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KÖTHE-BOCHNER SPACES THAT ARE BANACH ALGEBRAS WITH UNIT

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Abstract. In a previous paper of the first author, it was proved that the only Köthe (scalar) spaces which are also Banach algebras with unit are the $L^\infty(\mu)$ spaces. Using this result, in the present paper it is shown that the only Köthe-Bochner (vector) spaces which are also Banach algebras with unit are the $L^\infty(X, \mu)$ spaces.

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1. Introduction

The classical scalar *Köthe spaces* L_ρ are the natural generalizations of the Lebesgue spaces, Orlicz spaces and many other function spaces. The "ancestors" of the L_ρ spaces are the "gestufte Räume" introduced by G. Köthe and O. Toeplitz in the seminal paper [10] (G. Köthe continued their study in [9]). The general, natural setting of their theory within the framework of measurable functions is due to A. C. Zaanen and W. A. J. Luxemburg and, of course, to their pupils. The doctoral thesis of W. A. J. Luxemburg [12], under the supervision of A. C. Zaanen, was the first step into this direction (viewing the Köthe spaces as spaces of (classes of) measurable functions). Subsequently, A. C. Zaanen and W. A. J. Luxemburg wrote a long series of papers [13], concerning this subject and giving, practically, all the main results of the theory. A systematic presentation of the theory is contained in the monograph [15] of A. C. Zaanen (see also [3]). Notice that the name "Köthe spaces" was given by J. Dieudonné in [4] (in [14], some errors in Dieudonné's paper are corrected).

The theory of Köthe spaces continued to develop, especially in the direction of various generalizations. For instance, very recently, a non commutative theory appeared, with multiple Functional Analysis connections, see [6].

Perhaps the most natural generalization consists in considering vector valued measurable functions instead of scalar valued ones. The spaces $L_\rho(X)$ appearing in this way (X Banach space) are called *Köthe-Bochner spaces*, being intensively studied now. The monograph [11] is entirely dedicated to these spaces, containing a rich reference list.

The main goal of this paper is to characterize those $L_\rho(X)$ spaces which are at the same time Banach algebras with unit. Our proofs rely heavily on the results in the paper [2], where it was proved that, practically, the only Köthe spaces L_ρ which are also Banach algebras with unit are the spaces $L^\infty(\mu)$. Generalizing this fact, we show here that, considering a Banach algebra with unit X instead of the scalar field, the only spaces $L_\rho(X)$ which are Banach algebras with unit are the spaces $L^\infty(X,\mu)$. A practical example is given too.

2. Preliminary Part

Throughout the paper $\mathbb{N} = \{1, 2, ...\}$, $\mathbb{R}_+ = [0,\infty)$, $\overline{\mathbb{R}}_+ = [0,\infty] = \mathbb{R}_+ \cup \{\infty\}$ and $K = \mathbb{R}$ or $\mathbb{C}$.

All sequences $(x_n)_n$ are indexed with $\mathbb{N}$. When writing $(x_n)_n \subset A$, we mean $x_n \in A$ for any n.

For two non empty sets T and X and any element $x \in X$, we can consider the constant function $\underline{x} : T \to X$, acting via $\underline{x}(t) = x$ for any $t \in T$. Assume now that $(X, \| \ \|)$ is a normed space (if the norm $\| \ \|$ is understood, we write simply X). For any function $f : T \to X$ we can consider the function $|f| : T \to \mathbb{R}_+$, acting via $|f|(t) = \|f(t)\|$ for any $t \in T$. If $x \in X$ and $h : T \to K$, the function $hx : T \to X$ acts via $hx(t) = h(t)x$ for any $t \in T$. If f, $g : T \to X$, $\alpha \in K$, one defines pointwise $f + g : T \to X$, $\alpha f : T \to X$ and, in case X is an algebra, also $fg : T \to X$.

Two norms $\| \ \|_1$ and $\| \ \|_2$ on a vector space X are called *equivalent* if there exist two numbers $0 < a \le b$ such that $a\|x\|_1 \le \|x\|_2 \le b\|x\|_1$ for any $x \in X$ (i.e. $\| \ \|_1$ and $\| \ \|_2$ generate the same topology on X). Following this line, we say that two normed spaces $(X_1, \| \ \|_1)$ and $(X_2, \| \ \|_2)$ are *equivalent* if $X_1 = X_2$ and $\| \ \|_1$, $\| \ \|_2$ are equivalent norms.

For further purposes, we shall say that a normed (resp. Banach) space $(X, \| \ \|)$ is a *normed* (resp. *Banach*) *algebra with unit* if X is a non null algebra with unit e (hence $e \neq 0$) and the multiplication in X, denoted via $(x, y) \to xy$ is continuous (i.e. there exists a number $A > 0$ such that $\|xy\| \le A\|x\|\|y\|$ for any $x, y \in X$). This slightly more general definition is almost equivalent to the standard one, because, under the previous conditions, one can define on X a new norm $||| \ |||$ having the properties $|||e||| = 1$ and $|||xy||| \le |||x||| \, |||y|||$ for any x, y in X and such that $\| \ \|$ and $||| \ |||$ are equivalent norms. Indeed, $\mathcal{L}(X) = \{V : X \to X \mid V$ is linear and continuous} is a normed (resp. Banach) algebra with unit, with norm $\|V\|_o = \sup\{\|V(x)\| \mid x \in X, \|x\| \leq 1\}$, multiplication $(U,V) \to U \circ V$ and unit $e : X \to X$ acting via $e(x) = x$ for any $x \in X$. Using the injective algebra morphism (embedding) $\Omega : X \to \mathcal{L}(X)$ given via $\Omega(x) = V_x$ where $V_x(y) = xy$ for any $y \in X$, define $|||x||| = \|V_x\|_o$ for any $x \in X$.

Assume that X is an algebra and $\| \|_1$, $\| \|_2$ are equivalent norms on X. If $(X, \| \|_1)$ is a normed (resp. Banach) algebra with unit e, then $(X, \| \|_2)$ is a normed (resp. Banach) algebra with the same unit e and the same multiplication (if $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ and $\|xy\|_1 \leq A\|x\|_1\|y\|_1$, then $\|xy\|_2 \leq \frac{bA}{a^2}\|x\|_2\|y\|_2$).

A measure space is a triple $(T, \mathcal{T}, \mu)$, where T is a non empty set, $\mathcal{T} \subset \mathcal{P}(T) = \{A \mid A \subset T\}$ is a σ-algebra and $\mu : \mathcal{T} \to \overline{\mathbb{R}}_+$ is a non null σ-additive measure. We shall always assume that μ is *complete* (i.e. if $A \in \mathcal{T}$ with $\mu(A) = 0$ and $B \subset A$, then $B \in \mathcal{T}$) and *σ-finite* (i.e. there exists a sequence $(T_n)_n \subset \mathcal{T}$ such that $\bigcup_n T_n = T$ and $\mu(T_n) < \infty$ for any n). The last assumption is necessary because we deal with μ-measurable vector functions. The set of all μ-measurable functions $u : T \to \overline{\mathbb{R}}_+$ will be denoted by $M_+(\mu)$. For any $A \in \mathcal{T}$, we have $\varphi_A \in M_+(\mu)$, where φ_A is the characteristic (indicator) function of A.

A μ-function norm is a function $\rho : M_+(\mu) \to \overline{\mathbb{R}}_+$ having the following properties (for any $u, v \in M_+(\mu)$ and $\alpha \in \mathbb{R}_+$):

i) $\rho(u) = 0$ if and only if $u = 0$ $\mu - a.e.$ (i.e. μ-almost everywhere).

ii) $\rho(u) \leq \rho(v)$ whenever $u \leq v$.

iii) $\rho(u + v) \leq \rho(u) + \rho(v)$.

iv) $\rho(\alpha u) = \alpha\rho(u)$, with the convention $0 \cdot \infty = 0$.

Notice that, for $u, v \in M_+(\mu)$, we have:

a) if $\rho(u) < \infty$, then u is finite $\mu - a.e.$;

b) if $u = v$ $\mu - a.e.$, then $\rho(u) = \rho(v)$.

Let X be a non null Banach space. A function $f : T \to X$ is called μ-simple if it has the form $f = \sum_{i=1}^{n} \varphi_{A_i} x_i$, with $x_i \in X$ and $A_i \in \mathcal{T}$ mutually disjoint such that $\bigcup_{i=1}^{n} A_i = T$. A function $f : T \to X$ is called *μ-measurable* if there exists a sequence $(f_n)_n$ of μ-simple functions such that $f_n \underset{n}{\to} f$ $\mu - a.e.$ Let $M_X(\mu) \overset{def}{=} \{f : T \to X \mid f$ is μ-measurable$\}$. The vector space $M_X(\mu)$ has the property that, for any $f \in M_X(\mu)$, one has $|f| \in M_+(\mu)$.

Considering also a μ-function norm ρ, we define the vector space

$$\mathcal{L}_\rho(X) = \{f \in M_X(\mu) \mid \rho|f| \overset{def}{=} \rho(|f|) < \infty\}$$

which is seminormed with the seminorm given via $f \to \rho|f|$. The null space of this seminorm is

$$N_X(\mu) = \{f \in M_X(\mu) \mid \rho|f| = 0\} = \{f \in M_X(\mu) \mid f(t) = 0 \ \mu - a.e.\}.$$

We define $L_\rho(X) \stackrel{def}{=} \mathcal{L}_\rho(X)/N_X(\mu)$ and we see that $L_\rho(X)$ is a normed space called *Köthe-Bochner space* with norm given via $\tilde{f} \to \rho|f|$ (for any representative $f \in \tilde{f}$). It can be proved that $L_\rho(X)$ is a Banach space if and only if ρ has the *Riesz-Fischer property* (i.e. ρ has the property that $\rho(\sum_{n=1}^{\infty} u_n) \leq \sum_{n=1}^{\infty} \rho(u_n)$, for any $(u_n)_n \subset M_+(\mu)$).

In the particular case $X = K$, we write $\mathcal{L}_\rho$ instead of $\mathcal{L}_\rho(K)$, L_ρ instead of $L_\rho(K)$ and we say that L_ρ is a *Köthe space*. The Köthe spaces L_ρ generalize the Lebesgue spaces $L^p(\mu)$ (for $\rho = \| \ \|_p$, $1 \leq p \leq \infty$).

Recall that the function norm $\| \ \|_\infty$ (essential supremum) is defined as follows, for any $u \in M_+(\mu)$:

$$\|u\|_\infty = \inf\{A(u, N) \mid N \in \mathcal{T}, \ \mu(N) = 0\}$$

where $A(u, N) = \sup\{u(t) \mid t \in T \setminus N\}$. Then, for $\rho = \| \ \|_\infty$, if X is a Banach space, we write $L_\rho(X) \stackrel{def}{=} L^\infty(X, \mu)$. Hence, for $X = K$: $L^\infty(\mu) = L^\infty(K, \mu)$.

In the spirit of the definition accepted for Banach algebras with unit, we shall consider a μ-function norm ρ, the corresponding Köthe space L_ρ and we shall say that L_ρ is a *Köthe Banach algebra with unit* if the following conditions are fulfilled:

a) L_ρ is a Banach space. *b*) $\underline{1} \in \mathcal{L}_\rho$. *c*) For any $f, g \in \mathcal{L}_\rho$, one has $fg \in \mathcal{L}_\rho$. *d*) There exists a number $A > 0$ such that $\rho|fg| \leq A\rho|f|\rho|g|$ for any $f, g \in \mathcal{L}_\rho$. One can immediately see that, under these conditions, L_ρ becomes a commutative Banach algebra with unit $\tilde{\underline{1}}$ and multiplication defined on representatives as follows: $\tilde{f}\tilde{g} \stackrel{def}{=} \widetilde{fg}$ for any $\tilde{f}, \tilde{g}$ in L_ρ. This justifies the name of Köthe Banach algebra with unit. Clearly, $L^\infty(\mu)$ is a Köthe Banach algebra with unit.

In [2] (see also [3]) we proved the following result showing that, practically, $L^\infty(\mu)$ is the only Köthe Banach algebra with unit.

Theorem A. *Let ρ be a μ-function norm. The following assertions are equivalent:*

1. *L_ρ is a Köthe Banach algebra with unit.*
2. *The Banach spaces L_ρ and $L^\infty(\mu)$ are equivalent.*

Concerning the implication 2. $\Rightarrow$ 1. we feel obliged to notice that, due to equivalence, we have $L_\rho = L^\infty(\mu)$ as sets, consequently on L_ρ we consider the multiplication given by $L^\infty(\mu)$.

For the example at the end of the paper, we shall be concerned with the discrete measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), card)$, where $card : \mathcal{P}(\mathbb{N}) \to \overline{\mathbb{R}}_+$ is the counting measure, acting via $card(A)$ =the number of elements in A, if A is finite and $card(A) = \infty$, if A is infinite. The only negligible set is ϕ. A function $f : \mathbb{N} \to H$ is identified with a sequence $f \equiv (x_n)_n \subset H$, where $x_n = f(n)$ for any n. If X is a Banach space, any function $f : \mathbb{N} \to X$ is $card$-measurable. For any $card$-function norm ρ and any Banach space X, one has $L_\rho(X) \equiv \mathcal{L}_\rho(X)$ (equivalence classes in $L_\rho(X)$ contain only one element). It is clear that, in this case, if $u \equiv (u_n)_n \in M_+(card)$, one has $\|u\|_\infty = \sup_n u_n$. Hence, introducing the classical Banach space

$$l^\infty = \{x = (x_n)_n \mid x_n \in K, \sup_n |x_n| < \infty\}$$

with norm $\|x\| = \sup_n |x_n|$ as above, we have $L^\infty(card) = l^\infty$.

Assuming X itself is a Köthe space, $X = L_r$, for some $card$-function norm r, we have for any $f \in L_\rho(L_r)$: $f \equiv (f(m))_m$, where $f(m) \in L_r = \mathcal{L}_r$, hence we can identify $f(m) \equiv (x_{mn})_n \subset K$. Consequently, any $f \in L_\rho(L_r)$ can be identified with an infinite scalar matrix: $f \equiv (x_{mn})_{m,n}$.

For general measure theory, see [8]. For vector measurability, see [5]. For Functional Analysis, see [7]. For Banach algebras, see [1].

3. Results

In the sequel we shall consider a fixed measure space $(T, \mathcal{T}, \mu)$.

Definition 1. Two μ-function norms ρ_1 and ρ_2 are called *equivalent* if there exist two numbers $0 < a \leq b$ such that $a\rho_1(u) \leq \rho_2(u) \leq b\rho_2(u)$ for any $u \in M_+(\mu)$.

Lemma 1. *Let ρ_1 and ρ_2 be two μ-function norms. The following assertions are equivalent:*

1. *ρ_1 and ρ_2 are equivalent.*

2. *For any non null Banach space X, the normed spaces $L_{\rho_1}(X)$ and $L_{\rho_2}(X)$ are equivalent.*

3. *There exists a non null Banach space X such that the normed spaces $L_{\rho_1}(X)$ and $L_{\rho_2}(X)$ are equivalent.*

If 1. *(or* 2. *or* 3.*) is valid, one has $\mathcal{L}_{\rho_1}(X) = \mathcal{L}_{\rho_2}(X)$.*

Proof. One must prove only 1. ⇒ 2. and 3. ⇒ 1.

1. ⇒ 2. Let X be an arbitrary non null Banach space. Clearly $\mathcal{L}_{\rho_1}(X) = \mathcal{L}_{\rho_2}(X)$, hence $L_{\rho_1}(X) = L_{\rho_2}(X)$. Take $0 < a \leq b$ such that $a\rho_1(u) \leq \rho_2(u) \leq b\rho_1(u)$ for any $u \in M_+(\mu)$. Then, if $\tilde{f} \in L_{\rho_1}(X) = L_{\rho_2}(X)$, one has $a\left\|\tilde{f}\right\|_1 = a\rho_1|f| \leq \rho_2|f| = \left\|\tilde{f}\right\|_2 \leq b\rho_1|f| = b\left\|\tilde{f}\right\|_1$, $\|\ \|_i$ being the norm of $L_{\rho_i}(X)$, $i = 1, 2$.

3. $\Rightarrow$ 1. Let X be a non null Banach space such that $L_{\rho_1}(X)$ and $L_{\rho_2}(X)$ are equivalent. Let $0 < a \leq b$ such that, for any $\tilde{f} \in L_{\rho_1}(X) = L_{\rho_2}(X)$, one has $a \left\|\tilde{f}\right\|_1 = a\rho_1|f| \leq \left\|\tilde{f}\right\|_2 = \rho_2|f| \leq b \left\|\tilde{f}\right\|_2 = b\rho_2|f|$, $\| \, \|_i$ being the norm of $L_{\rho_i}(X)$, $i = 1, 2$. Let $u \in M_+(\mu)$. Either $\rho_1(u) < \infty$ or $\rho_1(u) = \infty$. In case $\rho_1(u) < \infty$, let $u' \in M_+(\mu)$ be such that u' is finite and $u' = u \ \mu - a.e.$ Let $x \in X$ with $\|x\| = 1$ (norm in X), hence $u'x \in \mathcal{L}_{\rho_1}(X)$, because $\rho_1|u'x| = \|x\| \, \rho_1|u'| = \rho_1(u') = \rho_1(u) < \infty$. Hence $\widetilde{u'x} \in L_{\rho_1}(X) = L_{\rho_2}(X)$ and $a \left\|\widetilde{u'x}\right\|_1 \leq \left\|\widetilde{u'x}\right\|_2 \leq b \left\|\widetilde{u'x}\right\|_1$, i.e. $a\rho_1(u) = a\rho_1(u') \leq \rho_2(u') = \rho_2(u) \leq b\rho_1(u') = b\rho_1(u)$. In case $\rho_1(u) = \infty$, one must have $\rho_2(u) = \infty$. Otherwise, one finds $u' \in M_+(\mu)$, u' finite, $u' = u \ \mu - a.e.$ and $\rho_2(u') < \infty$. This leads to $\rho_1(u') = \rho_1(u) < \infty$ as we have seen, contradiction. Finally, we proved that, for any $u \in M_+(\mu)$, one must have $a\rho_1(u) \leq \rho_2(u) \leq b\rho_1(u)$.

In the sequel, we shall consider a fixed Banach algebra $(X, \| \, \|)$ with unit e (and a number $A > 0$ such that $\|xy\| \leq A \|x\| \, \|y\|$ for any x, y in X). The norm of the space L_ρ will be denoted via $||| \, |||$ and the norm of the space $L_\rho(X)$ will be denoted via $||| \, |||_X$.

Definition 2. We shall say that $L_\rho(X)$ is a *Köthe-Bochner Banach algebra with unit* if the following conditions are fulfilled:

1. $(L_\rho(X), ||| \, |||_X)$ is a Banach space.
2. One has $\underline{e} \in \mathcal{L}_\rho(X)$.
3. For any f, g in $\mathcal{L}_\rho(X)$, one has $fg \in \mathcal{L}_\rho(X)$.
4. There exists a number $B > 0$ such that $\rho|fg| \leq B\rho|f|\rho|g|$, whenever f, g are in $\mathcal{L}_\rho(X)$.

Remark. From 3. it follows that, if $\tilde{f}, \tilde{g}$ are in $L_\rho(X)$, one can define the multiplication $\tilde{f}\tilde{g} \in L_\rho(X)$ via $\tilde{f}\tilde{g} \overset{def}{=} \widetilde{fg}$ (on representatives) and, in view of 4., this multiplication is continuous:

$$\left|\left|\left|\tilde{f}\tilde{g}\right|\right|\right|_X = \rho|fg| \leq B\rho|f|\rho|g| = B \left|\left|\left|\tilde{f}\right|\right|\right|_X |||\tilde{g}|||_X.$$

Hence, the normed algebra $(L_\rho(X), ||| \, |||_X)$ has unit $\underline{e}$ (see 2.) and is Banach (see 1.). This leads to the conclusion that $(L_\rho(X), ||| \, |||_X)$ is a Banach algebra with unit (which is commutative if X is commutative). Hence, the name Köthe-Bochner Banach algebra with unit is adequate.

Clearly, $L^\infty(X, \mu)$ is a Köthe-Bochner Banach algebra with unit.

Theorem 2. *The following assertions are equivalent:*

1. $(L_\rho(X), ||| \, |||_X)$ *is a Köthe-Bochner Banach algebra with unit.*
2. $(L_\rho, ||| \, |||)$ *is a Köthe Banach algebra with unit.*

3. *The Banach spaces* $L_\rho(X)$ *and* $L^\infty(X,\mu)$ *are equivalent.*
4. *The Banach spaces* L_ρ *and* $L^\infty(\mu)$ *are equivalent.*

Proof. The schema of proof is the following: 1. $\Rightarrow$ 2. $\Rightarrow$ 4. $\Rightarrow$ 3. $\Rightarrow$ 1.

1. $\Rightarrow$ 2. Because $L_\rho(X)$ is Banach, it follows that L_ρ is Banach. Because $\underline{e} \in \mathcal{L}_\rho(X)$ and $\underline{e} = \underline{1}e$, one has $\rho|\underline{e}| = \rho|\underline{1}e| = \|e\|\,\rho(\underline{1}) < \infty$, hence $\rho(\underline{1}) < \infty$ (because $e \neq 0$) and this implies $\underline{1} \in \mathcal{L}_\rho$. Take now f, g in $\mathcal{L}_\rho$. Then fe and ge are in $\mathcal{L}_\rho(X)$, hence $(fe)(ge) = fge \in \mathcal{L}_\rho(X)$ and there exists a number $B > 0$ such that $\rho|fge| \leq B\rho|fe|\rho|ge|$, i.e. $\|e\|\,\rho|fg| \leq \|e\|^2 B\rho|f|\rho|g|$, hence $\rho|fg| \leq B\,\|e\|\,\rho|f|\rho|g|$. This means that $fg \in \mathcal{L}_\rho$ and $\rho|fg| \leq H\rho|f|\rho|g|$, where $H = B\,\|e\|$. We proved that L_ρ is Banach algebra with unit.

2. $\Rightarrow$ 4. follows from Theorem A in the Preliminary Part.

4. $\Rightarrow$ 3. follows from Lemma 1.

3. $\Rightarrow$ 1. Because $L^\infty(X,\mu)$ is a Köthe-Bochner Banach algebra with unit and $L_\rho(X)$ is equivalent to $L^\infty(X,\mu)$, it follows that $L_\rho(X)$ is a Köthe-Bochner Banach algebra with unit.

Example 1 (Form of $L_\rho(L_r)$ spaces which are Köthe-Bochner Banach algebras with unit)**.** In case of the discrete measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), card)$, one can see that $L^\infty(\mu) = \mathcal{L}^\infty(\mu) = l^\infty$. Hence, a Köthe Banach algebra with unit L_r in this case must be equivalent to l^∞, i.e. $L_r = l^\infty$ with equivalent norms.

According to the preceding theorem, for this $L_r = l^\infty$, a space $L_\rho(L_r)$ is a Köthe-Bochner Banach algebra with unit if and only if L_ρ and l^∞ are equivalent Banach spaces, i.e. $L_\rho = l^\infty$ with equivalent norms.

An element $f \equiv (x_{mn})_{m,n} \in L_\rho(L_r)$ has the form $f(m) = (x_{mn})_n \in l^\infty$ for any m. We have, for any m:

$$a \sup_n |x_{mn}| \leq |f|(m) \leq b \sup_n |x_{mn}|$$

for some fixed $0 < a \leq b$ which do not depend upon m. Then, there exist $0 < A \leq B$ such that

$$A \sup_m |f(m)| \leq \rho|f| \leq B \sup_m |f(m)|$$

i.e.

$$Aa \sup_{m,n} |x_{mn}| \leq \|f\| \leq Bb \sup_{m,n} |x_{mn}|$$

(norm computed in $L_\rho(L_r)$).

Finally, it is seen that

$$L_\rho(L_r) \equiv \{(x_{mn})_{m,n} \subset K \mid \sup_{m,n} |x_{mn}| < \infty\}$$

equipped with a norm $\| \ \|$ having the property that there exist numbers $0 < L \leq M$ such that

$$L \sup_{m,n} |x_{mn}| \leq \|(x_{mn})_{m,n}\| \leq M \sup_{m,n} |x_{mn}|.$$

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