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Generalization on local property of absolute matrix summability of factored Fourier series

Şebnem Yıldız

Ahi Evran University/Department of Mathematicst, Kırşehir,Turkey

sebnemyildiz@ahievran.edu.tr; sebnem.yildiz82@gmail.com

Abstract: In this paper, a known theorem dealing with $\left|\bar{N}, p_n\right|_k$ summability methods of Fourier series is generalized to more general cases by taking normal matrices and by using local property of absolute matrix summability of factored Fourier series.

Keywords: Summability factors, absolute matrix summability, Fourier series, infinite series, Hölder inequality, Minkowski inequality.

MSC2010: 26D15; 42A24; 40F05; 40G99.

1 Introduction

Let (s_n) denote the n-th partial sum of the series $\sum a_n$. We write

$$R_n = \left\{ s_1 + \frac{1}{2} s_2 + ... + \frac{1}{n} s_n \right\} / logn.$$

Then the series $\sum a_n$ is said to be *absolutely summable* $(R, logn, 1)$ or *summable* $|R, logn, 1|$ if the sequence $\{R_n\}$ is of bounded variation, that is, the infinite series

$$\sum |R_n - R_{n+1}|$$

is convergent. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [8]).
The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \mid w_n - w_{n-1} \mid^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp.$k = 1$), $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C, 1|_k$ (resp.$\left|\bar{N}, p_n\right|$) summability. Also, if we take $k = 1$ and $p_n = 1/(n+1)$, $\left|\bar{N}, p_n\right|_k$ summability is equivalent to $|R, logn, 1|$ summability.
A lower triangular matrix of nonzero diagonal entries is said to be a normal matrix. Let $A = (a_{nv})$ be a normal matrix, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ with entries defined by,

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...$$

and

$$\hat{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv}, \quad n = 1, 2, ...$$

It should be noted that $\hat{A}$ and $\bar{A}$ are the well-known matrices of series to series and series to sequence transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v$$

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v$$

Let (θ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, (see [12],[20]) if

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

In the special case, if we take $a_{nv} = \frac{p_v}{P_n}$ and $\theta_n = \frac{P_n}{p_n}$, then we have $|\bar{N}, p_n|_k$ summability. Also, if we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$, then we have $|R, p_n|_k$ summability (see [5]).

2 The Known Results

Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality the constant term in the constant term in the Fourier series of f can be taken to be zero, so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t).$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) sin(nt) dt.$$

We write

$$\varphi(t) = \frac{1}{2} \left\{ f(x+t) + f(x-t) \right\}.$$

It is well known that the convergence of the Fourier series at $t = x$ is a local property of f (i.e., depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and so the summability of the Fourier series $t = x$ by any regular linear summability method is also a local property of f.
It has been pointed out by Bosanquet [1] that for the case $\lambda_n = logn$, the definition of *absolutely summable* $(R, logn, 1)$ or *summable* $|R, logn, 1|$ is equivalent to the definition of the summability $|R, \lambda_n, 1|$ used by Mohanty [11], λ_n being a monotonic increasing sequence tending to infinity with n.
Matsumoto [9] improved this result by replacing the series $\sum (logn)^{-1} C_n(t)$ by

$$\sum (loglogn)^{-p} C_n(t), \quad p > 1.$$

Bhatt [2] showed that the factor $(loglogn)^{-p}$ in the above series can be replaced by the more general factor $\gamma_n logn$ where (γ_n) is a convex sequence such that $\sum n^{-1}\gamma_n$ is convergent. Borwein [7] generalized Bhatt's result by proving that (λ_n) is a sequence for which

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty,$$

then the summability $|R, P_n, 1|$ of the factored Fourier series

$$\sum_{n=1}^{\infty} \lambda_n C_n(t)$$

at any point is a local property of f. On the other hand, Mishra [10] proved that if (γ_n) is as above, and if

$$P_n = O(np_n) \quad \text{and} \quad P_n \Delta p_n = O(p_n p_{n+1}),$$

the summability $|\bar{N}, p_n|$ of the series

$$\sum_{n=1}^{\infty} \gamma_n \frac{P_n}{np_n} C_n(t),$$

at any point is a local property of f. Bor [4] showed that $|\bar{N}, p_n|$ in Mishra's result can be replaced by a more general summability method $|\bar{N}, p_n|_k$, and introduced the following theorem on the local property of the summability $|\bar{N}, p_n|_k$ of the factored Fourier series, which generalizes most of the above results under more appropriate conditions then those given in them.
Theorem 2.1[6] Let $k \geq 1$ and the sequences (p_n) and (λ_n) be such that

$$\Delta X_n = O(1/n), \tag{1}$$

$$\sum_{n=1}^{\infty} n^{-1} \left\{ |\lambda_n|^k + |\lambda_{n+1}|^k \right\} X_n^{k-1} < \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} (X_n^k + 1) |\Delta\lambda_n| < \infty, \tag{3}$$

where $X_n = (np_n)^{-1} P_n$. Then the summability $|\bar{N}, p_n|_k$ $k \geq 1$ of the series $\sum_{n=1}^{\infty} \lambda_n X_n C_n(t)$ at a point can be ensured by a local property.

3 The Main Results

Many studies have been done for matrix generalization of Fourier series (see [13]-[28]). The aim of this paper is to extend Theorem 2.1 for $|A,\theta_n|_k$ summability method by taking normal matrices instead of weighted mean matrices.

Theorem 3.1 Let $A=(a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{n0}=1,\ n=0,1,..., \tag{4}$$

$$a_{n-1,v}\geq a_{nv},\ for\ n\geq v+1, \tag{5}$$

$$\sum_{v=1}^{n-1} a_{vv}\hat{a}_{n,v+1}=O(a_{nn}). \tag{6}$$

Let $(\theta_n a_{nn})$ be a non increasing sequence. If (λ_n) and (X_n) are sequences satisfying the following conditions:

$$\sum_{n=1}^{\infty}(\theta_n a_{nn})^{k-1} n^{-1}\left\{|\lambda_n|^k+|\lambda_{n+1}|^k\right\} X_n^{k-1}<\infty, \tag{7}$$

$$\sum_{n=1}^{\infty}(\theta_n a_{nn})^{k-1}(X_n^k+1)|\Delta\lambda_n|<\infty, \tag{8}$$

$$\Delta X_n=O(1/n), \tag{9}$$

where $X_n=(na_{nn})^{-1}$, and (θ_n) is any sequence of positive constants, then the summability $|A,\theta_n|_k$, $k\geq 1$ of the series

$$\sum \lambda_n X_n C_n(t),$$

at a point can be ensured by a local property.

We need the following lemma for the proof of Theorem 3.1.

Lemma 3.2 Let $(\theta_n a_{nn})$ be a non increasing sequence. Suppose that the matrix A and the sequences (λ_n) and (X_n) satisfy all the conditions of Theorem 3.1, and that (s_n) is bounded and (θ_n) is any sequence of positive constants. Then the series

$$\sum_{n=1}^{\infty}\lambda_n X_n a_n \tag{10}$$

is summable $|A,\theta_n|_k$, $k\geq 1$.

4 Proof of Lemma 3.2

Let (T_n) denotes the A-transform of the series (10). Then we have,

$$\bar{\Delta}T_n=\sum_{v=1}^{n}\hat{a}_{nv}a_v\lambda_v X_v,\quad X_0=0.$$

Applying Abel's transformation to this sum we have

$$\bar{\Delta}T_n=\sum_{v=1}^{n-1}\Delta(\hat{a}_{nv}\lambda_v X_v)s_v+a_{nn}\lambda_n X_n s_n.$$

By the formula for the difference of products of sequences (see [8], p.129) we have

$$\Delta(\hat{a}_{nv}\lambda_v X_v) = \lambda_v X_v \Delta\hat{a}_{nv} + \Delta(\lambda_v X_v)\hat{a}_{n,v+1} = \lambda_v X_v \Delta\hat{a}_{nv} + (X_v\Delta\lambda_v + \Delta X_v \lambda_{v+1})\hat{a}_{n,v+1},$$

$$\bar{\Delta}T_n = \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v \Delta\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1}\Delta X_v s_v + \sum_{v=1}^{n-1} \bar{\Delta}a_{nv}\lambda_v X_v s_v + a_{nn}\lambda_n X_n s_n$$
$$= T_n(1) + T_n(2) + T_n(3) + T_n(4).$$

To complete the proof of Lemma 3.2, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{11}$$

The elements $\hat{a}_{nv} \geq 0$ for each v, n. it is easily seen by using conditions (4) and (5) of Theorem 3.1. For detail (see [18]).

Also,

$$\sum_{v=1}^{n-1} |\bar{\Delta}a_{nv}| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) = \bar{a}_{n-1,0} - \bar{a}_{n0} + a_{n0} - a_{n-1,0} + a_{nn}$$
$$= a_{n0} - a_{n-1,0} + a_{nn} \leq a_{nn}. \tag{12}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v |\Delta\lambda_v||s_v| \right)^k$$
$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v^k |\Delta\lambda_v| \right) \left(\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v| \right)^{k-1},$$

and by taking account of (4) and (5), we have $\hat{a}_{n,v+1} \leq a_{nn}$, for $1 \leq v \leq n-1$ which implies that

$$\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v| \leq a_{nn} \sum_{v=1}^{n-1} |\Delta\lambda_v| = O(a_{nn}),$$

thus,

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,1}|^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v^k |\Delta\lambda_v|$$
$$= O(1) \sum_{v=1}^{m} X_v^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1} = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} X_v^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1}$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} X_v^k |\Delta\lambda_v|$$
$$= O(1) \quad \text{as} \quad m \to \infty,$$

in view of condition (8). Note that from (9) follows that $\Delta X_v = O(a_{vv}X_v)$. Also, we have

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k \leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}| |\Delta X_v| |s_v| \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}| a_{vv} X_v \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}|^k a_{vv} X_v^k \right) \left(\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} |\lambda_{v+1}|^k X_v^k \right)$$

$$= O(1) \sum_{v=1}^{m} |\lambda_{v+1}|^k a_{vv} X_v^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1} = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_{v+1}|^k a_{vv} X_v^k \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1}$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_{v+1}|^k a_{vv} X_v^{k-1} X_v = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_{v+1}|^k v^{-1} X_v^{k-1}$$

$$= O(1) \quad \text{as} \quad m \to \infty.$$

by virtue of the hypotheses of Lemma 3.2. On the other hand,we have

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| |\lambda_v| X_v \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left(\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| |\lambda_v|^k X_v^k \right) \left(\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| |\lambda_v|^k X_v^k$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} |\bar{\Delta} a_{nv}| = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_v|^k X_v^k \sum_{n=v+1}^{m+1} |\bar{\Delta} a_{nv}|$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_v|^k X_v^k a_{vv}$$

$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |\lambda_v|^k X_v^{k-1} v^{-1} = O(1) \quad \text{as} \quad m \to \infty.$$

by virtue of the hypotheses of Lemma 3.2. Finally, we have that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,4}|^k = O(1) \sum_{n=1}^{\infty} \theta_n^{k-1} |\lambda_n|^k X_n^k a_{nn}^k$$

$$= O(1) \sum_{n=1}^{\infty} (\theta_n a_{nn})^{k-1} |\lambda_n|^k X_n^k a_{nn}$$

$$= O(1) \sum_{n=1}^{\infty} (\theta_n a_{nn})^{k-1} |\lambda_n|^k X_n^{k-1} n^{-1} < \infty,$$

by virtue of the hypotheses of Lemma 3.2, This completes the proof of Lemma 3.2.

Proof of Theorem 3.1. Since the convergence of the Fourier series at a point is a local property of its generating function f, the theorem follows by formula (7.1) from Chapter II of the book (see [29]) and from Lemma 3.2.

5 APPLICATIONS

We can apply Theorem 3.1 to weighted mean $A = (a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$ when $0 \leq v \leq n$, where $P_n = p_0 + p_1 + ... + p_n$. We have that,

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad \text{and} \quad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.$$

The following results can be easily verified.
1. If we take $\theta_n = \frac{P_n}{p_n}$ in Theorem 3.1, then we have another theorem dealing with absolute matrix summability (see [18]).
2. If we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we have a theorem dealing with $\left|\bar{N}, p_n\right|_k$ summability (see [6]).
3. If we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we obtain a new result dealing with $|R, p_n|_k$ summability method.
4. If we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, then we have a result for $|C, 1|_k$ summability.

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