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EULER TYPE INTEGRAL OPERATOR INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. The main object of the present research note is to establish an Euler type integral operator involving generalized Mittag-Leffler function defined by Khan and Ahmed [9]. Furthermore, a variation of our main result and their special cases are also indicated.

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1. Introduction

Recently a number of authors namely, Ali [1], Agarwal et al. [2], Choi et al. [3, 4, 5], Khan and Ahmed [9] and Khan et al. [10, 11], Ghayasuddin et al. [7] have established some interesting integrals operators involving various kind of special function, which are potentially useful in many diverse field of physics and engineering sciences. In a sequel of such type of works mentioned above, in this paper, we further establish a new Euler type integral operator involving generalized Mittage -Leffler function. For the purposes our present study, we begin by recalling here the following definitions of some well known functions:

The Swedish mathematician Mittag-Leffler [14] introduced the function $E_\alpha(z)$ defined as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where $z \in \mathbb{C}$ and $\Gamma(s)$ is the Gamma function; $\alpha \geq 0$.

The Mittag-Leffler function is a direct generalization of $\exp(z)$ in which $\alpha = 1$. Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations (see [8]).

A generalization of $E_\alpha(z)$ was studied by Wiman [22] where he defined the function $E_{\alpha,\beta}(z)$ as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0$, which is also known as Mittag-Leffler function or Wiman's function.

Prabhakar [15] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form (see also Killbas et al. [12]):

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0$.

In (2007), Shukla and Prajapati [18] introduced and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.4}$$

where $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, q \in (0,1) \bigcup \mathbb{N}$ and $(\gamma)_{qn} = \dfrac{\Gamma(\gamma + qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol.

A new generalized Mittag-Leffler function was defined by Salim [21] as:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n}, \tag{1.5}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0$.

Afterward, Salim and Faraj [20] introduced the generalized Mittag-Leffler function $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ defined as:

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \tag{1.6}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}, \min\{\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0\} > 0; p, q > 0$ and $q < \Re(\alpha) + p$.

Very recently, Khan and Ahmed [9] introduced a further extension of Mittag-Leffler function defined as:

$$E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\nu)_{\sigma n}(\delta)_{pn}} z^n, \tag{1.7}$$

where $\alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C}$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0$; $p, q > 0$ and $q \leq \Re(\alpha) + p$.

Equation (1.7) is a generalization of equation (1.1)-(1.6).

• On setting $\mu = \nu$, $\rho = \sigma$, (1.7) reduces to the Mittag-Leffler function defined by (1.6) .

• On setting $\mu = \nu$, $\rho = \sigma$, and $p = q = 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.5) .

• On setting $\mu = \nu$, $\rho = \sigma$, and $\delta = p = 1$, (1.7) reduces to the Mittag-Leffler function defined by (1.4), which further for $q = 1$, gives the known generalization of Mittag-Leffler function gives by (1.3) .

• On setting $\mu = \nu$, $\rho = \sigma$, and $\delta = p = q = 1$, (1.7) reduces to (1.2), which further for $\beta = 1$, reduces to the Mittag-Leffler function by (1.1).

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [6] and Wright ([23, 24, 25]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [19, p.21]; see also [17]):

$${}_p\Psi_q \left[\begin{array}{cccc} (\alpha_1\ ,\ A_1), & \ldots.., & (\alpha_p\ ,\ A_p); & \\ & & & z \\ (\beta_1\ ,\ B_1), & \ldots.., & (\beta_q\ ,\ B_q); & \end{array} \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}\ , \tag{1.8}$$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

(i) $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$ and $0 < |z| < \infty;\ z \neq 0$.

(1.9)

(ii) $1+\sum_{j=1}^{q} B_j-\sum_{j=1}^{p} A_j = 0$ and $0 < |z| < {A_1}^{-A_1} \dots {A_p}^{-A_p} {B_1}^{B_1} \dots {B_q}^{B_q}$.

(1.10)

A special case of (1.8) is

$$ {}_p\Psi_q \left[\begin{array}{ccc} (\alpha_1\ ,\ 1), & ....., & (\alpha_p\ ,\ 1); \\ & & \\ (\beta_1\ ,\ 1), & ....., & (\beta_q\ ,\ 1); \end{array} z \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \, {}_pF_q \left[\begin{array}{ccc} \alpha_1, & ....., & \alpha_p\ ; \\ & & \\ \beta_1, & ....., & \beta_q\ ; \end{array} z \right], \tag{1.11} $$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [16])

$$ {}_pF_q \left[\begin{array}{ccc} \alpha_1, & ....., & \alpha_p\ ; \\ & & \\ \beta_1, & ....., & \beta_q\ ; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} $$

$$ = {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z), \tag{1.12} $$

where $(\lambda)_n$ is the Pochhammer's symbol [16].

Furthermore, we also recall here the following interesting and useful result due to MacRobert [13] :

$$ \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} dx = \frac{\Gamma(\alpha)\ \Gamma(\beta)}{a^\alpha b^\beta \Gamma(\alpha+\beta)}, \tag{1.13} $$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

2. Integral operator involving generalized Mittag-Leffler function

Theorem 2.1. If $\alpha, \beta, \gamma, \delta, \mu, \nu, \rho, \sigma, \eta, \lambda \in \mathbb{C}$, $\eta+\sigma+p-\rho-q > 0$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\lambda) > 0$, and $p, q > 0$ and $q \leq \Re(\alpha) + p$, then

$$ \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E^{\mu,\rho,\gamma,q}_{\eta,\lambda,\nu,\sigma,\delta,p} \left[\frac{2abx(1-x)}{[ax+b(1-x)]^2} \right] dx $$

$$ = \frac{\Gamma(\nu)\Gamma(\delta)}{a^\alpha b^\beta \Gamma(\mu)\Gamma(\gamma)} {}_5\Psi_4 \left[\begin{array}{c} (1,1), (\mu,\rho), (\gamma,q), (\beta,1), (\alpha,1); \\ \\ (\lambda,\eta), (\nu,\sigma), (\delta,p), (\alpha+\beta,2); \end{array} 2 \right]. \tag{2.1} $$

Proof. In order to establish our main result (2.1), we denote the left-hand side of (2.1) by I and then using (1.7), to get

$$I=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\eta n+\lambda)(\nu)_{\sigma n}(\delta)_{pn}}z^n. \quad (2.2)$$

Now changing the order of integration and summation (which is guaranteed under the given conditions), and then by using the result (1.13), we get

$$I=\frac{\Gamma(\nu)\Gamma(\delta)}{a^{\alpha}b^{\beta}\Gamma(\mu)\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(1+n)}{\Gamma(\lambda+\eta n)\Gamma(\nu+\sigma n)\Gamma(\delta+pn)\Gamma(\alpha+\beta+2n)}\frac{2^n}{n!}. \quad (2.3)$$

Finally, summing up the above series with the help of (1.8), we easily arrive at the right-hand side of (2.1). This completes the proof of our main result.

Next, we consider other variation of our main result, in which the L.H.S. of (2.1) is expressed in terms of generalized hypergeometric function ${}_pF_q$.

Variation of (2.1): Let the conditions of our main result be satisfied, then the following integral formula holds true:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E^{\mu,\rho,\gamma,q}_{\eta,\lambda,\nu,\sigma,\delta,p}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\lambda)\Gamma(\alpha+\beta)}$$

$$\times{}_{\rho+q+3}F_{\eta+\sigma+p+2}\left[\begin{array}{cccccc}\Delta(\rho;\ \mu), & \Delta(q;\gamma), & \alpha, & \beta, & 1; & \frac{\rho^{\rho}q^{q}}{2\eta^{\eta}p^{p}\sigma^{\sigma}}\\ \Delta(\eta;\lambda), & \Delta(\sigma;\nu), & \Delta(p;\delta), & \Delta(2;\alpha+\beta); & & \end{array}\right], \quad (2.4)$$

where $\Delta(m;\ l)$ abbreviates the array of m parameters $\frac{l}{m},\ \frac{l+1}{m},\ \cdots,\ \frac{l+m-1}{m}$, $m\geq 1$.

Proof: In order to prove the result (2.4), using the results

$$\Gamma(\alpha+n)=\Gamma(\alpha)(\alpha)_n$$

and

$$(l)_{kn}=\ k^{kn}\left(\frac{l}{k}\right)_n\left(\frac{l+1}{k}\right)_n\cdots\left(\frac{l+k-1}{k}\right)_n,$$

(Gauss multiplication theorem) in (2.3) and summing up the given series with the help of (1.12), we easily arrive at our required result (2.4).

3. Special Cases

(i). On setting $\mu=\nu$, $\rho=\sigma$ in (2.1) and then by using (1.6), we get the following interesting integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\eta,\lambda,p}^{\gamma,\delta,q}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$=\frac{\Gamma(\delta)}{a^\alpha b^\beta\Gamma(\gamma)}\ {}_4\Psi_3\left[\begin{array}{cccc} (\gamma,\ q), & (\alpha,\ 1), & (\beta,\ 1), & (1,\ 1); \\ & & & & 2 \\ (\lambda\ ,\ \eta), & (\delta,\ p), & (\alpha+\beta,2) & ; \end{array}\right], \tag{3.1}$$

where $\Re(\alpha)>0,\Re(\beta)>0,\Re(\lambda)>0,\Re(\gamma)>0,\Re(\delta)>0\ ;\ \eta,\alpha,\beta,\gamma,\delta\in\mathbb{C}$.

(ii). Further, On setting $\mu=\nu$, $\rho=\sigma$ in (2.4) and then by using (1.6), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\eta,\lambda,p}^{\gamma,\delta,q}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{a^\alpha b^\beta\Gamma(\lambda)\Gamma(\alpha+\beta)}\ {}_{q+3}F_{\eta+p+2}\left[\begin{array}{cccc} \Delta(q;\gamma,), & \alpha, & \beta, & 1; \\ & & & & \dfrac{q^q}{2\eta^\eta p^p} \\ \Delta(\eta;\lambda), & \Delta(p;\delta), & \Delta(2;\alpha+\beta) & ; \end{array}\right], \tag{3.2}$$

where $\Re(\alpha)>0,\Re(\beta)>0,\Re(\lambda)>0,\Re(\gamma)>0,\Re(\delta)>0\ ;\ \eta,\alpha,\beta,\gamma,\delta\in\mathbb{C}$.

(iii). On setting $\mu\ =\nu$, $\rho\ =\sigma$ and $p=q=1$ in (2.1) and then by using (1.5), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\eta,\lambda,}^{\gamma,\delta,}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$=\frac{\Gamma(\delta)}{a^\alpha b^\beta\Gamma(\gamma)}\ {}_4\Psi_3\left[\begin{array}{cccc} (1\ 1), & (\gamma,\ 1), & (\alpha,\ 1), & (\beta,1); \\ & & & & 2 \\ (\lambda,\eta), & (\delta,\ 1), & (\alpha+\beta,2) & ; \end{array}\right], \tag{3.3}$$

where $\Re(\alpha)>0,\Re(\beta)>0,\Re(\lambda)>0,\Re(\gamma)>0,\Re(\delta)>0;\ \alpha,\beta,\lambda,\eta,\gamma\in\mathbb{C}$.

(iv). On setting $\mu=\nu$, $\rho=\sigma$ and $p=q=1$ in (2.4) and then by using (1.5), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}E_{\eta,\lambda,}^{\gamma,\delta,}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\lambda)\Gamma(\alpha+\beta)}\ {}_4F_{\eta+3}\left[\begin{array}{ccccc} \gamma, & \alpha, & \beta, & 1 & ; \ \frac{1}{2\eta^{\eta}} \\ \Delta(\eta;\lambda), & \delta, & \Delta(2;\alpha+\beta), & & ; \end{array}\right], \tag{3.4}$$

where $\Re(\beta) > 0, \Re(\alpha) > 0, \Re(\lambda) > 0, \Re(\gamma) > 0, \Re(\delta) > 0$; $\alpha, \beta, \lambda, \eta, \gamma \in \mathbb{C}$.

(v). On setting $\mu = \nu$, $\rho = \sigma$ and $p = \delta = 1$ in (2.1) and then by using (1.4), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta,\lambda,}^{\gamma,q,}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{1}{a^{\alpha}b^{\beta}\Gamma(\gamma)}\ {}_3\Psi_2\left[\begin{array}{cccc} (\gamma,q), & (\alpha,\ 1), & (\beta,\ 1); & \\ & & & 2 \\ (\lambda,\ \eta), & (\alpha+\beta,\ 2); & & \end{array}\right], \tag{3.5}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0, \Re(\gamma) > 0$; $\alpha, \beta, \eta, \lambda, \gamma \in \mathbb{C}$ and $q \in (0,1) \bigcup N$.

(vi). On setting $\mu = \nu$, $\rho = \sigma$ and $p = \delta = 1$ in (2.4) and then by using (1.4), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta,\lambda,}^{\gamma,q}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\Gamma(\alpha)\ \Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\lambda)\ \Gamma(\alpha+\beta)}\ {}_{q+2}F_{\eta+2}\left[\begin{array}{cccc} \Delta(q;\gamma), & \alpha, & \beta; & \\ & & & \frac{q^q}{2\eta^{\eta}} \\ \Delta(\eta;\ \lambda), & \Delta(2;\ \alpha+\beta); & & \end{array}\right], \tag{3.6}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0, \Re(\gamma) > 0$; $\alpha, \beta, \eta, \lambda, \gamma \in \mathbb{C}$ and $q \in (0,1) \bigcup N$.

(vii). On setting $\mu = \nu$, $\rho = \sigma$ and $p = \delta = q = 1$ in (2.1) and then by using (1.3), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta,\lambda,}^{\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{1}{a^{\alpha}b^{\beta}\Gamma(\gamma)}\ {}_3\Psi_2\left[\begin{array}{cccc} (\gamma,1), & (\alpha,1), & (\beta,1); & \\ & & & 2 \\ (\lambda,\eta), & (\alpha+\beta,2) & ; & \end{array}\right], \tag{3.7}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0, \Re(\gamma) > 0$; $\alpha, \beta, \eta, \lambda, \gamma \in \mathbb{C}$.

(viii). On setting $\mu = \nu$, $\rho = \sigma$ and $p = \delta = q = 1$ in (2.4) and then by using (1.3), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta,\lambda,}^{\gamma}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\lambda)\Gamma(\alpha+\beta)}\ {}_3F_{\eta+2}\left[\begin{array}{cccc} \gamma, & \alpha, & \beta; & \\ & & & \frac{1}{2\eta^{\eta}} \\ \Delta(\eta;\lambda), & \Delta(2;\alpha+\beta) & ; & \end{array}\right], \tag{3.8}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0, \Re(\gamma) > 0$; $\alpha, \beta, \eta, \gamma, \lambda \in \mathbb{C}$.

(ix). On setting $\mu = \nu$, $\rho = \sigma$ and $p = q = \delta = \gamma = 1$ in (2.1) and then by using (1.2), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta,\lambda}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{1}{a^{\alpha}b^{\beta}}\, {}_3\Psi_2\left[\begin{array}{cccc} (1,1), & (\alpha,1), & (\beta,1); & \\ & & & 2 \\ (\lambda,\eta), & (\alpha+\beta,2) & ; & \end{array}\right], \tag{3.9}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$; $\alpha, \beta, \eta, \lambda \in \mathbb{C}$.

(x). On setting $\mu = \nu$, $\rho = \sigma$ and $p = q = \delta = \gamma = 1$ in (2.4) and then by using (1.2), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta,\lambda}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\lambda)\Gamma(\alpha+\beta)}\, {}_3F_2\left[\begin{array}{cccc} 1, & \alpha, & \beta; & \\ & & & \frac{1}{2\eta^{\eta}} \\ \Delta(\eta;\lambda), & \Delta(2;\alpha+\beta) & ; & \end{array}\right], \tag{3.10}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$; $\alpha, \beta, \eta, \lambda \in \mathbb{C}$.

(xi). On setting $\mu = \nu$, $\rho = \sigma$ and $p = q = \delta = \lambda = \gamma = 1$ in (2.1) and then by using (1.1), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{1}{a^{\alpha}b^{\beta}}\ {}_3\Psi_2\left[\begin{array}{cccc} (1,1), & (\alpha,1), & (\beta,1); & \\ & & & 2 \\ (1,\eta), & (\alpha+\beta,2) & ; & \end{array}\right], \tag{3.11}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0$; $\alpha, \beta, \eta \in \mathbb{C}$.

(xii). On setting $\mu = \nu$, $\rho = \sigma$ and $p = q = \delta = \gamma = \lambda = 1$ in (2.4) and then by using (1.1), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} E_{\eta}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{a^{\alpha}b^{\beta}\Gamma(\alpha+\beta)} \, {}_3F_{\eta+2}\left[\begin{array}{cccc} 1, & \alpha, & \beta; & \\ & & & \frac{1}{2\eta^{\eta}} \\ \Delta(\eta;\ 1), & \Delta(2;\alpha+\beta) & ; & \end{array}\right], \tag{3.12}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0;\ \alpha, \beta, \eta \in \mathbb{C}$.

(xiii). On setting $\mu = \nu$, $\rho = \sigma$ and $p = q = \delta = \lambda = \eta = \gamma = 1$ in (2.1), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\ \exp\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{1}{a^{\alpha}b^{\beta}} \, {}_2\Psi_1\left[\begin{array}{ccc} (\alpha,1), & (\beta,1); & \\ & & 2 \\ (\ \alpha+\beta,2); & & \end{array}\right], \tag{3.13}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0;\ \alpha, \beta \in \mathbb{C}$.

(xiv). On setting $\mu = \nu$, $\rho = \sigma$ and $p = q = \delta = \lambda = \eta = \gamma = 1$ in (2.1), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\ \exp\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\Gamma(\alpha)\ \Gamma(\beta)}{a^{\alpha}b^{\beta}\ \Gamma(\alpha+\beta)} \, {}_2F_2\left[\begin{array}{ccc} \alpha, & \beta\ ; & \\ & & \frac{1}{2} \\ \Delta(2;\ \alpha+\beta); & & \end{array}\right], \tag{3.14}$$

where $\Re(\alpha) > 0, \Re(\beta) > 0;\ \alpha, \beta \in \mathbb{C}$.

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