$$

2. In case $\mu.(T) \leq M$, one has

$$(S) \int_A f \mathrm{d}\mu = \sup_{\alpha \in [0, M]} \left(\alpha \wedge \mu\left(F_\alpha \cap A\right)\right)$$

because, for $\alpha > M$, one has

$$\alpha \wedge \mu\left(F_\alpha \cap A\right) = \mu\left(F_\alpha \cap A\right) \leq \mu\left(F_M \cap A\right) = M \wedge \mu\left(F_M \cap A\right).$$

3. In case μ is continuous, we have a practical device to compute the Sugeno integral. Namely, in this case the function φ (i.e. the function $\alpha \mapsto \mu\left(F_\alpha \cap A\right)$) is continuous, hence the function u given via $\alpha \mapsto \varphi(\alpha) - \alpha$ is strictly decreasing and continuous, with $u(0) \geq 0$ and $\lim_{\alpha \to \infty} u(\alpha) = -\infty$.

The unique zero of u, call it α_0, is exactly the Sugeno integral: $\alpha_0 = (S) \int_A \mathrm{d}\mu$.

Here are some properties of the Sugeno integral (f_1, f_2, f in $M(\mathcal{T})$, A, B in $\mathcal{T}$ and a in $[0, \infty)$).

Theorem 2 *1. If $\mu(A) = 0$, then $(S) \int_A f \mathrm{d}\mu = 0$.*

2. If μ is continuous and $(S) \int_A f \mathrm{d}\mu = 0$ then $\mu(A \cap \{t \in T \mid f(t) > 0\}) = 0$.

3. If $f_1 \leq f_2$, then $(S) \int_A f_1 \mathrm{d}\mu \leq (S) \int_A f_2 \mathrm{d}\mu$.

4. If $A \subset B$, then $(S) \int_A f \mathrm{d}\mu \leq (S) \int_B f \mathrm{d}\mu$.

5. $(S) \int_A a \mathrm{d}\mu = a \wedge \mu(A)$.

6. $(S) \int (f + a) \mathrm{d}\mu \leq (S) \int f \mathrm{d}\mu + (S) a \mathrm{d}\mu$.

Theorem 3 (Analogue of Beppo Levi's Theorem.) *Assume that $(T, \mathcal{T}, \mu)$ is a continuous monotone measure space.*

Let $(f_n)\, n$ be a monotone sequence in $M(\mathcal{T})$ and let $f = \lim\limits_{n} f_n$ (pointwise).

Then, for any $A \in \mathcal{T}$, one has

$$(S) \int_A f \mathrm{d}\mu = \lim_n (S) \int_A f_n \mathrm{d}\mu$$

(in case $(f_n)_n$ is increasing, this means

$$(S) \int_A f \mathrm{d}\mu = \sup_n (S) \int_A f_n \mathrm{d}\mu$$

and in case $(f_n)_n$ is decreasing, this means

$$(S) \int_A f \mathrm{d}\mu = \inf_n (S) \int_A f_n \mathrm{d}\mu.$$

Theorem 4 (Analogue of Fatou's Lemma.) *Assume that $(T, \mathcal{T}, \mu)$ is a continuous monotone measure space.*

For any sequence $(f_n)_n$ in $M(\mathcal{T})$, one has

$$(S) \int_A \liminf_n f_n \mathrm{d}\mu \leq \liminf_n (S) \int_A f_n \mathrm{d}\mu.$$

Theorem 5 (Uniform Convergence/) *Assume that $(f_n)_n$ is a sequence in $M(\mathcal{T})$ and that $f_n \overset{u}{\to} f$ (uniform convergence), and f_n, f take only finite values.*

Then, for any $A \in \mathcal{T}$, one has

$$(S) \int_A f \mathrm{d}\mu = \lim_n (S) \int_A f \mathrm{d}\mu.$$

B. The Choquet Integral

We consider $f \in M(\mathcal{T})$ and $A \in \mathcal{T}$. Again, we consider the decreasing functions $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(\alpha) = \mu(F_\alpha \cap A)$ and $\varphi_+ : [0, \infty) \to [0, \infty)$, $\varphi_+(\alpha) = \mu(F_{\alpha+} \cap A)$.

Let L be the Lebesgue measure on $[0, \infty)$. Then, we can integrate φ with respect to L and we shall write

$$\int \varphi \mathrm{d}L \overset{def}{=} \mu(F_\alpha \cap A) \, \mathrm{d}\alpha.$$

Definition 6 *The Choquet integral of f with respect to u on A is*

$$(C)\int_A f\mathrm{d}\mu = \int_0^\infty \mu\left(F_\alpha \cap A\right)\mathrm{d}\alpha.$$

In case $A = T$, we write only

$$(C)\int f\mathrm{d}\mu = \int_0^\infty \mu\left(F_\alpha\right)\mathrm{d}\alpha.$$

(this is the Choquet integral of f with respect to μ).

In case $(C)\int f\mathrm{d}\mu < \infty$, we say that f is the Choquet integrable with respect to μ.

Remarks.

1. We have the formula

$$(C)\int_A f\mathrm{d}\mu = (C)\int \varphi\mathrm{d}\mu \le (C)\int f\mathrm{d}\mu.$$

2. The definition of the Choquet integral is a generalization of the usual abstract Lebesgue integral. Indeed, if μ is a classic measure (i.e. μ is $\sigma-$additive), then we have the equality

$$(C)\int_A f\mathrm{d}\mu = \int_A f\mathrm{d}\mu$$

the last integral being classic.

3. We have the formula

$$(C)\int_A f\mathrm{d}\mu = \int_0^\infty \mu\left(F_{\alpha+} \cap A\right)\mathrm{d}\alpha.$$

Here are some properties of the Choquet integral (f_1, f_2 in $M(\mathcal{T})$, A, B in $\mathcal{T}$ and a in $[0,\infty)$).

Theorem 6 *1. If $\mu(A) = 0$, then $(C)\int_A f\mathrm{d}\mu = 0$.*

2. If $\mu(A \cap \{t \in T || f(t) > 0\}) = 0$, then $(C)\int_A f\mathrm{d}\mu = 0$. Conversely, if μ is continuous and $(C)\int_A f\mathrm{d}\mu = 0$, then $\mu(A \cap \{t \in T || f(t) > 0\}) = 0$.

3. If $f_1 \le f_2$, then $(C)\int_A f_1\mathrm{d}\mu \le \int_A f_2\mathrm{d}\mu$.

4. If $A \subset B$, then $\int_A f\mathrm{d}\mu \le \int_B f\mathrm{d}\mu$.

5. $\int_A 1\mathrm{d}\mu = \mu(A)$.

6. $(C)\int_A af\mathrm{d}\mu = a(C)\int_A f\mathrm{d}\mu$.

7. $(C)\int_A (f + a)\mathrm{d}\mu = (C)\int_A f\mathrm{d}\mu + a\mu(A)$.

Theorem 7 (Analogue of Beppo Levi's Theorem) *Assume that $(T, \mathcal{T}, \mu)$ is a continuous monotone measure space.*

Let $(f_n)_n$ be a monotone sequence in $M(\mathcal{T})$ and let $f = \lim_n f_n$ (pointwise).

Then, for any $A \in \mathcal{T}$, one has

$$(C) \int_A f \mathrm{d}\mu = \lim_n (C) \int_A f_n \mathrm{d}\mu$$

(see also Theorem 3).

Theorem 8 (Analogue of Lebesgue's Dominated Convergence Theorem) *Assume that $(T, \mathcal{T}, \mu)$ is a continuous monotone measure space.*

Let $(f_n)_n$ be a sequence in $M(\mathcal{T})$, f and g in $M(\mathcal{T})$ and $A \in \mathcal{T}$. Assume that $f_n \underset{n}{\to} f$ pointwise, g is Choquet integrable with respect to μ and $f_n \leq g$ for any n. Then f_n and f are Choquet integrable with respect to μ and

$$(C) \int_A f \mathrm{d}\mu = \lim_n (C) \int_A f_n \mathrm{d}\mu.$$

Theorem 9 (Uniform Convergence) *Let $f \in M(\mathcal{T})$ be Choquet integrable with respect to μ. Assume that $(f_n)_n$ is a sequence in $M(\mathcal{T})$ such that $f_n \overset{u}{\to} f$ (uniform convergence) and f_n, f take only finite values.*

Then:

1. There exists n_0 such that f_n is Choquet integrable with respect to μ for any $n \geq n_0$.

2. For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that, for any $n \geq n_0$, one has

$$\left| (C) \int f \mathrm{d}\mu - (C) \int f_n \mathrm{d}\mu \right| \leq \varepsilon \mu(T).$$

4 Computing Devices for Choquet Integrals

Throughout this paragraph we shall deal with a continuous monotone measure space $(T, \mathcal{T}, \mu)$ and a function $f \in M(\mathcal{T})$. We shall try to compute $(C) \int f \mathrm{d}\mu$ with a preassigned precision.

It is known that, in case f is a simple function, one can write f in the form

$$f = \sum_{i=1}^{n} a_i \varphi_{A_i}$$

with $a_i \in (\mathbb{R}_+)$ and $A_i \in \mathcal{T}$ mutually disjoint this form is unique in case f is not null and we stipulate to have the numbers a_i distinct). Moreover, we can rewrite f such that $a_1 \leq a_2 \leq \ldots \leq a_n$.

Then, we have the formula

$$(C) \int f \mathrm{d}\mu = \sum_{i=1}^{n} (a_i - a_{i-1}) \mu \left(\bigcup_{p=i}^{n} A_p \right) \tag{4.1}$$

with the convention $a_0 = 0$. Notice that the result in formula (4.1) does not depend upon the representation of f.

We shall extend formula (4.1) for more general functions, namely elementary functions, which are of the form (pointwise convergence)

$$f = \sum_{i=1}^{\infty} a_i \varphi_{A_i}$$

with $A_i \in \mathcal{T}$ mutually disjoint and $a_i \in \mathbb{R}_+$.

In case $0 \leq a_1 < a_2 < a_3 < \ldots$ and all A_i are non empty (hence f is not null) we say that f is a canonical elementary function. In this case, one has the formula (same convention $a_0 = 0$)

$$(C) \int f \mathrm{d}\mu = \sum_{i=1}^{\infty} (a_i - a_{i-1}) \, \mu \left(\bigcup_{p=i}^{\infty} A_p \right). \tag{4.2}$$

This formula (4.2) remains valid even in case some A_i are empty (one can skip the respective φ_{A_i} in the representation of the non null f).

We shall present in the sequel the fundamental devices for computing $(C) \int f \mathrm{d}\mu$. These procedures are: discretization and truncation.

Discretization

For our general $f \in M(\mathcal{T})$, let us fix $i \in N$. We shall construct a positive elementary function $u(i)$ as follows.

For any $n = 0, 1, 2, 3, \ldots$, and any $p \in \{1, 2, \ldots, i\}$, write

$$B(n, i, p) = \left[n + \frac{p-1}{i}, n + \frac{p}{i} \right)$$

$$\alpha(n, i, p) = n + \frac{p-1}{i} = \min B(n, i, p)$$

(the disjoint intervals $B(n, i, p)$ with length $\dfrac{1}{i}$ have union equal to $[0, \infty)$).

Then we get the disjoint sets

$$A(n, i, p) = f^{-1}(B(n, i, p)) \in \mathcal{T}$$

having union T (some $A(n, i, p)$ can be empty). Now, it is possible to define (pointwise convergence) the elementary function $u(i) : T \to \mathbb{R}_+$

$$u(i) = \sum_{n=0}^{\infty} \sum_{p=1}^{i} a(n, i, p) \varphi_{A(n,i,p)}.$$

Clearly, $u(i)$ is an elementary function whose (possible) values can be arranged in strictly increasing order: $a(0, i, 1) = 0 < a(0, i, 2) = \frac{1}{i} < \ldots < a(0, i, i) = \frac{i-1}{i} < a(1, i, 1) = 1 < a(1, i, 2) = 1 + \frac{1}{i} < \ldots$

The sequence $(u(i))_i$ converges uniformly to f, because, for any $t \in T$ and any $i \in \mathbb{N}$:

$$|f(t) - u(i)(t)| = u(t) - u(i)(t) \leq \frac{1}{i}.$$

We call $u(i)$ *the $i-$discretization* of f.

Using previous results, we get, for any $i \in \mathbb{N}$

$$\left| (C) \int f \mathrm{d}\mu - (C) \int u(i) \mathrm{d}\mu \right| \leq \frac{1}{i} \mu(T)$$

hence

$$\lim_i (C) \int u(i) \mathrm{d}\mu = (C) \int f \mathrm{d}\mu.$$

Truncation

Again fix $i \in N$ and let us define another new function $u(i)$.

Namely $u(i) = f \wedge i$, i.e. $u(i)(t) = f(t)$, if $f(t) \leq i$ and $u(i)(t) = i$, if $f(t) > i$. We shall call $u(i)$ *the $i-$truncation of f.*

Assume that f is Choquet integrable. Then we have

$$\left| (C) \int f \mathrm{d}\mu - (C) \int u(i) \mathrm{d}\mu \right| = \int\limits_i^\infty \mu\left(F_\alpha\right) \mathrm{d}\alpha$$

which implies that

$$\lim_i (C) \int u(i) \mathrm{d}\mu = (C) \int f \mathrm{d}\mu.$$

Unified notations

From now on we shall consider that f is Choquet integrable with respect to μ. Write

$$v \stackrel{def}{=} (C) \int f \mathrm{d}\mu$$

(v is our "target").

We shall construct the numerical sequence $(v(i))_i$ and then, for any $i \in \mathbb{N}$, the numerical sequence $(v(i,n))_n$ as follows (two types of construction):

Strategy Truncation Discretization (STD)

$$v(i) \stackrel{def}{=} (C) \int f(i) \mathrm{d}\mu$$

where $f(i)$ is the $i-$truncation of f;

$$v(i,n) \stackrel{def}{=} (C) \int f(i,n) \mathrm{d}\mu$$

where $f(i,n)$ is the $n-$discretization of $f(i)$.

Strategy Discretization-Truncation (SDT)

$$v(i) \overset{def}{=} (C) \int f(i) \mathrm{d}\mu$$

where $f(i)$ is the $i-$discretization of f;

$$v(i,n) \overset{def}{=} (C) \in f(i,n) \mathrm{d}\mu$$

where $f(i,n)$ is the $n-$truncation of $f(i)$.

For STD:

$$|v - v(i)| \leq \int_i^\infty \mu\left(F_\alpha\right) \mathrm{d}\alpha \ \text{ and } \ |v(i) - v(i,n)| \leq \frac{1}{n}\mu(T). \tag{4.3}$$

For SDT:

$$|v - v(i)| \leq \frac{1}{i}\mu(T) \ \text{ and } \ |v(i) - v(i,n)| \leq \int_n^\infty \mu\left(F_\alpha\right) \mathrm{d}\alpha. \tag{4.4}$$

Hence, for both strategies (use (4.3) and (4.4):

$$\lim_n v(i,n) = v(i) \ \text{ for any } i \ \text{ and } \ \lim_i v(i) = v. \tag{4.5}$$

Relations (4.5) are the key of the forthcoming computations.

Now we have the formulae of future approximation.

Theorem 10 *1. We work for STD. One has, if i and n are in $\mathbb{N}$:*

$$v(i,n) = \frac{1}{n}\sum_{p=1}^{i \cdot n} \mu\left(f^{-1}\left(\left[\frac{p}{n}, \infty\right)\right)\right). \tag{4.6}$$

In case $\min\limits_{t \in T} f(t) = 1$, (4.6) becomes

$$v(i,n) = \mu(T) + \frac{1}{n}\sum_{p=1}^{(i-1) \cdot n} \mu\left(f^{-1}\left(\left[1 + \frac{p}{n}, \infty\right)\right)\right). \tag{3.6}'$$

2. Let $\varepsilon > 0$ and assume that $i(\varepsilon)$ and $n(\varepsilon)$ in $\mathbb{N}$ are such that (see (4.3)):

$$\frac{1}{n(\varepsilon)}\mu(T) < \frac{\varepsilon}{2} \ \text{ and } \ \int_{i(\varepsilon)}^\infty \mu\left(F_\alpha\right) \mathrm{d}\alpha < \frac{\varepsilon}{2}. \tag{4.7}$$

Then, for any $i \geq i(\varepsilon)$ and $n \geq n(\varepsilon)$, one has

$$|v - v(i,n)| = v - v(i,n) < \varepsilon$$

i.e. $v(i(\varepsilon), n(\varepsilon))$ is a "good" approximation of $v = (C) \int f \mathrm{d}\mu$.

Theorem 11 *We work for SDT.*

1. One has, for i and n in $\mathbb{N}$:

$$v(i, n) = \frac{1}{i} \sum_{p=1}^{n \cdot i} \mu \left(f^{-1} \left(\left[\frac{p}{i}, \infty \right) \right) \right). \tag{4.8}$$

(dual of formula (4.6)).

In case $\min_{t \in T} f(t) = 1$, (4.8) becomes

$$v(i, n) = \mu(T) + \frac{1}{i} \sum_{p=1}^{(n-1) \cdot i} \mu \left(f^{-1} \left(\left[1 + \frac{p}{i}, \infty \right) \right) \right) \tag{3.8'}$$

(dual of the formula (3.6)').

2. Let $\varepsilon > 0$ and assume that $i(\varepsilon)$ and $n(\varepsilon)$ in $\mathbb{N}$ are such that (see (4.4)):

$$\frac{1}{i(\varepsilon)} \mu(T) < \frac{\varepsilon}{2} \quad and \quad \int_{n(\varepsilon)}^{\infty} \mu\left(F_\alpha\right) d\alpha < \frac{\varepsilon}{2}. \tag{4.9}$$

Then, for any $i \geq i(\varepsilon)$ and $n \geq n(\varepsilon)$, one has

$$|v - v(i, n)| = |v - v(i, n)| < \varepsilon$$

i.e. $v(i(\varepsilon), n(\varepsilon))$ is a "good" approximation of $v = (C) \int f d\mu$.

We want to close with a numerical exemplification. To this end, we shall use the following result.

Theorem 12 (Transfer of Integrability) *We use the notations of Theorem 1.5'.*

Assume that f is $m-$integrable. Then, for any $0 \neq \lambda \in \left(-\frac{1}{A}, \infty \right)$, the function f is Choquet integrable with respect to $\mu = u(\lambda, m)$.

Numerical Exemplification

Take $T = (0, 1]$ and $\mathcal{T} =$ the Borel sets of T.

The measure $\mu : \mathcal{T} \to \mathbb{R}_+$ will be the $1-$Sugeno measure defined via

$$\mu(A) = 2^{m(A)} - 1$$

for any $A \in \mathcal{T}$, where $m : \mathcal{T} \to \mathbb{R}_+$ is the Lebesgue measure. Hence $\mu = u(1, m) = h_1 \circ m$, with previous notations.

The function f will be defined with the aid of a fixed $0 < \theta < 1$. Namely $f : (0, 1] \to \mathbb{R}_+$ acts via

$$f(t) = t^{-\theta}.$$

In order to see that f is Choquet integrable with respect to μ, one can use Theorem 12, because f is $m-$integrable.

A short computation shows that

$$(C) \int d\mu = 1 + \int_{1}^{\infty} \left(2^{\alpha^{-1/\theta}} - 1 \right) d\alpha$$

hence one cannot compute $(C) \int f \mathrm{d}\mu$ exactly.

We have the evaluation

$$1 + \frac{\theta}{1 - \theta} \ln 2 \le (C) \int f \mathrm{d}\mu <\le 1 + \frac{1}{1 - \theta} \ln 4$$

which gives, for $\theta = \dfrac{1}{2}$:

$$1.6931471 \le (C) \int f \mathrm{d}\mu \le 2.386294.$$

In the sequel, we shall work for $\theta = \dfrac{1}{2}$, computing an approximate value for $(C) \int \mathrm{d}\mu = \nu$. The error will be less than $\dfrac{1}{100}$, so we shall work for $\varepsilon = \dfrac{1}{100}$.

First, we use STD.

One can prove that, for $i \ge 1$, one has

$$\int\limits_{i}^{\infty} \mu\left(F_{\alpha}\right) \mathrm{d}\alpha \le (\ln 4) \int\limits_{i}^{\infty} \alpha^{-1/\theta} \mathrm{d}\alpha = (\ln 4) \cdot \frac{\theta}{1 - \theta} \cdot \frac{1}{i^{\frac{1-\theta}{\theta}}} \, .$$

So, in order to have (4.7) (write n instead of $n(\varepsilon)$ and i instead of $i(\varepsilon)$), it will be sufficient to have (because $\mu(T) = 1$):

$$\frac{1}{n} < \frac{\varepsilon}{2} \quad \text{and} \quad (\ln 4) \cdot \frac{\theta}{1 - \theta} \cdot \frac{1}{i^{\frac{1-\theta}{\theta}}} < \frac{\varepsilon}{2}.$$

In our case, this means

$$\frac{1}{n} \le \frac{1}{200} \quad \text{and} \quad (\ln 4) \cdot \frac{1}{i} < \frac{1}{200}$$

i.e. $n > 200$ and $i > 200 \cdot \ln 4 \approx 277.2588$.

We take $n = 201$ and $i = 278$. The approximate value $v(i, n)$ of $(C) \int f \mathrm{d}\mu$ is (see formula (3.6)' because $\min\limits_{t \in T} f(t) = 1$):

$$v(i, n) = 1 + \frac{1}{n} \sum_{p=1}^{(i-1) \cdot n} \left(2^{\left(1 + \frac{p}{n}\right)^{-1/\theta}} - 1 \right) = 2 - i + \frac{1}{n} \sum_{p=1}^{(i-1) \cdot n} 2^{\left(1 + \frac{p}{n}\right)^{-1/\theta}}$$

and we have

$$|v - v(278, 201)| < \frac{1}{100}.$$

Using a JavaScript program, we get the desired approximation:

$$v(278, 201) = 2 - 278 + \frac{1}{201} \sum_{p=1}^{201 \cdot (278-1)} 2^{\left(1 + \frac{p}{201}\right)^{-2}} = 1.780885.$$

Now, we use SDT.

As previously, for $n > 1$ one has

$$\int\limits_{n}^{\infty} \mu(F_{\alpha}) \mathrm{d}\alpha < (\ln 4) \cdot \frac{\theta}{1 - \theta} \cdot \frac{1}{n^{\frac{1-\theta}{\theta}}}.$$

So, in order to have (4.9) (again write n instead of $n(\varepsilon)$ and i instead of $i(\varepsilon)$), it will be sufficient to have

$$\frac{1}{i} < \frac{\varepsilon}{2} \text{ and } (\ln 4) \cdot \frac{\theta}{1-\theta} \cdot \frac{1}{n^{\frac{1-\theta}{\theta}}} < \frac{\varepsilon}{2}.$$

In our case, this means

$$\frac{1}{i} < \frac{1}{200} \text{ and } (\ln 4) \cdot \frac{1}{n} < \frac{1}{200}.$$

Mutatis mutandis we get $i = 201$ and $n = 278$. The good approximation is $\nu(201, 278)$:

$$[\nu - \nu(201, 278)] < \frac{1}{100},$$

with $v(201, 278) = 1.780885$ (see formula (3.8)').

In the paper "Computing Choquet Integrals" (jointly with A. Plăviţu), which recently appeared in the Journal Fuzzy Sets and Systems, vol. 327, 2017, p. 46-68, we develop the computing devices for Choquet integrals presented here and we introduce some new ones.

References

[1] I. Chitescu. Why λ-additive (fuzzy) measures? Kybernetika, 51, no.2 (2014), 246-254.

[2] I. Chitescu. Parametric continuity of Choquet and Sugeno Integrals. Current Trends in Analysis and Its Applications. Proceedings of the 9th ISSAC Congress, Krakow 2013 (2015), 341-346.

[3] I. Chitescu, D. Stanica, A. Plavitu. Existence and uniqueness of countable λ-measures with preassigned values. Fuzzy Sets and Systems, 244 (2014), 1-19.

[4] G. Choquet. Theory of capacities. Annales de l'lnstitut Fourier, 5(1954), 131-295.

[5] D. Denneberg. Non-Additive Measures and Integral. Kluwer Academic Publishers, Dordrecht, 1994.

[6] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap. Aggregation Functions. Encyclopedia of Mathematics and Its Applications, 2009.

[7] M. Grabisch, T. Murofushi, M. Sugeno (editors). Fuzzy Measures and Integrals. Theory and Applications. Physica Verlag, Heidelberg, 2000.

[8] P. R. Halmos. Measure Theory. D. Van Nostrand Company, Inc. (eleventh printing). Princeton, Toronto, New York, London, 1950.

[9] J. L. Kelley. General Topology. American Book- Van Nostrand- Reinhold (sixth printing). New York, Toronto, London, Melbourne, 1969.

[10] R. Kruse. On the construction of fuzzy measures. Fuzzy Sets and Systems, 8, no. 3 (1982), 323-327.

[11] E. Pap. Null Additive Set Functions. Kluwer Academic Publishers, Dordrecht, 1995.

[12] E. Pap (editor). Handbook of Measure Theory (two volumes). North Holland, Elsevier, 2002.

[13] D. Schmeidler. Integral Representations without Additivity. Proceed Amer. Math. Soc., 97, no. 2, (1986), 255-261.

[14] M. Sugeno. Theory of Fuzzy Integrals and Its Applications. Ph. D. Dissertation. Tokyo Institute of Technology, 1974.

[15] M. Sugeno. A note on derivatives of functions with respect to fuzzy measures. Fuzzy Sets and Systems, 222 (2013), 1-17.

[16] M. Sugeno. A way to Choquet Calculus. IEEE Transactions on Fuzzy Systems, 23, no. 5 (2015), 1439-1457.

[17] Z. Wang. Une classe de mesures floues - les quasimesures. BUSEFAL, 6 (1981), 28-37.

[18] Z. Wang, G. Klir. Generalized Measure Theory. Springer, 2009.

[19] Z. Wang, R. Yang, K. - S. Leung. Nonlinear Integrals and Their Applications in Data Mining. World Scientific, 2010.