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A GENERALIZATION OF CHEBYSHEV POLYNOMIALS WITH WELL-KNOWN KINDS AND TRANSITION RELATIONS

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Abstract. In this study, we define a generalized formulae for well-known types of Chebyshev polynomials and give Binet's like formulae for these polynomials. Afterwards, we show the relations between our proposed generalization polynomials and different kinds of Chebyshev polynomials.

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1. Introduction

In recent years, many researchers focus their attentions on the study of orthogonal polynomials, especially Chebyshev polynomials that have many useful analytical and algebraic properties [1, 3, 6]. These polynomials can also be effectively used in the applied branches, for example, the solution of ordinary differential equations with mixed boundary conditions is proposed by the Chebyshev approximations in [2, 4, 8]. Four kinds of those polynomials are, mainly, well-known and have widespread applications. These kinds can be defined as the following recurrence relations for $n \geq 1$ respectively [1, 3]:

$$
\begin{array}{rcl}
\cdot Kind\ I & : & T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \quad T_0(x) = 1, T_1(x) = x, \\
\cdot Kind\ II & : & U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0, \quad U_0(x) = 1, U_1(x) = 2x, \\
\cdot Kind\ III & : & V_{n+1}(x) - 2xV_n(x) + V_{n-1}(x) = 0, \quad V_0(x) = 1, V_1(x) = 2x - 1, \\
\cdot Kind\ IV & : & W_{n+1}(x) - 2xW_n(x) + W_{n-1}(x) = 0, \quad W_0(x) = 1, W_1(x) = 2x + 1,
\end{array}
\tag{1}
$$

Even though the relations between generalized Chebyshev polynomial with these types of Chebyshev polynomials have been shown and studied, no generalization can be drawn to include four kinds in a single recurrence formula. Also, the relations between these types of Chebyshev polynomials have been shown in eqn. (17), eqn. (18) and eqn. (19). In this paper, we first present a generalization of Chebyshev polynomials for well-known four kinds. Then we obtain relations between this new generalized polynomials and their kinds.

2. Main results

In this section, we define a generalized sequence $\{G_n(x)\}$ of Chebyshev polynomials for well-known four kinds. We also give some equalities showing the relations between Chebyshev polynomials of the four kinds and our proposed generalization on Chebyshev polynomials. We define the following polynomials sequence.

Definition 1. *For $|x| \leq 1$, Generalized Chebyshev polynomials sequence $\{G_n(x)\}$ is defined by the recurrence relation*

$$G_{n+1}(x) = 2xG_n(x) - G_{n-1}(x),\ n \geq 1 \tag{2}$$

$\{G_n(x)\} = \left\{1, ax+b, 2ax^2+2bx-1, 4ax^3+4bx^2-(a+2)\,x-b, \cdots\right\}$, *with the initial conditions $G_0(x) = 1,\ G_1(x) = ax+b$ where the a and b are integers.*

We call each term $G_n(x)$ of this sequence as Generalized Chebyshev polynomials for $n \geq 1$. We also indicate that this sequence can be transformed into the other kinds of Chebyshev polynomials for the special choices of a and b. Namely, in the sequence of $\{G_n(x)\}$;

·If $a = 1$ and $b = 0$, then it turns into the first kind Chebyshev polynomial sequence known as

$$\{T_n(x)\} = \left\{1, x, 2x^2-1, 4x^3-3x, \cdots\right\},$$

·If $a = 2$ and $b = 0$, then it turns into the second kind Chebyshev polynomial sequence known as

$$\{U_n(x)\} = \left\{1, 2x, 4x^2-1, 8x^3-4x, \cdots\right\},$$

·If $a = 2$ and $b = -1$, then it turns into the third kind Chebyshev polynomial sequence known as

$$\{V_n(x)\} = \left\{1, 2x-1, 4x^2-2x-1, 8x^3-4x^2-4x+1, \cdots\right\},$$

·If $a = 2$ and $b = 1$, then it turns into the fourth kind Chebyshev polynomial sequence known as

$$\{W_n(x)\} = \left\{1, 2x+1, 4x^2+2x-1, 8x^3+4x^2-4x-1, \cdots\right\}.$$

The characteristic equation of eqn. (2) is

$$r^2 = 2xr - 1, \tag{3}$$

hence, we can easily write the following four properties of r_1 and r_2 for eqn. (3):

$$\begin{array}{ll} 1. & r_1 = x + \sqrt{x^2-1}, r_2 = x - \sqrt{x^2-1} \\ 2. & r_2 < 0 < r_1, \qquad |r_2| < r_1, \\ 3. & r_1 + r_2 = 2x,\ r_1 r_2 = 1,\ r_1 - r_2 = 2\sqrt{x^2-1}, \\ 4. & \lim_{n\to\infty} \frac{G_{n+1}(x)}{G_n(x)} = r_1, \quad for\ all\ x \in \mathbb{R}. \end{array} \tag{4}$$

We can give the following Binet's-like formula for $G_n(x)$.

Theorem 1. *For $n \geq 1$, we have*

$$G_n(x) = \frac{\frac{a}{2}\left(r_1^{n+1} - r_2^{n+1} + r_1^{n-1} - r_2^{n-1}\right) + b\left(r_1^n - r_2^n\right) - \left(r_1^{n-1} - r_2^{n-1}\right)}{r_1 - r_2}. \tag{5}$$

Proof. To show above equality, we use induction method on m. It is obvious that $G_0(x) = 1,\ G_1(x) = ax + b$, for $m = 0$ and $m = 1$, respectively. Assume that

$$\begin{aligned} G_{m-1}(x) &= \frac{\frac{a}{2}\left(r_1^m - r_2^m + r_1^{m-2} - r_2^{m-2}\right) + b\left(r_1^{m-1} - r_2^{m-1}\right) - \left(r_1^{m-2} - r_2^{m-2}\right)}{r_1 - r_2} \\ G_m(x) &= \frac{\frac{a}{2}\left(r_1^{m+1} - r_2^{m+1} + r_1^{m-1} - r_2^{m-1}\right) + b\left(r_1^m - r_2^m\right) - \left(r_1^{m-1} - r_2^{m-1}\right)}{r_1 - r_2} \end{aligned} \tag{6}$$

is true for every integer m. Then, using eqn. (2) and the property $r_1 + r_2 = 2x$ of eqn.(4), we can write

$$(r_1 + r_2)\, G_m(x) - G_{m-1}(x) = G_{m+1}(x). \tag{7}$$

When we write eqn. (6) in the eqn.(7), we obtain

$$G_{m+1}(x) = \frac{\frac{a}{2}\left(r_1^{m+2} - r_2^{m+2} + r_1^m - r_2^m\right) + b\left(r_1^{m+1} - r_2^{m+1}\right) - \left(r_1^m - r_2^m\right)}{r_1 - r_2}. \tag{8}$$

Thus, the proof is completed.

Corollary 2. *For $n \geq 1$, we can verify the following famous Binet's formulas for different kinds from eqn. (5), these are,*

· *For $a = 1$ and $b = 0$, Binet's formula for the first kind Chebyshev polynomials is*

$$T_n(x) = \frac{r_1^n + r_2^n}{2} \tag{9}$$

· *For $a = 2$ and $b = 0$, Binet's formula for the second kind Chebyshev polynomials is*

$$U_n(x) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} \tag{10}$$

· *For $a = 2$ and $b = -1$, Binet's formula for the third kind Chebyshev polynomials is*

$$V_n(x) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} - \frac{r_1^n - r_2^n}{r_1 - r_2} = U_n(x) - U_{n-1}(x) \tag{11}$$

· *For $a = 2$ and $b = 1$, Binet's formula for the fourth kind Chebyshev polynomials is*

$$W_n(x) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2} + \frac{r_1^n - r_2^n}{r_1 - r_2} = U_n(x) + U_{n-1}(x) \tag{12}$$

as given in [5, 7].

We can give the following generating function for n th degree polynomial $G_n(x)$.

Theorem 3. *For $p > r_1$, the equality*

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{G_j(x)}{p^j} = \frac{(ax+b)p - 1}{p^2 - 2px + 1} \tag{13}$$

holds.

Proof. From eqn. (5), we have

$$\begin{aligned} & \lim_{n\to\infty} \sum_{j=1}^{n} \frac{G_j(x)}{p^j} \qquad (14) \\ = \; & \lim_{n\to\infty} \sum_{j=1}^{n} \frac{\frac{a}{2}\left(r_1^{j+1} - r_2^{j+1} + r_1^{j-1} - r_2^{j-1}\right) + b\left(r_1^j - r_2^j\right) - \left(r_1^{j-1} - r_2^{j-1}\right)}{p^j\,(r_1 - r_2)} \\ = \; & \frac{1}{(r_1 - r_2)} \frac{a}{2} \sum_{j=1}^{n} \left[r_1 \left(\frac{r_1}{p}\right)^j - r_2 \left(\frac{r_2}{p}\right)^j + r_2 \left(\frac{r_1}{p}\right)^j - r_1 \left(\frac{r_2}{p}\right)^j \right] \\ & + \frac{1}{(r_1 - r_2)} b \sum_{j=1}^{n} \left[\left(\frac{r_1}{p}\right)^j - \left(\frac{r_2}{p}\right)^j \right] - \frac{1}{(r_1 - r_2)} \sum_{j=1}^{n} \left[r_2 \left(\frac{r_1}{p}\right)^j - r_1 \left(\frac{r_2}{p}\right)^j \right]. \end{aligned}$$

If necessary arrangements are made on the eqn. (14), we obtain

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^{n}\frac{G_j(x)}{p^j} &= \frac{a}{2\left(r_1-r_2\right)}\left[r_1\left(\frac{r_1}{p}\right)\frac{p}{(p-r_1)}-r_2\left(\frac{r_2}{p}\right)\frac{p}{(p-r_2)}\right] \\
&+\frac{a}{2\left(r_1-r_2\right)}\left[r_2\left(\frac{r_1}{p}\right)\frac{p}{(p-r_1)}-r_1\left(\frac{r_2}{p}\right)\frac{p}{(p-r_2)}\right] \\
&+\frac{b}{\left(r_1-r_2\right)}\left[\left(\frac{r_1}{p}\right)\frac{p}{(p-r_1)}-\left(\frac{r_2}{p}\right)\frac{p}{(p-r_2)}\right] \\
&-\left[r_2\left(\frac{r_1}{p}\right)\frac{p}{(p-r_1)}-r_1\left(\frac{r_2}{p}\right)\frac{p}{(p-r_2)}\right].
\end{aligned} \tag{15}
$$

Therefore, simplifying expression in the eqn. (15) and using properties of the characteristic roots

$$
\lim_{n\to\infty}\sum_{j=1}^{n}\frac{G_j(x)}{p^j}=\frac{(ax+b)p-1}{p^2-2px+1} \tag{16}
$$

holds. The proof is completed.

It looks like the following: the transition relation to the another kind from other kind of n th degree Chebyshev polynomials [5, 7].

$$
U_n(x)=\frac{T_n(x)-T_{n+2}(x)}{2(1-x^2)}, \tag{17}
$$

$$
T_n(x)=\frac{V_n(x)-V_{n-1}(x)}{2}, \tag{18}
$$

$$
T_n(x)=\frac{W_n(x)-W_{n-1}(x)}{2}, \tag{19}
$$

In the same way, we can give the following two theorems that show the relations between our proposed polynomial $G_n(x)$ and four kinds of Chebyshev polynomials.

Theorem 4. *The polynomial $G_n(x)$ can be attached to the first kind and second kind of Chebyshev polynomials as*

$$
G_n(x)=U_n(x)+\left[(a-2)\,x+b\right]U_{n-1}(x) \tag{20}
$$

and

$$
G_n(x)=\frac{\left(1-bx-ax^2\right)T_n(x)+\left[(a-1)\,x+b\right]T_{n-1}(x)}{1-x^2}. \tag{21}
$$

Proof. From eqn. (5), we have

$$\begin{aligned} G_n(x) &= \frac{\frac{a}{2}\left(r_1^{n+1}-r_2^{n+1}+r_1^{n-1}-r_2^{n-1}\right)+b\left(r_1^{n}-r_2^{n}\right)-\left(r_1^{n-1}-r_2^{n-1}\right)}{r_1-r_2} \\ &= \frac{a}{2}U_n(x)+bU_{n-1}(x)+(\frac{a}{2}-1)U_{n-2}(x). \end{aligned} \tag{22}$$

By substituting $U_{n-2}(x)=2xU_{n-1}(x)-U_n(x)$ in the eqn.(22), we obtain

$$G_n(x)=U_n(x)+\left[(a-2)\,x+b\right]U_{n-1}(x). \tag{23}$$

This completes the proof of eqn. (20). For the proof of eqn. (21), when use eqn. (17), eqn.(20) and the relation $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$, it is easy to see

$$G_n(x)=\frac{\left(1-bx-ax^2\right)T_n(x)+\left[(a-1)\,x+b\right]T_{n-1}(x)}{1-x^2}. \tag{24}$$

Thus, the proof is completed.

Theorem 5. *The polynomial $G_n(x)$ can be attached to the third kind and fourth kind of Chebyshev polynomials as*

$$\begin{aligned} G_n(x) &= \frac{\left[(1-b)+(1-a-b)\,x-ax^2\right]V_n(x)}{2\left(1-x^2\right)} \\ &\quad+\frac{\left[(1+b)+(a+b-1)x+(a-2)x^2\right]V_{n-1}(x)}{2\left(1-x^2\right)} \end{aligned} \tag{25}$$

and

$$\begin{aligned} G_n(x) &= \frac{\left[(1+b)+(a-b-1)\,x-ax^2\right]W_n(x)}{2\left(1-x^2\right)} \\ &\quad+\frac{\left[(b-1)+(a-b-1)x+(2-a)x^2\right]W_{n-1}(x)}{2\left(1-x^2\right)}. \end{aligned} \tag{26}$$

Proof. It is easy to see that the first and second equalities in the theorem (5) are obvious when the polynomial $T_n(x)$ is replace by (18) and (19) in the equation (21) respectively.

We give another important property of n th degree generalized polynomial.

Theorem 6. *The equality*

$$G_n(x) + G_{-n}(x) = 2T_n(x) \tag{27}$$

is satisfied for all $x \in \mathbb{R}$.

Proof. If we put $-n$ instead of integer n in (5), then we can write

$$G_{-n}(x) = \frac{\frac{a}{2}\left(r_1^{-n+1} - r_2^{-n+1} + r_1^{-n-1} - r_2^{-n-1}\right) + b\left(r_1^{-n} - r_2^{-n}\right) - \left(r_1^{-n-1} - r_2^{-n-1}\right)}{r_1 - r_2} \tag{28}$$

Since $r_1 = r_2^{-1}$ and $r_2 = r_1^{-1}$, we can write

$$\begin{aligned} G_{-n}(x) &= \left[\frac{\frac{a}{2}(r_1^{n+1} - r_2^{n+1}) + \frac{a}{2}(r_1^{n-1} - r_2^{n-1}) + b\left(r_1^{n} - r_2^{n}\right) - \left(r_1^{n-1} - r_2^{n-1}\right)}{r_1 - r_2}\right] \\ &\quad + \frac{\left(r_1^{n+1} - r_2^{n+1}\right) - \left(r_1^{n-1} - r_2^{n-1}\right)}{r_1 - r_2}. \end{aligned} \tag{29}$$

Therefore, using (5) and (9), it is clear that

$$G_n(x) + G_{-n}(x) = 2T_n(x) \tag{30}$$

holds. The proof is completed.

In the following theorem, we give D'ocagne-like formula for the polynomial $G_n(x)$.

Theorem 7. *For $n, m \in \mathbb{R}$, we have*

$$G_n(x)G_{m+1}(x) - G_m(x)G_{n+1}(x) = \left[G_1^2(x) - 2xG_1(x) + 1\right] U_{n-m-1}(x). \tag{31}$$

Proof. From (5), we write

$$
\begin{aligned}
& G_n(x)G_{m+1}(x) \qquad\qquad (32)\\
= & \left[\frac{\frac{a}{2}(r_1^{n+1}-r_2^{n+1})+\frac{a}{2}(r_1^{n-1}-r_2^{n-1})+b\left(r_1^{n}-r_2^{n}\right)-\left(r_1^{n-1}-r_2^{n-1}\right)}{r_1-r_2}\right]\\
& \times\left[\frac{\frac{a}{2}(r_1^{m+2}-r_2^{m+2})+\frac{a}{2}(r_1^{m}-r_2^{m})+b\left(r_1^{m+1}-r_2^{m+1}\right)-(r_1^{m}-r_2^{m})}{r_1-r_2}\right]\\
= & [\frac{a^2}{4}(r_1^{n+m+3}-r_1^{n-m-1}+r_1^{n+m+1}-r_1^{n-m+1}-r_2^{n-m-1}+r_2^{n+m+3}\\
& -r_2^{n-m+1}+r_2^{n+m+1}+r_1^{n+m+1}-r_1^{n-m-3}+r_1^{n+m-1}-r_1^{n-m-1}\\
& -r_2^{n-m-3}+r_2^{n+m+1}-r_2^{n-m-1}+r_2^{n+m-1})+\frac{ab}{2}(r_1^{n+m+2}-r_1^{n-m}\\
& -r_2^{n-m}+r_2^{n+m+2}+r_1^{n+m}-r_1^{n-m-2}-r_2^{n-m-2}+r_2^{n+m})\\
& -\frac{a}{2}(r_1^{n+m+1}-r_1^{n-m+1}-r_2^{n-m+1}+r_2^{n+m+1}+r_1^{n+m-1}-r_1^{n-m-1}\\
& -r_2^{n-m-1}+r_2^{n+m-1})+\frac{ab}{2}(r_1^{n+m+2}-r_1^{n-m-2}+r_1^{n+m}-r_1^{n-m}\\
& -r_2^{n-m-2}+r_2^{n+m+2}-r_2^{n-m}+r_2^{n+m})+b^2(r_1^{n+m+1}-r_1^{n-m-1}\\
& -r_2^{n-m-1}+r_2^{n+m+1})-b(r_1^{n+m}-r_1^{n-m}-r_2^{n-m}+r_2^{n+m})\\
& -\frac{a}{2}(r_1^{n+m+1}-r_1^{n-m-3}+r_1^{n+m-1}-r_1^{n-m-1}-r_2^{n-m-3}+r_2^{n+m+1}\\
& -r_2^{n-m-1}+r_2^{n+m-1})-b(r_1^{n+m}-r_1^{n-m-2}-r_2^{n-m-2}+r_2^{n+m})\\
& +(r_1^{n+m-1}-r_1^{n-m-1}-r_2^{n-m-1}+r_2^{n+m-1})]\frac{1}{(r_1-r_2)^2}
\end{aligned}
$$

In the same manner, we can obtain the following equality:

$$
\begin{aligned}
& G_m(x)G_{n+1}(x)\\
= & \left[\frac{\frac{a}{2}(r_1^{m+1}-r_2^{m+1})+\frac{a}{2}(r_1^{m-1}-r_2^{m-1})+b\left(r_1^{m}-r_2^{m}\right)-\left(r_1^{m-1}-r_2^{m-1}\right)}{r_1-r_2}\right]\\
& \times\left[\frac{\frac{a}{2}(r_1^{n+2}-r_2^{n+2})+\frac{a}{2}(r_1^{n}-r_2^{n})+b\left(r_1^{n+1}-r_2^{n+1}\right)-(r_1^{n}-r_2^{n})}{r_1-r_2}\right]
\end{aligned}
$$

$$\begin{aligned}
&= [\frac{a^2}{4}(r_1^{n+m+3} - r_1^{m-n-1} + r_1^{n+m+1} - r_1^{m-n+1} - r_2^{m-n-1} + r_2^{n+m+3} \\
&-r_2^{m-n+1} + r_2^{n+m+1} + r_1^{n+m+1} - r_1^{m-n-3} + r_1^{n+m-1} - r_1^{m-n-1} - r_2^{m-n-3} \\
&+r_2^{n+m+1} - r_2^{m-n-1} + r_2^{n+m-1}) + \frac{ab}{2}(r_1^{n+m+2} - r_1^{m-n} - r_2^{m-n} \\
&+r_2^{n+m+2} + r_1^{n+m} - r_1^{m-n-2} - r_2^{m-n-2} + r_2^{n+m}) - \frac{a}{2}(r_1^{n+m+1} \\
&-r_1^{m-n+1} - r_2^{m-n+1} + r_2^{n+m+1} + r_1^{n+m-1} - r_1^{m-n-1} - r_2^{m-n-1} + r_2^{n+m-1}) \\
&+\frac{ab}{2}(r_1^{n+m+2} - r_1^{m-n-2} + r_1^{n+m} - r_1^{m-n} - r_2^{m-n-2} + r_2^{n+m+2} \\
&-r_2^{m-n} + r_2^{n+m}) + b^2(r_1^{n+m+1} - r_1^{m-n-1} - r_2^{m-n-1} + r_2^{n+m+1}) \\
&-b(r_1^{n+m} - r_1^{m-n} - r_2^{m-n} + r_2^{n+m}) - \frac{a}{2}(r_1^{n+m+1} - r_1^{m-n-3} \\
&+r_1^{n+m-1} - r_1^{m-n-1} - r_2^{m-n-3} + r_2^{n+m+1} - r_2^{m-n-1} + r_2^{n+m-1}) \\
&-b(r_1^{n+m} - r_1^{m-n-2} - r_2^{m-n-2} + r_2^{n+m}) \\
&+(r_1^{n+m-1} - r_1^{m-n-1} - r_2^{m-n-1} + r_2^{n+m-1})]\frac{1}{(r_1-r_2)^2}
\end{aligned} \tag{33}$$

By subtracting eqn. (33) from eqn. (32), we obtain

$$\begin{aligned}
&G_n(x)G_{m+1}(x) - G_m(x)G_{n+1}(x) \\
= \ &\left[(ax+b)^2 - 2x\,(ax+b) + 1\right]\left(\frac{r_1^{n-m} - r_2^{n-m}}{r_1 - r_2}\right).
\end{aligned} \tag{34}$$

And when eqn. (10) and eqn. (34) are combined, the proof is completed.

Next theorem shows a remarkably important property of polynomial $G_n(x)$ of n th degree that can considered as Catalan-like formula.

Theorem 8. *For $n, m \in \mathbb{R}$, we write*

$$G_{n-m}(x)G_{n+m}(x) - \left[G_n^2(x) - G_m^2(x)\right] = G_1(x)U_{2m-1}(x) - U_{2m-2}(x). \tag{35}$$

Proof. By using eqn. (5), we obtain the following two equations:

$$\begin{aligned}
& G_{n-m}(x)G_{n+m}(x) \\
= & \left[\frac{\frac{a}{2}(r_1^{n-m+1}-r_2^{n-m+1})+\frac{a}{2}(r_1^{n-m-1}-r_2^{n-m-1})}{r_1-r_2}\right. \\
& \left.\frac{+b\left(r_1^{n-m}-r_2^{n-m}\right)-\left(r_1^{n-m-1}-r_2^{n-m-1}\right)}{r_1-r_2}\right]\times \\
& \left[\frac{\frac{a}{2}(r_1^{n+m+1}-r_2^{n+m+1})+\frac{a}{2}(r_1^{n+m-1}-r_2^{n+m-1})}{r_1-r_2}\right. \\
& \left.\frac{+b\left(r_1^{n+m}-r_2^{n+m}\right)-\left(r_1^{n+m-1}-r_2^{n+m-1}\right)}{r_1-r_2}\right]
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
G_n^2(x)-G_m^2(x) = & \left[\frac{\frac{a}{2}(r_1^{n+1}-r_2^{n+1})+\frac{a}{2}(r_1^{n-1}-r_2^{n-1})}{r_1-r_2}\right. \\
& \left.\frac{+b\left(r_1^{n}-r_2^{n}\right)-\left(r_1^{n-1}-r_2^{n-1}\right)}{r_1-r_2}\right]^2\times \\
& \left[\frac{\frac{a}{2}(r_1^{m+1}-r_2^{m+1})+\frac{a}{2}(r_1^{m-1}-r_2^{m-1})}{r_1-r_2}\right. \\
& \left.\frac{+b\left(r_1^{m}-r_2^{m}\right)-\left(r_1^{m-1}-r_2^{m-1}\right)}{r_1-r_2}\right]^2.
\end{aligned} \tag{37}$$

Therefore, when we subtract eqn. (37) from eqn. (36), we can see

$$\begin{aligned}
& G_{n-m}(x)G_{n+m}(x)-\left[G_n^2(x)-G_m^2(x)\right] \\
= & (ax+b)\left(\frac{r_1^{2m}-r_2^{2m}}{r_1-r_2}\right)-\left(\frac{r_1^{2m-1}-r_2^{2m-1}}{r_1-r_2}\right).
\end{aligned} \tag{38}$$

Thus, the proof is completed by combining eqn. (10) and eqn. (38).

Finally, we can give the following Honsberger-like formula for the polynomial $G_n(x)$.

Theorem 9. *For $n, m \in \mathbb{R}$, the equality:*

$$\begin{aligned} & G_n(x)G_{m+1}(x) + G_{n-1}(x)G_m(x) \\ = \; & \frac{x\left[G_1^2(x)-1\right]T_{n+m}(x) - 2x\left[G_1(x)-x\right]T_{n+m-1}(x)}{x^2-1} \\ & -\frac{\left[G_1^2(x)-2xG_1(x)+1\right]T_{n-m-1}(x)}{x^2-1} \end{aligned} \tag{39}$$

holds.

Proof. The proof of Theorem 9 is done easily by using eqn. (5) as in the proofs of Theorem 7 and Theorem 8.

3. Conclusion

In this study, we present a generalization of Chebyshev polynomials for well-known four kinds. Then we obtain relations between generalized Chebyshev polinomial $G_n(x)$ and well-known four kinds using characteristic roots of generalized Chebyshev polinomial $G_n(x)$. Therewithal, we give the generating function for the polynomial $G_n(x)$. Also, we give Binet's-like formula, D'ocagne-like formula, Catalan-like formula and Honsberger-like formula for the polynomial $G_n(x)$. In addition, we are continuing our studies on orthogonality of Chebyshev polynomials which can be obtained while $a, b \notin \{-1, 0, 1, 2\}$.

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