Acta Universitatis Apulensis No. 57/2019
ISSN: 1582-5329 pp. 1-17
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.57.01

CERTAIN NEW INTEGRAL FORMULAS ASSOCIATED WITH SPECIAL FUNCTIONS

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Abstract. In this paper, we establish four theorems in order to evaluate integrals of special or generalized functions and polynomials. The generality of these integrals yields many new and known formulas of a number of special functions. The examples involving Wright function,Mittag-Leffler function,zeta function,Hermite and Bernoulli polynomials given in this paper show the potential of the newly defined theorems which can help to find a large number of integrals involving various types of special functions.

2010 *Mathematics Subject Classification*: 33C20, 33B15, 33E12.

Keywords: Mittag-Leffler function, Gamma function, generalized hypergeometric function ${}_pF_q$, generalized Wright functions ${}_p\Psi_q$, multiindex Mittag-Leffler function.

1. Introduction and Preliminaries

The generalization of the generalized hypergeometric series ${}_pF_q$ due to Wright [13, 14, 15] who defined and studied the generalized Wright Hypergeometric function given by (see[1],p.21 and [6])

$$
{}_p\Psi_q\left[z\right] = {}_p\Psi_q\left[\begin{array}{ll}(\alpha_1,A_1),\cdots,(\alpha_p,A_p); & \\ & z\\ (\beta_1,B_1),\cdots,(\beta_q,B_q); & \end{array}\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jk)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jk)}\frac{z^k}{k!}, \quad (1)
$$

where the coefficients $A_1,...,A_p$ and $B_1,...,B_q$ are positive real numbers such that

$$
1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j \geqq 0.
$$

A special case of (1) is

$$
{}_p\Psi_q\left[\begin{array}{c}(\alpha_1,1),\ldots,(\alpha_p,1);\\(\beta_1,1),\ldots,(\beta_q,1);\end{array} z\right]=\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,\ldots,\alpha_p;\\\beta_1,\ldots,\beta_q;\end{array} z\right]. \tag{2}
$$

where ${}_pF_q$ is the generalized hypergeometric series (see [12]) and $(a)_n=\Gamma(a+n)/\Gamma(a)$.

Kiryakova [8] defined the multiple (multiindex) Mittag-Leffler function as follows. Let $m>1$ be an integer, $\rho_1,\rho_2,...,\rho_m>0$ and $\mu_1,\mu_2,...,\mu_m$ be arbitrary real numbers. By means of "multiindices", $(\rho_i),(\mu_i)$, $i=1,....,m$, we introduce the so-called multiindex (m-tuple,multiple) Mittag-Leffler functions

$$
E_{\left(\frac{1}{\rho i}\right),(\mu_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\mu_1+\frac{k}{\rho_1}\right)...\Gamma\left(\mu_m+\frac{k}{\rho_m}\right)}. \tag{3}
$$

The following are interesting relation of this function to other special functions
(i) For $m=2$, if we put $\frac{1}{\rho_1}=\alpha,\frac{1}{\rho_2}=0$ and $\mu_1=1,\mu_2=1$ in (3) we have

$$
E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)}. \tag{4}
$$

(ii) For $m=2,$ if we put $\frac{1}{\rho_1}=\alpha,\frac{1}{\rho_2}=0$ and $\mu_1=\beta,\mu_2=1$ in (3) we have

$$
E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)}. \tag{5}
$$

(iii) For $m=2,$ if we put $\frac{1}{\rho_1}=1,\frac{1}{\rho_2}=1$ and $\mu_1=v+1,\mu_2=1$, and replacing z by $\frac{-z^2}{4}$ in (3) we have (see [8])

$$
E_{(1,1),(1+v,1)}\left(\frac{-z^2}{4}\right)=\left(\frac{2}{z}\right)^v J_v(z). \tag{6}
$$

where $J_v(z)$ is a Bessel function of first kind (see [12, 2]).

(iv) For $m=2,$ if we put $\frac{1}{\rho_1}=1,\frac{1}{\rho_2}=1$ and $\mu_1=\frac{3-v+\mu}{2},\mu_2=\frac{3+v+\mu}{2}$, and replacing z by $\frac{-z^2}{4}$ in (3) we have (see [8])

$$
E_{(1,1),\left(\frac{3-v+\mu}{2},\frac{3+v+\mu}{2}\right)}\left(\frac{-z^2}{4}\right)=\frac{4}{z^{\mu+1}}S_{\mu,v}(z). \tag{7}
$$

where $S_{\mu,v}(z)$ is the Lommel function (see [12, 2]).

(v) For $m=2$, if we put $\frac{1}{\rho_1}=1, \frac{1}{\rho_2}=1$ and $\mu_1=\frac{3}{2}, \mu_2=\frac{3+2v}{2}$, and replacing z by $\frac{-z^2}{4}$ in (3) we have (see [8])

$$E_{(1,1),\left(\frac{3}{2},\frac{3+2v}{2}\right)}\left(\frac{-z^2}{4}\right)=\frac{4}{z^{\mu+1}}\mathcal{H}_v(z). \tag{8}$$

where $\mathcal{H}_v(z)$ is the Struve function (see [12, 2]).

The Hurwitz (or generalized) zeta function $\zeta(s,a)$ is defined by [2]and [7]

$$\zeta(s,a)=\sum_{m=0}^{\infty}\frac{1}{(a+m)^s},\ \mathcal{R}(s)>1, a\neq\{0,-1,-2,...\}$$

which,just as Riemann zeta function $\zeta(s)$ can be continued meromorphically everywhere in the complex s plane except for a simple pole (with residue 1). From this definition, we have

$$\zeta(s,1)=\zeta(s)=\frac{1}{2^{s-1}}\zeta(s,\frac{1}{2})$$

for the Riemann zeta function $\zeta(s)$. A generalization of Hurwitz (or generalized) zeta function $\zeta(s,a)$ is given by Goyal and Laddha [7] in the form

$$\phi_\mu^*(z,s,a)=\sum_{m=0}^{\infty}\frac{(\mu)_m z^m}{(a+m)^s m!} \tag{9}$$

where $a\neq\{0,-1,-2,...\}, \mu\geq 1$ and either $|z|<1, \mathcal{R}(s)>0$, or $z=1$ and $\mathcal{R}(s)>\mu$.

The 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]) are defined as

$$H_n(x,y)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \tag{10}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ [12] when $y=-1$ and x is replaced by $2x$.

The generalized Hermite-Bernoulli polynomials ${}_HB_n^{[\alpha,m-1]}(x,y)$, $m \geq 1$ for a real or complex parameter α defined by Pathan and Waseem A Khan [11] by means of the generating function defined in a suitable neighborhood of $t = 0$

$$G^{[\alpha,m-1]}(x,y,t) = e^{yt^2}G^{[\alpha,m-1]}(x,t) = \left(\frac{t^m}{e^t - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt+yt^2}$$
$$= G^{[\alpha,m-1]}(t)e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{[\alpha,m-1]}(x,y)\frac{t^n}{n!}, \tag{11}$$

contain as its special cases not only generalized Bernoulli polynomials $B_n^{[\alpha,m-1]}(x)$

$$G^{[\alpha,m-1]}(x,t) = G^{[\alpha,m-1]}(t)e^{xt} = \sum_{n=0}^{\infty} B_n^{[\alpha,m-1]}(x)\frac{t^n}{n!} \tag{12}$$

but also Kampe de Feriet generalization of the $H_n(x,y)$ (c.f.Eq.(10)). For $\alpha = 1$, (12) reduces to a known result of Pathan [10].

For $m = 1$, we obtain from (11)

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{13}$$

which is a generalization of the generating function (1.6) of Dattoli et al [4] in the form

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!} \tag{14}$$

In view of (13),the special case m=1 of (11) may be written in the form

$${}_HB_n^{(\alpha)}(x,y) = \sum_{s=0}^{n} \binom{n}{s} B_{n-s}^{(\alpha)} H_s(x,y) \tag{15}$$

where ${}_HB_n^{(\alpha)}(x,y)$ are generalized Hermite- Bernoulli polynomials and $B_n^{(\alpha)}$ are generalized Bernoulli numbers.

It is possible to define generalized Hermite-Bernoulli numbers ${}_HB_n^{[\alpha,m-1]}$ assuming that

$${}_HB_n^{[\alpha,m-1]}(0,0) = {}_HB_n^{[\alpha,m-1]} \tag{16}$$

For the present investigation, we also need the following two integral formulae (see [9]):

$$\int_0^1 \frac{t^{\mu-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\lambda}}dt = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{17}$$

provided $0 < \Re(\mu) < \Re(\lambda) < 0$.

$$\int_0^\infty t^{\lambda}e^{-at^2}\ln(bt)dt = \frac{\Gamma(\frac{\lambda+1}{2})}{4a^{\frac{\lambda+1}{2}}}[\ln\frac{b^2}{a} + \Psi(\frac{\lambda+1}{2})], \tag{18}$$

where $0 < \Re(\lambda)$, $0 < \Re(a)$ and Ψ function is the logarithmic derivative of the Gamma function (see [12]).

2. Main Theorems

Consider a two variable generating function $F(x,y,t)$ which possesses a formal (not necessarily convergent for t not equal to zero) power series expansion in t such that

$$F(x,y,t) = \sum_{n=0}^{\infty} C_n f_n(x,y)\, t^n, \tag{19}$$

where each member of the generalized set $f_n(x,y)$ is independent of t,and the coefficient set C_n may contain the parameters of the set $f_n(x,y)$but is independent of t, x and y.

Theorem 1. *Let the generating function $F(x,y,t)$ defined by* (19)*be such that*

$$F\left(x,y,\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}\right)$$

remains uniformly convergent for $t \in (0,1)$ and $0 < \Re(\alpha) < \Re(\beta)$.Then

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}}F\left(x,y,\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}\right)dt$$
$$= 2a^{\alpha-\beta}\Gamma(\beta-\alpha)\sum_{n=0}^{\infty} C_n f_n(x,y)\frac{\Gamma(1+n+\beta)}{\Gamma(n+\beta)}\frac{\Gamma(2\alpha+2n)}{\Gamma(1+\alpha+\beta+2n)}. \tag{20}$$

Proof. Replace t by $\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}$ in (19) to get

$$F\left(x,y,\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}\right)=\sum_{n=0}^{\infty}C_n f_n\left(x,y\right)\left(\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}\right)^n. \tag{21}$$

Now multiplying both the sides of (21) by

$$\frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}},$$

integrating with respect to t between the limits 0 and 1 and using the integral (17) and

$$(n+\beta)=\frac{\Gamma\left(1+n+\beta\right)}{\Gamma\left(n+\beta\right)}, \tag{22}$$

we get the required result.

The next theorem gives a further interesting consequences of the generating function (19). Theorem 1 will play an essential role in the derivation of our later results.

Theorem 2. *Let the generating function* $F\left(x,y,t\right)$ *defined by* (19) *be such that*

$$F\left(x,y,\frac{xt}{\left[t+a+\sqrt{t^2+2at}\right]}\right)$$

remains uniformly convergent for $t\in(0,1)$ *and* $0<\Re\left(\alpha\right)<\Re\left(\beta\right)$. *Then*

$$\int_0^1\frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}}F\left(x,y,\frac{xt}{\left[t+a+\sqrt{t^2+2at}\right]}\right)dt$$
$$=\ 2a^{\alpha-\beta}\Gamma\left(\beta-\alpha\right)\sum_{n=0}^{\infty}C_n x^n f_n\left(x,y\right)\frac{\Gamma\left(1+n+\beta\right)}{\Gamma\left(n+\beta\right)}\frac{\Gamma\left(2\alpha+2n\right)}{\Gamma\left(1+\alpha+\beta+2n\right)}. \tag{23}$$

Proof. First replace t by tx in (19) and then replace t by $\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}$ to get

$$F\left(x,y,\frac{tx}{\left[t+a+\sqrt{t^2+2at}\right]}\right)=\sum_{n=0}^{\infty}C_n x^n f_n\left(x,y\right)\left(\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}\right)^n. \tag{24}$$

The proof now parallels the above theorem 1.

Theorem 3. *Let the generating function $F(x,y,t)$ defined by (19) be such that $F(x,y,t)$ remains uniformly convergent for $t \in (0,\infty)$, $0 < \Re(\nu)$ and $0 < \Re(a)$. Then*

$$\int_0^\infty t^\lambda e^{-at^2} \ln(bt) F(x,y,t)\,dt = \sum_{n=0}^\infty C_n f_n(x,y) \frac{\Gamma(\frac{\lambda+n+1}{2})}{4a^{\frac{\lambda+n+1}{2}}}[\ln\frac{b^2}{a} + \Psi(\frac{\lambda+n+1}{2})] \tag{25}$$

where Ψ function is the logarithmic derivative of the Gamma function (see [12]).

Proof. The proof of this theorem is based on (11) and runs parallel to that of theorem 2 as given above.The assertion (25) follows readily from (19) and we omit the details involved.

Now we consider more briefly a different type of approach to special functions for the function $F(t,s,b)$ which possesses a formal (not necessarily convergent for t not equal to zero) power series expansion in t such that

$$F(t,s,b) = \sum_{n=0}^\infty C_n f_n(s,b)\, t^n, \tag{26}$$

where $\mathcal{R}(s) > 1, b \neq \{0,-1,-2,...\}$, each member of the generalized set $f_n(s,b)$ is independent of t,and the coefficient set C_n may contain the parameters of the set $f_n(s,b)$ but is independent of t,s and b.

Theorem 4. *Let the function $F(t,s,b)$ defined by (eqn-int1b) be such that*

$$F\left(\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}, s, b\right)$$

remains uniformly convergent for $t \in (0,1)$, $\mathcal{R}(s) > 1, b \neq \{0,-1,-2,...\}$ and $0 < \Re(\alpha) < \Re(\beta)$. Then

$$\begin{aligned}
&\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^\beta} F\left(\frac{t}{\left[t+a+\sqrt{t^2+2at}\right]}, s, b\right) dt \\
&= 2a^{\alpha-\beta}\Gamma(\beta-\alpha) \sum_{n=0}^\infty C_n f_n(s,b) \frac{\Gamma(1+n+\beta)}{\Gamma(n+\beta)} \frac{\Gamma(2\alpha+2n)}{\Gamma(1+\alpha+\beta+2n)}.
\end{aligned} \tag{27}$$

Proof. The proof of this theorem is based on (26) and runs parallel to that of theorem 1 as given above.

3. EXAMPLES

Example 1. *If we take* $f_n(x,y) = H_n(x,y)$, $C_n = \frac{1}{n!}$ *then*

$$F(x,y,t) = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!} = e^{xt+yt^2}. \quad (28)$$

where $H_n(x,y)$ *is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]).*
On the other hand,by choosing the following bilinear generating function which is known as Mehler's formula (see [12])

$$F(x,y,t) = \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-1/2} exp(\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}. \quad (29)$$

we get $C_n = \frac{1}{n!}$ *and* $f_n(x,y) = H_n(x)H_n(y)$.

Corollary 5. *By considering the generating functions defined in* (28), (29) *and theorem 1, we have the following integral formulae:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} e^{xT+yT^2} dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha) a^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{H_n(x,y)}{n!} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \quad (30)$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$ *and* $H_n(x,y)$ *is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]).*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} (1-4T^2)^{-1/2} exp(\frac{4xyT - 4(x^2+y^2)T^2}{1-4T^2}) dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha) a^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \quad (31)$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$ *and* $H_n(x)$ *is Hermite polynomial [12].*

Corollary 6. *By considering the generating function defined in* (28) *when x is replaced by $2x$ and y=-1, we have the following integral formula:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} e^{2xT-T^2} dt$$
$$= \quad 2^{1-\alpha}\Gamma(\beta-\alpha)\, a^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \tag{32}$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$ and $H_n(x)$ is the Hermite polynomial [12].

As can be seen from the above equation and the reduction $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$,the result (30) *for $x = 0$ yields*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} e^{-T^2} dt = 2^{1-\alpha}\Gamma(\beta-\alpha)\, a^{\alpha-\beta}$$

$$\times \quad {}_2\Psi_2 \left[\begin{array}{c} (1+\beta,2), (2\alpha,4); \\ (\beta,2), (\alpha+\beta+1,4); \end{array} \frac{-1}{4} \right]. \tag{33}$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$.

It follows easily from theorem 2 and (28) *that*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} e^{x^2(T+yT^2)} dt$$
$$= \quad 2^{1-\alpha}\Gamma(\beta-\alpha)\, a^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{x^n H_n(x,y)}{n!} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \tag{34}$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$ and $H_n(x,y)$ is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]). Note that (30) *is not a consequence of* (32)*.*

Corollary 7. *Consider the generating functions defined in* (28) *and* (29) *together with theorem 3.Then for $0 < \Re(\lambda)$ and $0 < \Re(a)$ we have*

$$\int_0^{\infty} t^{\lambda} exp(xt+(y-a)t^2) \ln(bt) dt = \sum_{n=0}^{\infty} \frac{H_n(x,y)}{n!} \frac{\Gamma(\frac{\lambda+n+1}{2})}{4a^{\frac{\lambda+n+1}{2}}} [\ln \frac{b^2}{a} + \Psi(\frac{\lambda+n+1}{2})] \tag{35}$$

where Ψ function is the logarithmic derivative of the Gamma function (see [12]) and $H_n(x,y)$ is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]).

$$\int_0^\infty t^\lambda(1-4t^2)^{-1/2}exp(-at^2+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2})\ln(bt)dt$$
$$=\sum_{n=0}^\infty \frac{H_n(x)H_n(y)}{n!}\frac{\Gamma(\frac{\lambda+n+1}{2})}{4a^{\frac{\lambda+n+1}{2}}}[\ln\frac{b^2}{a}+\Psi(\frac{\lambda+n+1}{2})] \tag{36}$$

where Ψ function is the logarithmic derivative of the Gamma function (see [12]) and $H_n(x,y)$ is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]).

Example 2. *Making use of* (11) *and taking $F(x,y,t)=G^{[\alpha,m-1]}(x,y,t)=e^{yt^2}G^{[\alpha,m-1]}(x,t)$ and $C_n=\frac{1}{n!}$, we can write $f_n(x,y)={}_HB_n^{[\alpha,m-1]}(x,y)$ where ${}_HB_n^{[\alpha,m-1]}(x,y)$ are generalized Hermite-Bernoulli polynomials.*

Corollary 8. *By considering the generating function defined in* (11) *and theorem 1, we have the following integral formula:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^\beta}G^{[\alpha,m-1]}(T)e^{xT+yT^2}dt$$
$$=2^{1-\alpha}\Gamma(\beta-\alpha)\,a^{\alpha-\beta}\sum_{n=0}^\infty\frac{{}_HB_n^{[\alpha,m-1]}(x,y)}{n!}\frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \tag{37}$$

where $T=\frac{t}{t+a+\sqrt{t^2+2at}}$, ${}_HB_n^{[\alpha,m-1]}(x,y)$ are generalized Hermite-Bernoulli polynomials and $H_n(x,y)$ is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]).

First we observe that for $\alpha=0$, (34) reduces to (29). In case $x=y=0$,we use (16) to get the following interesting result involving Hermite-Bernoulli numbers

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^\beta}G^{[\alpha,m-1]}(T)dt$$
$$=2^{1-\alpha}\Gamma(\beta-\alpha)\,a^{\alpha-\beta}\sum_{n=0}^\infty\frac{{}_HB_n^{[\alpha,m-1]}}{n!}\frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \tag{38}$$

where $T=\frac{t}{t+a+\sqrt{t^2+2at}}$ and ${}_HB_n^{[\alpha,m-1]}$ are generalized Hermite-Bernoulli numbers.

Corollary 9. *By considering the generating function defined in (1.11) and theorem 2, we have the following integral formula:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} G^{[\alpha,m-1]}(T) e^{x^2(T+yT^2)} dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha)\, a^{\alpha-\beta} \sum_{n=0}^{\infty} x^n \frac{{}_HB_n^{[\alpha,m-1]}(x,y)}{n!} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \quad (39)$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$*,* ${}_HB_n^{[\alpha,m-1]}(x,y)$ *are generalized Hermite-Bernoulli polynomials and* $H_n(x,y)$ *is 2-variable Kampé de Fériet generalization of the Hermite polynomials (see [5]).*

Note that for $\alpha = 0$, (36) *reduces to* (32) *and for m=1,we can use* (13) *to get*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} T^{\alpha} e^{-\alpha T+x^2(T+yT^2)} dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha)\, a^{\alpha-\beta} \sum_{n=0}^{\infty} x^n \frac{{}_HB_n^{(\alpha)}(x,y)}{n!} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \quad (40)$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$ *and* ${}_HB_n^{(\alpha)}(x,y)$ *is given by* (13).

Corollary 10. *Consider the generating function defined in* (11) *together with theorem 3. Then for* $0 < \Re(\lambda)$ *and* $0 < \Re(a)$ *we have*

$$\int_0^{\infty} T^{\lambda} e^{-aT^2+xT+yT^2} \ln(bT) G^{[\alpha,m-1]}(T) dt$$
$$= \sum_{n=0}^{\infty} \frac{{}_HB_n^{[\alpha,m-1]}(x,y)}{n!} \frac{\Gamma(\frac{\lambda+n+1}{2})}{4a^{\frac{\lambda+n+1}{2}}} [\ln \frac{b^2}{a} + \Psi(\frac{\lambda+n+1}{2})] \quad (41)$$

where $G^{[\alpha,m-1]}(t)$ *is given by (1.11),*Ψ *function is the logarithmic derivative of the Gamma function (see [12]),*$T = \frac{t}{t+a+\sqrt{t^2+2at}}$ *and* ${}_HB_n^{[\alpha,m-1]}(x,y)$ *are generalized Hermite-Bernoulli polynomials.*

Example 3. *In* (19)*,we choose* $y = 0$ *and take* $f_n(x) = x^n$, $C_n = \frac{1}{\Gamma\left(\mu_1+\frac{n}{\rho_1}\right)\Gamma\left(\mu_2+\frac{n}{\rho_2}\right)}$ *so that*

$$F(x,t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{\Gamma\left(\mu_1+\frac{n}{\rho_1}\right)\Gamma\left(\mu_2+\frac{n}{\rho_2}\right)} = E_{\left(\frac{1}{\rho_1},\frac{1}{\rho_2}\right),(\mu_1,\mu_2)}(xt). \quad (42)$$

where $E_{\left(\frac{1}{\rho 1},\frac{1}{\rho 2}\right),(\mu_1,\mu_2)}(x)$ *is multi-index Mittag-Leffler function, in* (3).

Corollary 11. *Let the conditions of theorem 1 satisfies, then the following integral formula holds true*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} E_{\left(\frac{1}{\rho 1},\frac{1}{\rho 2}\right),(\mu_1,\mu_2)}\left(\frac{xt}{t+a+\sqrt{t^2+2at}}\right)dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha)\,a^{\alpha-\beta}\ {}_3\Psi_4\left[\begin{array}{c}(1+\beta,1),(2\alpha,2),(1,1);\\ \left(\mu_1,\frac{1}{\rho_1}\right),\left(\mu_2,\frac{1}{\rho_2}\right),(\beta 1,1),(\alpha+\beta+1,2);\end{array}\ \frac{x}{2}\right] \quad (43)$$

Proof. Consider the generating function $F(x,t)$ defined in (42) and integrating with respect to t between the limits 0 to 1, we have

$$\mathcal{L}_1 = \int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} E_{\left(\frac{1}{\rho 1},\frac{1}{\rho 2}\right),(\mu_1,\mu_2)}\left(\frac{xt}{t+a+\sqrt{t^2+2at}}\right)dt$$
$$= \int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} \sum_{n=0}^{\infty} \frac{(xt)^n}{\Gamma\left(\mu_1+\frac{n}{\rho_1}\right)\Gamma\left(\mu_2+\frac{n}{\rho_2}\right)\left(t+a+\sqrt{t^2+2at}\right)^n}dt$$

Interchanging the integration and summation, we get

$$\mathcal{L}_1 = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma\left(\mu_1+\frac{n}{\rho_1}\right)\Gamma\left(\mu_2+\frac{n}{\rho_2}\right)} \int_0^1 \frac{t^{\alpha+n-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta+n}}dt,$$

Solving the inner integral using (17)

$$\mathcal{L}_1 = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma\left(\mu_1+\frac{n}{\rho_1}\right)\Gamma\left(\mu_2+\frac{n}{\rho_2}\right)} \frac{2(\beta+n)\left(\frac{a}{2}\right)^{\alpha+n}\Gamma(2\alpha+2n)\Gamma(\beta-\alpha)}{2^n\Gamma(1+\alpha+\beta+2n)},$$

The use of (22) gives

$$\mathcal{L}_1 = 2^{1-\alpha}\Gamma(\beta-\alpha)\,a^{\alpha-\beta}\sum_{n=0}^{\infty} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma\left(\mu_1+\frac{n}{\rho_1}\right)\Gamma\left(\mu_2+\frac{n}{\rho_2}\right)\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}\left(\frac{x}{2}\right)^n.$$

In view of (1), we obtain the desired result.

Example 4. *Take* $f_n(x) = x^n$, $C_n = \frac{1}{\Gamma\left(\mu+\frac{n}{\rho}\right)}$ *so that*

$$F(x,t) = \sum_{n=0}^{\infty} \frac{x^n t^n}{\Gamma\left(\mu+\frac{n}{\rho}\right)} = E_{\left(\frac{1}{\rho}\right),(\mu)}(xt). \quad (44)$$

Corollary 12. *Consider the generating function defined in* (44)*, equations* $(1),(22)$ *and integrating with respect to t between the limits 0 to 1 together with theorem 1, we get*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^\beta} E_{\left(\frac{1}{\rho}\right),(\mu)}\left(\frac{xt}{t+a+\sqrt{t^2+2at}}\right) dt$$
$$= 2^{1-\alpha} a^{\alpha-\beta} \Gamma(\beta-\alpha)\ {}_3\Psi_3\left[\begin{array}{c} (1+\beta,1),(2\alpha,2),(1,1); \\ \left(\mu,\frac{1}{\rho}\right),(\beta\,1),(\alpha+\beta+1,2); \end{array} \frac{x}{2}\right]. \quad (45)$$

Example 5. *Take* $f_n(x) = \left(\frac{-x^2}{4}\right)^n$, $C_n = \frac{1}{n!\Gamma(\rho+n+1)}$ *so that*

$$F(x,t) = \sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{4}\right)^n t^n}{n!\Gamma(\rho+n+1)} = E_{(1,1),(1+\rho,1)}\left(\frac{-x^2 t}{4}\right). \quad (46)$$

Corollary 13. *The following integral formula holds:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^\beta} E_{(1,1),(1+\rho,1)}\left(\frac{-x^2 t}{4\left[t+a+\sqrt{t^2+2at}\right]}\right) dt$$
$$= 2^{1-\alpha} \Gamma(\beta-\alpha) a^{\alpha-\beta}\ {}_2\Psi_3\left[\begin{array}{c} (1+\beta,1),(2\alpha,2); \\ (\rho+1,1),(\beta,1),(\alpha+\beta+1,2); \end{array} \frac{x}{2}\right]. \quad (47)$$

Example 6. *If we take* $f_n(x) = \left(\frac{-x^2}{4}\right)^n$, $C_n = \frac{1}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\rho+\frac{3}{2}\right)}$ *then*

$$F(x,t) = \sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{4}\right)^n t^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\rho+\frac{3}{2}\right)} = E_{(1,1),\left(\frac{3}{2},\frac{3}{2}+\rho\right)}\left(\frac{-x^2 t}{4}\right). \quad (48)$$

Corollary 14. *By considering* (48) *and theorem 1, we have the following formula*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^\beta} E_{(1,1),\left(\frac{3}{2},\frac{3}{2}+\rho\right)}\left(\frac{-x^2 t}{4\left[t+a+\sqrt{t^2+2at}\right]}\right) dt$$
$$= 2^{1-\alpha} a^{\alpha-\beta} \Gamma(\beta-\alpha)$$
$$\times\ {}_3\Psi_4\left[\begin{array}{c} (1+\beta,1),(2\alpha,2),(1,1); \\ \left(\frac{3}{2},1\right),\left(\rho+\frac{3}{2},1\right),(\beta,1),(\alpha+\beta+1,2); \end{array} \frac{-x^2}{8}\right]. \quad (49)$$

Example 7. *If we take* $f_n(x) = \left(\frac{-x^2}{4}\right)^n$, $C_n = \frac{1}{\Gamma\left(\frac{3-v+\mu}{2}+n\right)\Gamma\left(\frac{3+v+\mu}{2}+n\right)}$ *then*

$$F(x,t) = \sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{4}\right)^n t^n}{\Gamma\left(\frac{3-v+\mu}{2}+n\right)\Gamma\left(\frac{3+v+\mu}{2}+n\right)} = E_{(1,1),\left(\frac{3-v+\mu}{2},\frac{3+v+\mu}{2}\right)}\left(\frac{-x^2t}{4}\right). \quad (50)$$

Corollary 15. *By considering the generating function defined in* (50)*, we have the following integral formula:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} E_{(1,1),\left(\frac{3-v+\mu}{2},\frac{3+v+\mu}{2}\right)}\left(\frac{-x^2t}{4\left[t+a+\sqrt{t^2+2at}\right]}\right)dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha)\,a^{\alpha-\beta}$$
$$\times\ {}_3\Psi_4\left[\begin{matrix}(1+\beta,1),(2\alpha,2),(1,1); \\ \left(\frac{3-v+\mu}{2},1\right),\left(\frac{3+v+\mu}{2},1\right),(\beta,1),(\alpha+\beta+1,2);\end{matrix}\ \frac{-x^2}{8}\right]. \quad (51)$$

Corollary 16. *If we take* $f_n(x) = \left(\frac{x}{2}\right)^{2n+v+1}$, $C_n = \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{1}{2}\right)}$, $\alpha = \mu - n$ *and* $\beta = \lambda+v+1+n$ *in theorem 1, then we obtain the integrals involving Struve function as:*

$$\int_0^1 \frac{t^{\mu-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\lambda}}\mathcal{H}_v\left(\frac{x}{t+a+\sqrt{t^2+2at}}\right)dt$$
$$= 2^{-v-\mu}x^{v+1}a^{-(\lambda+v+1-\mu)}\Gamma(2\mu)$$
$$\times{}_3\Psi_4\left[\begin{matrix}(v+\lambda,2),(\lambda+v-\mu+1,2),(1,1); \\ \left(\frac{3}{2},1\right),(v+\lambda+1,2),\left(v+\frac{1}{2},1\right),(\lambda+v+\mu+2,2);\end{matrix}\ -\frac{x^2}{4a^2}\right]. \quad (52)$$

Corollary 17. *If we take* $f_n(x) = \left(\frac{x}{2}\right)^{2n+v+1}$, $C_n = \frac{(-1)^n}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{1}{2}\right)}$, $\alpha = \mu+v+n+1$ *and* $\beta = \lambda+v+1+n$ *in theorem 2 , then we have:*

$$\int_0^1 \frac{t^{\mu-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\lambda}}\mathcal{H}_v\left(\frac{xt}{t+a+\sqrt{t^2+2at}}\right)dt$$
$$= 2^{-2v-\mu-1}x^{v+1}a^{\mu-v}\Gamma(\lambda-\mu)$$
$$\times{}_3\Psi_4\left[\begin{matrix}(v+\lambda+2,2),(2\mu+2v,4),(1,1); \\ \left(\frac{3}{2},1\right),(v+\lambda+1,2),\left(v+\frac{1}{2},1\right),(\lambda+\mu+2v+2,4);\end{matrix}\ -\frac{x^2}{16}\right]. \quad (53)$$

Corollary 18. *Notice that a substitution $\nu = \frac{1}{2}$ in (52) and (53) yields the following results*

$$\int_0^1 \frac{t^{\mu-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\lambda}} \mathcal{H}_{\frac{1}{2}}\left(\frac{x}{t+a+\sqrt{t^2+2at}}\right) dt$$
$$= 2^{-\mu-\frac{1}{2}} x^{\frac{3}{2}} a^{\mu-\lambda-\frac{3}{2}} \Gamma(2\mu)$$
$$\times {}_2\Psi_3\left[\begin{array}{c}\left(\lambda+\frac{5}{2},2\right),\left(\lambda-\mu+\frac{3}{2},2\right);\\ \left(\frac{3}{2},1\right),\left(\lambda+\frac{3}{2},2\right),\left(\lambda+\mu+\frac{5}{2},2\right);\end{array} -\frac{x^2}{4a^2}\right], \tag{54}$$

and

$$\int_0^1 \frac{t^{\mu-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\lambda}} \mathcal{H}_{\frac{1}{2}}\left(\frac{xt}{t+a+\sqrt{t^2+2at}}\right) dt$$
$$= x^{\frac{3}{2}} \frac{a^{\mu-\lambda}}{4} \Gamma(\lambda-\mu)$$
$$\times {}_2\Psi_3\left[\begin{array}{c}\left(\lambda+\frac{5}{2},2\right),(2\mu+3,4);\\ \left(\frac{3}{2},1\right),\left(\lambda+\frac{3}{2},2\right),(\lambda+\mu+4,4);\end{array} -\frac{x^2}{16}\right]. \tag{55}$$

where $\mathcal{H}_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}(1-\cos z)$.

Example 8. *If we take* $f_n(s,b) = (b+n)^{-s}$, $C_n = \frac{(\mu)_n}{n!}$, *then*

$$F(t,s,b) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}(b+n)^{-s} = \phi^*_\mu(t,s,b) \tag{56}$$

where $\phi^*_\mu(t,s,b)$ *given by* (9) *is Hurwitz (or generalized) zeta function defined by Goyal and Laddha [7]and is a generalization of Reimann zeta function* $\zeta(s,b)$.

Corollary 19. *By considering the generating function defined in* (56) *and theorem 4, we have the following integral formula:*

$$\int_0^1 \frac{t^{\alpha-1}}{\left[t+a+\sqrt{t^2+2at}\right]^{\beta}} \phi^*_\mu(T,s,b) dt$$
$$= 2^{1-\alpha}\Gamma(\beta-\alpha) a^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}(b+n)^{-s} \frac{\Gamma(1+n+\beta)\Gamma(2\alpha+2n)}{\Gamma(n+\beta)\Gamma(1+\alpha+\beta+2n)}. \tag{57}$$

where $T = \frac{t}{t+a+\sqrt{t^2+2at}}$, $\mathcal{R}(s) > 1, b \neq \{0,-1,-2,...\}$ *and* $\phi^*_\mu(t,s,b)$ *is the Hurwitz (or generalized) zeta function defined by Goyal and Laddha [7].*

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