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COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this work, we represent two new subclasses of the bi-univalent functions (by using convolution) which both $f(z)$ and $f^{-1}(z)$ are m-fold symmetric analytic functions. Among other results, for this new subclasses, bounds on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ are given in this study.

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1. Introduction

Let $\mathcal{A}$ indicate an *analytic* function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\mathbb{D} = \{z : z \in \mathbb{C} \ |z| < 1\}$, and are in the form of following equation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Furthermore, let $\mathcal{S}$ indicate a subclass in $\mathcal{A}$, being univalent in $\mathbb{D}$ (see [4]).

For $f(z)$ defined by (1) and $\Theta(z)$ defined by

$$\Theta(z) = z + \sum_{n=2}^{\infty} \Theta_n z^n, \qquad (\Theta_n \geq 0),$$

the Hadamard product $(f \star \Theta)(z)$ of the functions $f(z)$ and $\Theta(z)$ defined by

$$(f \star \Theta)(z) = z + \sum_{n=2}^{\infty} a_n \Theta_n z^n.$$

For $0 \leq \beta < 1$ and $\lambda \in \mathbb{C}$, we let $Q_\lambda(h,\beta)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1) and functions $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n, \qquad (h_n > 0), \tag{2}$$

and satisfying the analytic criterion:

$$\Theta_\lambda(h,\beta) = \left\{ f \in \mathcal{A} : \Re\left\{ (1-\lambda)\frac{(f \star h)(z)}{z} + \lambda (f \star h)'(z) \right) > \beta, \qquad 0 \leq \beta < 1, \qquad z \in \mathbb{D} \right\}.$$

From the Koebe 1/4 Theorem (for details, see [4]) each univalent f has an inverse f^{-1} fulfilling

$$f^{-1}\left(f(z)\right) = z \qquad (z \in \mathbb{D})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

On the other hand, f^{-1} is represented by

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

$$= w + \sum_{n=2}^{\infty} b_n w^n.$$

When both of f and f^{-1} are univalent, $f \in \mathcal{A}$ is known to be *bi-univalent* in $\mathbb{D}$. The equation given by (1) shows the class of all bi-univalent functions in $\mathbb{D}$, and this class is represented by Σ. For detailed information about the class of Σ was given in the references [2], [6], [7], [9], [10] and [11].

Let $m \in \mathbb{N}$. A domain $\mathbb{E}$ is known as *m-fold symmetric* if a rotation of $\mathbb{E}$ around origin with an angle $2\pi/m$ maps $\mathbb{E}$ on itself. It is then seen that, an analytic $f(z)$ in $\mathbb{D}$ being m-fold symmetric ($m \in \mathbb{N}$) satisfies the following condition

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).$$

Especially, each $f(z)$ and odd $f(z)$ are one-fold symmetric and two-fold symmetric respectively. m-fold symmetric univalent functions in $\mathbb{D}$ are represented by $\mathcal{S}_m$. In this case $f \in \mathcal{S}_m$ has the following form

$$f(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1} \qquad \left(z \in \mathbb{D},\ m \in \mathbb{N}\right). \tag{3}$$

Srivastava et al, in [8] defined m-fold symmetric bi-univalent function. They showed that each $f(z)$ derivatives an m-fold symmetric bi-univalent function for each ($m \in \mathbb{N}$) and also they brought out the results of such derivations. In addition, the following expansion of f^{-1} was acquired by them.

$$\begin{aligned} F(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &= - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots \end{aligned}$$

where $f^{-1} = F$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{D}$.

A whole treatment of this problem is given in books of several authours [13] by Oldham and Spanier, and [14] by Miller and Ross. However, our study is based on Srivastava [10] who provide more information for the concept of bi-univalent functions and we can find further detailed information in [1]. The object of the present paper is to introduce, by using convolution, new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

2. Coefficient Estimates for the function class $\mathcal{H}_{\Sigma_m}(\lambda, h, \beta, \alpha)$

In this section, we introduce by using convolution the function class $\mathcal{H}_{\Sigma_m}(\lambda, h, \beta, \alpha)$ by means of the following definition.

Definition 1. A function $f(z) \in \Sigma_m$ given by (2) is said to be in the class $\mathcal{H}_{\Sigma_m}(\lambda, h, \beta, \alpha)$ $(0 < \alpha \leq 1,\ \lambda \geq 0, m \in \mathbb{N})$ if the following conditions are satisfied:

$$f \in \Sigma_m \text{ and } \left| arg(1-\lambda)\frac{(f \star h)(z)}{z} + \lambda(f \star h)'(z)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \qquad (4)$$

and

$$\left| arg(1-\lambda)\frac{(f \star h)^{-1}(w)}{w} + \lambda((f \star h)^{-1})'(w)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \qquad (5)$$

where the function $(f \star h)^{-1}(w)$ defined as follows

$$(f \star h)^{-1}(w) = w - a_2h_2w^2 + (2a_2^2h_2^2 - a_3h_3)w^3 - (5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4)w^4 + \cdots .$$

We can easily see that for one fold case, when we take $\lambda = 1$, $m = 2$ and $h(z) = \frac{z}{1-z}$, the class $\mathcal{H}_{\Sigma}(\lambda, h, \beta)$ reduce to the class $\mathcal{H}_{\Sigma}(\beta)$ studied by Srivastava et al.[8]. So as to calculate our main result, we need to following lemma.

Lemma 1.[12] Let the function $\psi(z) = 1 + \sum_{k=1}^{\infty} h_k z^k,\ z \in \mathbb{U}$, such that $\psi \in P_m(\beta)$. Then

$$|h_k| \leq n(1-\beta),\ \ k \geq 1.$$

Theorem 1. Let $f \in \mathcal{H}_{\Sigma_m}(\lambda, h, \beta, \alpha)$ $(0 < \alpha \leq 1,\ \lambda \geq 0,\ 0 \leq \beta < 1,\ m \in \mathbb{N})$ be given by (3) where the function $h(z)$ is given by (2). If $h_{m+1}, h_{2m+1} \neq 0$ and $\lambda \in \mathbb{C} \setminus \{\frac{-1}{m}; \frac{-1}{2m}\}$ then

$$|a_{m+1}| \leq min\left\{\sqrt{\frac{2\alpha^2 n(1-\beta)}{[(1+m\lambda)^2 + \alpha m(1+2m\lambda - m\lambda^2)]|h_{m+1}|^2}}, \frac{\alpha n(1-\beta)}{(1+m\lambda)|h_{m+1}|}\right\} \tag{6}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{\alpha n(1-\beta) + \frac{\alpha(\alpha-1)}{2}n^2(1-\beta)^2}{(1+2m\lambda)|h_{2m+1}|}, \frac{\alpha n(1-\beta)}{(1+2m\lambda)|h_{2m+1}|} + \frac{n^2(1-\beta)^2\alpha^2(m+1)}{2(1+m\lambda)^2|h_{2m+1}|}\right\}. \tag{7}$$

Proof. Let $f \in \mathcal{H}_{\Sigma_m}(\lambda, h, \beta, \alpha)$. Then

$$(1-\lambda)\frac{(f \star h)(z)}{z} + \lambda(f \star h)'(z) = [p(z)]^{\alpha} \tag{8}$$

and for its inverse map, $(f \star h)^{-1}$, we have

$$(1-\lambda)\frac{(f \star h)^{-1}(w)}{w} + \lambda((f \star h)^{-1})(w) = [q(w)]^{\alpha} \tag{9}$$

where $p(z), q(w) \in P_n(\beta)$. Using the both functions $p(z)$ and $q(w)$ have the following forms.

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{10}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{11}$$

Now, equating the coefficients in equation (8) and (9), we get

$$(1+m\lambda)a_{m+1}h_{m+1} = \alpha p_m \tag{12}$$

$$(1+2m\lambda)a_{2m+1}h_{2m+1} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2 \tag{13}$$

$$-(1+m\lambda)a_{m+1}h_{m+1} = \alpha q_m \tag{14}$$

$$(1+2m\lambda)\left[(m+1)a_{m+1}^2h_{m+1}^2 - a_{2m+1}h_{2m+1}\right] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \tag{15}$$

Since $p, q \in P_n(\beta)$, according to Lemma 1, the next inequalities hold:

$$|p_t| \leq n(1-\beta), \qquad t \geq 1 \tag{16}$$

$$|q_t| \leq n(1-\beta), \qquad t \geq 1. \tag{17}$$

From (12) and 14, we get

$$p_m = -q_m \tag{18}$$

and

$$2(1+m\lambda)^2a_{m+1}^2h_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{19}$$

Furthermore from (13), (15) and 22, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{[(1+m\lambda)^2+\alpha m(1+2m\lambda-m\lambda^2)]|h_{m+1}|^2} \tag{20}$$

$$\leq \frac{2\alpha^2 n(1-\beta)}{[(1+m\lambda)^2+\alpha m(1+2m\lambda-m\lambda^2)]|h_{m+1}|^2}. \tag{21}$$

From (12), by using (16) we get

$$a_{m+1} \leq \frac{\alpha n(1-\beta)}{(1+m\lambda)h_{m+1}}.$$

From (13), by using (16) we obtain

$$a_{2m+1} \leq \frac{\alpha n(1-\beta) + \frac{\alpha(\alpha-1)}{2}n^2(1-\beta)^2}{(1+2m\lambda)h_{2m+1}}.$$

Also, substracting (15) from (13), we get

$$(1+2m\lambda)\left[2a_{2m+1}h_{2m+1} - (m+1)a_{m+1}^2h_{m+1}^2\right] = \alpha(p_{2m}-q_{2m}) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right).$$

and using (12), (16) and (17) we finally have

$$(1+2m\lambda)\left[2a_{2m+1}h_{2m+1}-(m+1)a_{m+1}^2h_{m+1}^2\right]=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{22}$$

From (22) we can obtain the equality given as

$$a_{m+1}^2h_{m+1}^2=\frac{\alpha^2\left(p_m^2+q_m^2\right)}{2(1+m\lambda)^2} \tag{23}$$

when we write this equality given as (23) in (22) and by using (16) and (17), and also observing that $p_m^2=q_m^2$, it follows that

$$a_{2m+1}=\frac{\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2)+(1+2m\lambda)(m+1)\frac{\alpha^2(p_m^2+q_m^2)}{2(1+m\lambda)^2}}{2(1+2m\lambda)h_{2m+1}}.$$

$$a_{2m+1}=\frac{\alpha(p_{2m}-q_{2m})}{2(1+2m\lambda)h_{2m+1}}+\frac{\alpha(\alpha-1)(p_m^2-q_m^2)}{4(1+2m\lambda)h_{2m+1}}+\frac{(m+1)\alpha^2(p_m^2+q_m^2)}{4(1+m\lambda)^2h_{2m+1}} \tag{24}$$

Taking the absolute value of (24) and applying (16) and (17), and also taking into consideration that $p_m^2=q_m^2$, we obtain

$$|a_{2m+1}|\leq\frac{\alpha n(1-\beta)}{(1+2m\lambda)|h_{2m+1}|}+\frac{n^2(1-\beta)^2\alpha^2(m+1)}{2(1+m\lambda)^2|h_{2m+1}|}. \tag{25}$$

which completes the proof of the Theorem 1.

3. Coefficient Estimates for the function class $\mathcal{H}_{\Sigma_m}(\lambda,h,\xi)$

Definition 2. A function $f(z)\in\Sigma_m$ given by (3) is said to be in the class $\mathcal{H}_{\Sigma_m}(\lambda,h,\xi)$ $(0\leq\xi<1,\ \lambda\geq0, m\in\mathbb{N})$ if the following conditions are satisfied:

$$f\in\Sigma_m \text{ and } Re\left\{(1-\lambda)\frac{(f\star h)(z)}{z}+\lambda(f\star h)'(z)\right\}>\xi\ \ (z\in\mathbb{U}) \tag{26}$$

and

$$Re\left\{(1-\lambda)\frac{(f\star h)^{-1}(w)}{w}+\lambda((f\star h)^{-1})'(w)\right\}>\xi\ \ (w\in\mathbb{U}) \tag{27}$$

where the function $(f\star h)^{-1}(w)$ defined as follows

$$(f\star h)^{-1}(w)=w-a_2h_2w^2+(2a_2^2h_2^2-a_3h_3)w^3-(5a_2^3h_2^3-5a_2h_2a_3h_3+a_4h_4)w^4+\cdots.$$

Theorem 2. Let $f \in \mathcal{H}_{\Sigma_m}(\lambda, h, \xi)$ $(\lambda \geq 0,\ 0 \leq \xi < 1,\ m \in \mathbb{N})$ be given by (3) where the function $h(z)$ is given by (2). If $h_{m+1}, h_{2m+1} \neq 0$ and $\lambda \in \mathbb{C} \setminus \{\frac{-1}{m}; \frac{-1}{2m}\}$ then

$$|a_{m+1}| \leq \sqrt{\frac{2n(1-\xi)}{(m+1)(1+2m\lambda)|h_{m+1}|^2}} \tag{28}$$

and

$$|a_{2m+1}| \leq \frac{n(1-\xi)}{(1+2m\lambda)|h_{2m+1}|} + \frac{(m+1)n^2(1-\xi)^2}{2(1+m\lambda)^2|h_{2m+1}|}. \tag{29}$$

Proof. Let $f \in \mathcal{H}_{\Sigma_m}(\lambda, h, \xi)$. From the Definition 2 we obtain

$$(1-\lambda)\frac{(f \star h)(z)}{z} + \lambda(f \star h)'(z) = p(z) \tag{30}$$

and

$$(1-\lambda)\frac{(f \star h)^{-1}(w)}{w} + \lambda((f \star h)^{-1})'(w) = q(w) \tag{31}$$

where $p(z), q(w) \in P_n(\xi)$.

Using the fact that $p(z), q(w)$ have the Taylor expansions given by (10) and (11), respectively. Equating coefficients (30) and (31) yields

$$(1+m\lambda)a_{m+1}h_{m+1} = p_m \tag{32}$$

$$(1+2m\lambda)a_{2m+1}h_{2m+1} = p_{2m} \tag{33}$$

$$-(1+m\lambda)a_{m+1}h_{m+1} = q_m \tag{34}$$

$$(1+2m\lambda)\left[(m+1)a_{m+1}^2h_{m+1}^2 - a_{2m+1}h_{2m+1}\right] = q_{2m}. \tag{35}$$

Since $p(z), q(w) \in P_n(\xi)$, with respect to Lemma 1, the inequalities given (16) and (17) and thus, from (33) and (35), by using the inequalities (16) and (17) we get

$$|a_{m+1}|^2 \leq \frac{(|p_{2m}| + |q_{2m}|)(1-\xi)}{(m+1)|(1+2m\lambda)||h_{m+1}|^2} \leq \frac{2n(1-\xi)}{(m+1)|(1+2m\lambda)||h_{m+1}|^2},\ for\ \lambda \in \mathbb{C} \setminus \left\{\frac{-1}{2m}\right\}. \tag{36}$$

From (32), by using (16) we obtain

$$a_{m+1} \leq \frac{n(1-\xi)}{|(1+m\lambda)||h_{m+1}|} \;\; for \;\; \lambda \in \mathbb{C} \setminus \left\{ \frac{-1}{m} \right\}.$$

Also from (33) and by using (16) we obtain

$$a_{2m+1} \leq \frac{n(1-\xi)}{|(1+2m\lambda)||h_{2m+1}|} \;\; for \;\; \lambda \in \mathbb{C} \setminus \left\{ \frac{-1}{2m} \right\}.$$

Also, subtracting (13)from (33), we obtain

$$(1+2m\lambda)\left[2a_{2m+1}h_{2m+1} - (m+1)a_{m+1}^2 h_{m+1}^2\right] = p_{2m} - q_{2m},$$

and using (32), (16) and (17) in the above equality, we have

$$|a_{2m+1}| \leq \frac{n(1-\xi)}{|(1+2m\lambda)||h_{2m+1}|} + \frac{(m+1)n^2(1-\xi)^2}{2|(1+m\lambda)^2||h_{2m+1}|} \;\; for \;\; \lambda \in \mathbb{C} \setminus \left\{ \frac{-1}{m}; \frac{-1}{2m} \right\}$$

which completes the proof of Theorem 2.

Taking some special values of parameters we can obtain some corollories given below.

Corollary 1. Taking $\lambda = 0$ in Theorem 1 we obtain

$$|a_{m+1}| \leq \min\left\{ \sqrt{\frac{2\alpha^2 n(1-\beta)}{(1+\alpha m)|h_{m+1}|^2}}, \frac{\alpha n(1-\beta)}{|h_{m+1}|} \right\} \;\; for \;\; h_{2m+1} \neq 0$$

and

$$|a_{2m+1}| \leq \min\left\{ \frac{\alpha n(1-\beta) + \frac{\alpha(\alpha-1)}{2} n^2 (1-\beta)^2}{|h_{2m+1}|}, \frac{\alpha n(1-\beta)}{|h_{2m+1}|} + \frac{n^2(1-\beta)^2 \alpha^2 (m+1)}{2|h_{2m+1}|} \right\}, \;\; for \;\; h_{2m+1} \neq 0.$$

Corollary 2. Taking $\lambda = 1$ Theorem 1, we obtain

$$|a_{m+1}| \leq \min\left\{ \sqrt{\frac{2\alpha^2 n(1-\beta)}{[(1+m)^2 + \alpha m(1+m)]|h_{m+1}|^2}}, \frac{\alpha n(1-\beta)}{(1+m)|h_{m+1}|} \right\} \;\; for \;\; h_{2m+1} \neq 0$$

and

$$|a_{2m+1}| \leq \min\left\{ \frac{\alpha n(1-\beta) + \frac{\alpha(\alpha-1)}{2} n^2 (1-\beta)^2}{(1+2m)|h_{2m+1}|}, \frac{\alpha n(1-\beta)}{(1+2m)|h_{2m+1}|} + \frac{n^2(1-\beta)^2 \alpha^2 (m+1)}{2(1+m)^2|h_{2m+1}|} \right\},$$

$for \;\; h_{2m+1} \neq 0.$

When we put $\lambda = 1$, $n = 2$ and $h(z) = \frac{z}{1-z}$ in Theorem 1 we can easily obtain the next Corollary. **Corollary 3.** For $f \in H_\Sigma(\beta)$ we obtain next inequalities

$$|a_{m+1}| \leq \min\left\{\sqrt{\frac{4\alpha^2(1-\beta)}{(1+m)^2+\alpha m(1+m)}}, \frac{2\alpha(1-\beta)}{(1+m)}\right\}$$

and

$$|a_{2m+1}| \leq \min\left\{\frac{2\alpha(1-\beta)+\frac{4\alpha(\alpha-1)(1-\beta)^2}{2}}{(1+2m)}, \frac{2\alpha(1-\beta)}{(1+2m)}+\frac{4(1-\beta)^2\alpha^2(m+1)}{2(1+m)^2}\right\}.$$

For one-fold case and $\alpha = 1$ in Corollory 5 the last Corollary is obtained as follows:

Corollary 4. For one-fold case and $\alpha = 1$ in Corollory 3 the last Corollary is obtained as follows [3]:

For $f \in H_\Sigma(\beta)$ we obtain next inequalities

$$|a_2| \leq \left\{\sqrt{\frac{2(1-\beta)}{3}}\right\} \quad 0 \leq \beta \leq \frac{1}{3}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{3} \quad 0 \leq \beta \leq \frac{1}{3}$$

$$|2a_2^2 - a_3| \leq \frac{2(1-\beta)}{3} \quad 0 \leq \beta \leq \frac{1}{3}.$$

Remark. For 1-fold symmetric bi-univalent functions, Theorem 1 and Theorem 2 reduce to results given by Frasin and Aouf [5]. Also,for 1-fold symmetric bi-univalent functions,if we put $\lambda = 1$ in our Theorems, we obtain to results which were given by Srivastava et al.[8]. Furthermore, for m-fold symmetric bi-univalent functions, if we put $\lambda = 1$ in Theorem 1 and Theorem 2, we obtain to results which were given by Srivastava et al.[8].

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