Acta Universitatis Apulensis No. 57/2019
ISSN: 1582-5329 pp. 31-39
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.57.03

SOME RESULTS ON $Q-$ STARLIKE FUNCTIONS OF COMPLEX ORDER

A. Çetinkaya, Y. Kahramaner, Y. Polatoğlu

Abstract. In this paper, we define a new subclass of starlike functions by using quantum calculus. For every $q \in (0,1)$, $q-$ difference operator is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad z \in \mathbb{D}.$$

Using the above operator and definition of starlike functions of complex order, we introduce the class of $q-$ starlike functions of complex order denoted by $S_q^*(1-b)$. In the present paper, we will investigate coefficient inequality, growth and distortion theorems and radius of $q-$ starlikeness for the class $S_q^*(1-b)$.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: $q-$ starlike functions of complex order, coefficient inequality, distortion theorem, growth theorem, radius of $q-$ starlikeness.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0, f'(0) = 1$ for every $z \in \mathbb{D}$. We say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$, if there exists a Schwarz function ϕ which is analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f_1(z) = f_2(\phi(z))$. In particular, when f_2 is univalent, then the above subordination is equivalent to $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ (Subordination principle [4]).

A function $f \in \mathcal{A}$ is said to be starlike function of complex order $b, (b \in \mathbb{C}\diagdown\{0\})$ denoted by $S^*(1-b)$ if and only if $f(z)/z \neq 0$ and

$$Re\left[1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right)\right] > 0$$

for all $z \in \mathbb{D}$. This class was introduced and studied by Nasr and Aouf (see [8]).

In 1909 and 1910 Jackson [5, 6] initiated a study of $q-$ difference operator by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad \text{for} \quad z \in B\diagdown\{0\}, \tag{2}$$

where B is a subset of complex plane $\mathbb{C}$, called $q-$ geometric set if $qz \in B$, whenever $z \in B$. Note that if a subset B of $\mathbb{C}$ is $q-$ geometric, then it contains all geometric sequences $\{zq^n\}_0^\infty$, $zq \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. The $q-$ difference operator (2) is also called Jackson $q-$ difference operator. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [1, 3, 7]).

Also, note that $D_q f(0) \to f'(0)$ as $q \to 1^-$ and $D_q^2 f(z) = D_q(D_q f(z))$. In fact, $q-$ calculus is ordinary classical calculus without the notion of limits. Recent interest in $q-$ calculus is because of its applications in various branches of mathematics and physics. For definitions and properties of $q-$ difference operator and $q-$ calculus, one may refer to [1, 3, 7].

Under the hypothesis of the definition of $q-$ difference operator, we have the following rules:

(1) For a function $f(z) = z^n$, we observe that

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1}.$$

Therefore we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n \frac{1-q^n}{1-q} z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. Clearly, as $q \to 1^-$, $[n]_q \to n$.

(2) If functions f and g are defined on a $q-$ geometric set $B \subset \mathbb{C}$ such that $q-$ derivatives of f and g exist for all $z \in B$, then

(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$ where a and b are real or complex constants.

(ii) $D_q(f(z).g(z)) = g(z)D_qf(z) + f(qz)D_qg(z)$.

(iii) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_qf(z) - f(z)D_qg(z)}{g(z)g(qz)}, \quad g(z)g(qz) \neq 0.$

(iv) As a right inverse, Jackson introduced the $q-$ integral

$$\int_0^z f(\zeta)d_q\zeta = z(1-q)\sum_{n=0}^{\infty} q^n f(zq^n)$$

provided that the series converges.

Denote by $\mathcal{P}_q$ the family of functions of the form $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots$, analytic in $\mathbb{D}$ and satisfying the condition

$$\left|p(z) - \frac{1}{1-q}\right| \leq \frac{1}{1-q},$$

where $q \in (0,1)$ is a fixed real number.

Lemma 1. *[2] $p \in \mathcal{P}_q$ if and only if $p(z) \prec \frac{1+z}{1-qz}$. This result is sharp for the functions $p(z) = \frac{1+\phi(z)}{1-q\phi(z)}$, where ϕ is a Schwarz function.*

Using definitions of starlike functions of complex order [8] and $q-$ difference operator, we define $q-$ starlike functions of complex order as below:

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $S_q^*(1-b)$ such that*

$$Re\left[1 + \frac{1}{b}\left(z\frac{D_qf(z)}{f(z)} - 1\right)\right] > 0, \tag{3}$$

where $q \in (0,1)$, $b \in \mathbb{C}\diagdown\{0\}$, $z \in \mathbb{D}$, then f is called $q-$ starlike function of complex order. Clearly, as $q \to 1^-$, this class reduces to the class of starlike functions of complex order.

The aim of this paper is to give coefficient inequality, growth and distortion theorems and radius of $q-$ starlikeness for the class $q-$ starlike function of complex order.

2. Main results

We first prove coefficient inequality for the class $S_q^*(1-b)$. For our main theorem, we need the following lemma.

Lemma 2. *If p is an element of $\mathcal{P}_q$, then $|p_n| \leq 1+q$. This result is sharp for all $n \geq 1$.*

Proof. In view Lemma 1, we have $p(z) = \frac{1+\phi(z)}{1-q\phi(z)}$. Therefore we get

$$p(z) - 1 = (1 + qp(z))\phi(z). \tag{4}$$

Thus the equality in (4) can be written in the form

$$\sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} d_k z^k = q\phi(z) \sum_{k=1}^{n} p_k z^k \Rightarrow F(z) = q\phi(z)G(z), \tag{5}$$

it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |G(re^{i\theta})|^2 d\theta \tag{6}$$

for each $r, (0 < r < 1)$. By expressing (6) in terms of the coefficients in (5), we obtain the inequality

$$\sum_{k=1}^{n} |p_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \leq q^2 \sum_{k=1}^{n} |p_k|^2 r^{2k}. \tag{7}$$

Taking limit in (7) as $r \to 1$, we conclude that

$$|p_n|^2 \leq (1+q)^2 + (q^2 - 1)\sum_{k=1}^{n} |p_k|^2. \tag{8}$$

Since $(q^2 - 1) \leq 0$, inequality in (8) gives

$$|p_n| \leq 1 + q, \quad n \geq 1.$$

This completes the proof.

Theorem 3. *If a function f of the form (1) belongs to $S_q^*(1-b)$, then*

$$|a_n| \leq \frac{1}{([n]_q - 1)!} \prod_{k=1}^{n-1} \Big(([k]_q - 1) + |b|(1+q) \Big), \tag{9}$$

where $q \in (0,1)$, $b \in \mathbb{C} \diagdown \{0\}$, $z \in \mathbb{D}$. This inequality is sharp for every $n \geq 2$.

Proof. In view of definition of the class $S_q^*(1-b)$ and subordination principle, we can write

$$1+\frac{1}{b}\left(z\frac{D_q f(z)}{f(z)}-1\right)=p(z),$$

where $p\in\mathcal{P}_q$ with $p(0)=1$. Since $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $p(z)=1+p_1z+p_2z^2+...$, then we have

$$z+[2]_qa_2z^2+[3]_qa_3z^3+...=z+(a_2+bp_1)z^2+(a_3+bp_1a_2+bp_2)z^3+$$
$$(a_4+bp_1a_3+bp_2a_2+bp_3)z^4+... \quad .$$

Comparing the coefficients of z^n on both sides, we obtain

$$[n]_qa_n=a_n+bp_1a_{n-1}+bp_2a_{n-2}+...+bp_{n-2}a_2+bp_{n-1}$$

for all integer $n\geq 2$. In view of Lemma 2, we get

$$([n]_q-1)|a_n|\leq|b|(1+q)(|a_{n-1}|+...+|a_2|+1),$$

or equivalently

$$|a_n|\leq\frac{1}{[n]_q-1}|b|(1+q)\sum_{k=1}^{n-1}|a_k|, \quad |a_1|=1. \tag{10}$$

In order to prove (9), we will use process of iteration. Let $c=|b|(1+q)$ and use our assumption $|a_1|=1$ in (10), we obtain successively

$$\begin{aligned}
&\text{for} \quad n=2, \quad |a_2|\leq\frac{1}{[2]_q-1}c,\\
&\text{for} \quad n=3, \quad |a_3|\leq\frac{1}{([3]_q-1)!}c(([2]_q-1)+c),\\
&\text{for} \quad n=4, \quad |a_4|\leq\frac{1}{([4]_q-1)!}c(([2]_q-1)+c)(([3]_q-1)+c).
\end{aligned}$$

Hence induction shows that for n, we obtain

$$|a_n|\leq\frac{1}{([n]_q-1)!}c(([2]_q-1)+c)(([3]_q-1)+c)...(([n-1]_q-1)+c).$$

This proves (9).

Extremal function is the solution of the $q-$ differential equation

$$1+\frac{1}{b}\left(z\frac{D_qf(z)}{f(z)}-1\right)=\frac{1+z}{1-qz}.$$

Corollary 4. *Taking $q \to 1^-$ and choosing $b = 1$ in (9), we get $|a_n| \leq n$ for every $n \geq 2$. This is the well known coefficient inequality for starlike functions.*

We now introduce growth and distortion theorems and radius of $q-$ starlikeness for the class $S_q^*(1-b)$.

Theorem 5. *If a function f of the form (1) belongs to $S_q^*(1-b)$, then*

$$\Big(rF_2(q, Reb, |b|, r)\Big)^{\frac{1-q}{logq^{-1}}} \leq |f(z)| \leq \Big(rF_1(q, Reb, |b|, r)\Big)^{\frac{1-q}{logq^{-1}}}, \tag{11}$$

where

$$F_1(q, Reb, |b|, r) = \frac{1}{\big(1-qr\big)^{\frac{1}{2q}(1+q)(Reb+|b|)}\big(1+qr\big)^{\frac{1}{2q}(1+q)(Reb-|b|)}},$$

$$F_2(q, Reb, |b|, r) = \frac{1}{\big(1-qr\big)^{\frac{1}{2q}(1+q)(Reb-|b|)}\big(1+qr\big)^{\frac{1}{2q}(1+q)(Reb+|b|)}}.$$

These bounds are sharp.

Proof. In view of Lemma 1 and subordination principle, we write

$$\left[1 + \frac{1}{b}\left(z\frac{D_q f(z)}{f(z)} - 1\right)\right] \prec \frac{1+z}{1-qz}.$$

The linear transformation $(\frac{1+z}{1-qz})$ maps $|z| = r$ onto the disc with the centre $C(r) = \frac{1+qr^2}{1-q^2r^2}$ and the radius $\rho(r) = \frac{(1+q)r}{1-q^2r^2}$, therefore we obtain

$$\left|z\frac{D_q f(z)}{f(z)} - \frac{1 + (b(q+q^2) - q^2)r^2}{1-q^2r^2}\right| \leq \frac{|b|(1+q)r}{1-q^2r^2}. \tag{12}$$

From inequality in (12), we get

$$Re\left(z\frac{D_q f(z)}{f(z)}\right) \leq \frac{1 + (1+q)|b|r + ((q^2+q)Reb - q^2)r^2}{(1-qr)(1+qr)}, \tag{13}$$

$$Re\left(z\frac{D_q f(z)}{f(z)}\right) \geq \frac{1 - (1+q)|b|r + ((q^2+q)Reb - q^2)r^2}{(1-qr)(1+qr)}. \tag{14}$$

On the other hand we have

$$Re\left(z\frac{D_q f(z)}{f(z)}\right) = r\frac{\partial_q}{\partial r}log|f(z)|. \tag{15}$$

Considering (13), (14) and (15), we get

$$\frac{\partial_q}{\partial r} log|f(z)| \leq \frac{1}{r} + \frac{\frac{1}{2}(1+q)(Reb+|b|)}{(1-qr)} - \frac{\frac{1}{2}(1+q)(Reb-|b|)}{(1+qr)},$$

and

$$\frac{\partial_q}{\partial r} log|f(z)| \geq \frac{1}{r} + \frac{\frac{1}{2}(1+q)(Reb-|b|)}{(1-qr)} - \frac{\frac{1}{2}(1+q)(Reb+|b|)}{(1+qr)}.$$

Taking $q-$ integral on both sides of the last inequalities, we get (11).

Corollary 6. *Since* $\lim_{q\to 1} \frac{1-q}{logq^{-1}} = 1$, *for special case* $b = 1$, *we obtain*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

This is the well known growth theorem for starlike functions [4].

Corollary 7. *Inequality in (12) can be written in the following form*

$$\left| z\frac{D_q f(z)}{f(z)} \right| \leq \frac{|1+(b(q+q^2)-q^2)r^2| + |b|(1+q)r}{1-q^2r^2},$$

and

$$\left| z\frac{D_q f(z)}{f(z)} \right| \geq \frac{|1+(b(q+q^2)-q^2)r^2| - |b|(1+q)r}{1-q^2r^2}.$$

Therefore, using (11) we obtain

$$|D_q f(z)| \leq \frac{1+|b|(1+q)r+|b(q+q^2)-q^2|r^2}{r(1-q^2r^2)} (rF_1(q, Reb, |b|, r))^{\frac{1-q}{logq^{-1}}},$$

and

$$|D_q f(z)| \geq \frac{1-|b|(1+q)r+|b(q+q^2)-q^2|r^2}{r(1-q^2r^2)} (rF_2(q, Reb, |b|, r))^{\frac{1-q}{logq^{-1}}}.$$

Corollary 8. *Since* $\lim_{q\to 1} \frac{1-q}{logq^{-1}} = 1$, *for* $b = 1$, *Corollary 7 gives*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

This is the well known distortion theorem for starlike functions [4].

Corollary 9. *The radius of $q-$ starlikeness of the class $S_q^*(1-b)$ is*

$$R_q^* = \frac{2}{|b|(1+q)+\sqrt{|b|^2(1+q)^2-4((q+q^2)Reb-q^2)}}.$$

Proof. The inequality

$$\left| z\frac{D_q f(z)}{f(z)} - \frac{1+(b(q^2+q)-q^2)r^2}{1-q^2r^2} \right| \leq \frac{|b|(1+q)r}{1-q^2r^2}$$

can be written in the form

$$Re\left(z\frac{D_q f(z)}{f(z)} \right) \geq \frac{1-|b|(1+q)r+((q+q^2)Reb-q^2)r^2}{1-q^2r^2}.$$

Hence for $r < R_q^*$ the right side of the preceding inequality is positive if

$$R_q^* = \frac{2}{|b|(1+q)+\sqrt{|b|^2(1+q)^2-4((q+q^2)Reb-q^2)}}.$$

SOME SPECIAL CASES

(i) For $b = 1$, we obtain $R_q^* = \dfrac{1}{q}$. This is the radius of $q-$ starlikeness of $q-$ starlike functions.

(ii) For $q \to 1^-$, $R_1^* = \frac{1}{|b|+\sqrt{|b|^2-2Reb+1}}$. This is the radius of starlikeness of starlike functions of complex order (see [8]).

It should be noticed that by giving specific values to b, we obtain growth theorems, distortion theorems and radius of $q-$ starlikeness of the following important subclasses.

(1) $b = 1-\alpha, \quad 0 \leq \alpha < 1, \quad S_q^*(1-b) = S_q^*(\alpha)$ is the class of $q-$ starlike functions of order α.

(2) $b = e^{-i\lambda}\cos\lambda, \quad |\lambda| < \frac{\pi}{2}, \quad S_q^*(1-b) = S_q^*(1-e^{-i\lambda}\cos\lambda)$ is the class of $q-\lambda-$ spirallike functions.

(3) $b = (1-\alpha)e^{-i\lambda}\cos\lambda, \quad 0 \leq \alpha < 1, \quad |\lambda| < \frac{\pi}{2}$,
$S_q^*(1-b) = (1-(1-\alpha)e^{-i\lambda}\cos\lambda)$ is the class of $q-\lambda-$ spirallike functions of order α.

References

[1] G. E. Andrews, *Applications of the basic hypergeometric functions*, SIAM Rev. 16 (1974), 441-484.

[2] A. Çetinkaya, O. Mert, *A certain class of harmonic mappings related to functions of bounded boundary rotation*, Proc. of 12^{th} Int. Sym. on Geometric Function Theory and Applications, (2016), 67-76.

[3] N. J. Fine, *Basic hypergeometric series and applications*, Math. Surveys Monogr. 1988.

[4] A. W. Goodman, *Univalent functions volume I and II*, Polygonal Pub. House, 1983.

[5] F. H. Jackson, *On $q-$ functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, 46 (1909), 253-281.

[6] F. H. Jackson, *On $q-$ definite integrals*, Quart. J. Pure Appl. Math. 41 (1910), 193-203.

[7] V. Kac, P. Cheung, *Quantum calculus*, Springer, 2002.

[8] M. A. Nasr, M. K. Aouf, *Starlike functions of complex order*, J. Natur. Sci. 25 (1985), 1-12.

Asena Çetinkaya
Department of Mathematics and Computer Sciences, Istanbul Kültür University
Istanbul, Turkey
email: *asnfigen@hotmail.com*

Yasemin Kahramaner
Department of Mathematics, Istanbul Ticaret University
Istanbul, Turkey
email: *ykahra@gmail.com*

Yaşar Polatoğlu
Department of Mathematics and Computer Sciences, Istanbul Kültür University
Istanbul, Turkey
email: *y.polatoglu@iku.edu.tr*