Acta Universitatis Apulensis No. 58/2019
ISSN: 1582-5329 pp. 35-43
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.58.03

SOME DOUBLE INTEGRALS INVOLVING MULTIVARIABLE I-FUNCTION

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Abstract. A remarkably large number of integral formulas involving diverse special functions have been presented. In this sequel, we aim to establish two double definite integral formulas whose integrands include the multivariable I-function. The integral formulas presented here, being very general, are found to reduce to yield a large number of relatively simple integral formulas whose integrands contain various special functions deducible from the multivariable I-function, just two of which are demonstrated.

2010 *Mathematics Subject Classification*: Primary: 33C60, 33C99; Secondary: 44A20

Keywords: gamma function, beta function, Legendre polynomial, hypergeometric function, multivariable I-function, multivariable H-function, double definite integral formulas.

1. Introduction and Preliminaries

Prasad [11] introduced the multivariable I-function defined in terms of multiple Mellin-Barnes type integrals

$$I\left(z_1, z_2, \ldots, z_r\right) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \tag{1}$$

$$\left[\begin{array}{c|c} z_1 & \left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2};\cdots;\quad \left(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj}\right)_{1,p_r}:\left(a^{(1)}_j,\alpha^{(1)}_j\right)_{1,p^{(1)}};\cdots;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \vdots & \\ z_r & \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};\cdots;\quad \left(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj}\right)_{1,q_r}:\left(b^{(1)}_j,\beta^{(1)}_j\right)_{1,q^{(1)}};\cdots;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array}\right] \tag{2}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_r}\phi\left(s_1,\cdots,s_r\right)\prod_{i=1}^{r}\phi_i\left(s_i\right)\ z_i^{s_i}\,\mathrm{d}s_1\cdots\mathrm{d}s_r\ \ (r\in\mathbb{N},\ \omega=\sqrt{-1}), \tag{3}$$

where

$$\phi_i\left(s_i\right)=\frac{\prod_{j=1}^{m^{(i)}}\Gamma\left(b_j^{(i)}-\beta_j^{(i)}s_i\right)\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}s_i\right)}{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b_j^{(i)}+\beta_j^{(i)}s_i\right)\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)}\ (i=1,\ldots,r) \tag{4}$$

and

$$\begin{aligned}&\phi\left(s_1,\cdots,s_r\right)\\ &=\frac{\prod_{j=1}^{n_2}\Gamma\left(1-a_{2j}+\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i\right)\prod_{j=1}^{n_3}\Gamma\left(1-a_{3j}+\sum_{i=1}^{3}\alpha_{3j}^{(i)}s_i\right)\cdots}{\prod_{j=n_2+1}^{p_2}\Gamma\left(a_{2j}-\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i\right)\prod_{j=n_3+1}^{p_3}\Gamma\left(a_{3j}-\sum_{i=1}^{3}\alpha_{3j}^{(i)}s_i\right)\cdots}\\ &\quad\frac{\cdots\prod_{j=1}^{n_r}\Gamma\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i\right)}{\cdots\prod_{j=n_r+1}^{p_r}\Gamma\left(a_{rj}-\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i\right)\prod_{j=1}^{q_2}\Gamma\left(1-b_{2j}-\sum_{i=1}^{2}\beta_{2j}^{(i)}s_i\right)}\\ &\quad\times\frac{1}{\prod_{j=1}^{q_3}\Gamma\left(1-b_{3j}+\sum_{i=1}^{3}\beta_{3j}^{(i)}s_i\right)\cdots\prod_{j=1}^{q_r}\Gamma\left(1-b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}s_i\right)}.\end{aligned} \tag{5}$$

Here Γ is the familiar gamma function (see, e.g., [17, Section 1.1] and, in the following, let $\mathbb{N}$ and $\mathbb{R}^+$ be the sets of positive integers and positive real numbers, respectively. For the existence and convergence conditions of the multivarible I-function (1), we refer to [11]. For example, the condition for absolute convergence of multiple Mellin-Barnes type contour integral (1) can be obtained by extension of the corresponding conditions for multivariable H-function as follows:

$$|\arg z_i|<\frac{1}{2}\Omega_i\pi,$$

where, for $i=1,\ldots,r$,

$$\begin{aligned}\Omega_i=&\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}\\ &+\left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)}-\sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right)+\cdots+\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)\\ &-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right).\end{aligned} \tag{6}$$

We note that the multivariable I-function (1) generalizes the multivariable H-function (see [19, 20]), which has been investigated by many authors (see, e.g., [2, 3, 4, 9]).

In this paper, we aim to establish two double integral formulas involving the multivariable I-function (1). We also consider some special cases of the two integral formulas.

For simplicity, throughout this paper, we use the following notations:

$$U := p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}. \tag{7}$$

$$V := 0, n_2; 0, n_3; \cdots ; 0, n_{r-1}. \tag{8}$$

$$X := m^{(1)}, n^{(1)}; \cdots ; m^{(r)}, n^{(r)}. \tag{9}$$

$$Y := p^{(1)}, q^{(1)}; \cdots ; p^{(r)}, q^{(r)}. \tag{10}$$

$$\mathbb{A} := \left(a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}\right)_{1,p_2}; \cdots ; \left(a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots, \alpha_{(r-1)k}^{(r-1)}\right)_{1,p_{r-1}}. \tag{11}$$

$$A := \left(a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)}\right)_{1,p_r}. \tag{12}$$

$$\mathbf{A} := \left(a_k^{(1)}, \alpha_k^{(1)}\right)_{1,p^{(1)}}; \cdots ; \left(a_k^{(r)}, \alpha_k^{(r)}\right)_{1,p^{(r)}}. \tag{13}$$

$$\mathbb{B} := \left(b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}\right)_{1,q_2}; \cdots ; \left(b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots, \beta_{(r-1)k}^{(r-1)}\right)_{1,q_{r-1}}. \tag{14}$$

$$B := \left(b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots, \beta_{rk}^{(r)}\right)_{1,q_r}. \tag{15}$$

$$\mathbf{B} := \left(b_k^{(1)}, \beta_k^{(1)}\right)_{1,q^{(1)}}; \cdots ; \left(b_k^{(r)}, \beta_k^{(r)}\right)_{1,q^{(r)}}. \tag{16}$$

2. Required Integrals

Some known integral formulas are recalled in the following lemma.

Lemma 1. *Each of the following integral formulas holds.*

$$\int_0^1 x^\rho\, P_\upsilon(x)\, \mathrm{d}x = \frac{\sqrt{\pi}2^{-\rho-1}\Gamma(\rho+1)}{\Gamma\left(1+\frac{\rho-\upsilon}{2}\right)\Gamma\left(\frac{\rho+\upsilon+3}{2}\right)} \quad (\Re(\rho) > -1); \tag{17}$$

$$\int_{-1}^1 (x+1)^\rho\, P_\upsilon(x)\, \mathrm{d}x = \frac{2^{\rho+1}\Gamma^2(\rho+1)}{\Gamma(\rho+\upsilon+2)\Gamma(\rho-\upsilon+1)} \quad (\Re(\rho) > -1); \tag{18}$$

$$\int_0^1 x^{\mu-1}\left(1-x^\lambda\right)^{\upsilon-1} \mathrm{d}x = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \upsilon\right) \quad \left(\min\{\Re(\mu), \Re(\nu)\} > 0,\ \lambda \in \mathbb{R}^+\right); \tag{19}$$

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\,{}_2F_1\left[\begin{matrix} a,\ b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\ x\right]\mathrm{d}x$$
$$=\frac{\pi\Gamma(\rho)\,2^{-2\rho+1}\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\rho-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\rho-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\rho-\frac{1}{2}b+\frac{1}{2}\right)} \tag{20}$$
$$\left(\Re(\rho)>0,\ \Re\Big(\rho+\frac{1-a-b}{2}\Big)>0\right),$$

where $P_v(x)$ is the Legendre polynomial, $B(\alpha,\beta)$ is the beta function, and ${}_2F_1$ is the hypergeometric function.

Proof. For (17), (18), (19), and (20), we refer, for example, respectively, to [7, p. 1187, Entry 7.126-1], [7, p. 1187, Entry 7.127-1], [7, p. 500, Entry 3.251-1], and [10, p. 73, Eq. (3.1.22)].

3. Main Integrals

We present two double definite integrals involving the multivariable I-function (1) asserted by Theorems 2 and 3.

Theorem 2. *Let $h_i,\,k_i\in\mathbb{R}^+$ $(i=1,\ldots,r)$, $\Re(\rho)>0$, $\Re\left(\rho+\frac{1-a-b}{2}\right)>0$, and $\Re(\sigma)>-1$. Also let*

$$\Re(1+\sigma)+\sum_{i=1}^{r}h_i\min_{1\le j\le m^{(i)}}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0,\ \ \Re(\rho)+\sum_{i=1}^{r}k_i\min_{1\le j\le m^{(i)}}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0,$$

and

$$\Re\left(\frac{1-a-b}{2}+\rho\right)+\sum_{i=1}^{r}k_i\min_{1\le j\le m^{(i)}}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0.$$

Further let $\left|\arg\left(z_i x^{h_i}y^{k_i}(1-y)^{k_i}\right)\right|<\frac{1}{2}\Omega_i\pi$ $(i=1,\,\ldots,\,r)$ with Ω_i the same as in (6). Then

$$\int_0^1\int_0^1 x^{\sigma}y^{\rho-1}(1-y)^{\rho-1}\ P_v(x)\ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\ x\right]\ I\begin{pmatrix} z_1x^{h_1}y^{k_1}(1-y)^{k_1} \\ \vdots \\ z_rx^{h_r}y^{k_r}(1-y)^{k_r}\end{pmatrix}\mathrm{d}x\,\mathrm{d}y$$
$$=\frac{\sqrt{\pi^3}\,\Gamma\left(\frac{a+b+1}{2}\right)}{2^{2\rho+\sigma-1}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\,I^{V;0,n_r+3;X}_{U:p_r+3,q_r+4;Y}\left(\begin{matrix} z_1 2^{-2k_1-h_1} \\ \vdots \\ z_r 2^{-2k_r-h_r}\end{matrix}\ \middle|\ \begin{matrix}\mathbb{A}:\ A_1,A_2,A_3,A:\ \mathbf{A} \\ \vdots \\ \mathbb{B}:\ B,B_1,B_2,B_3,B_4:\ \mathbf{B}\end{matrix}\right), \tag{21}$$

where $A_1 =: (-\sigma; h_1, \ldots, h_r)$, $A_2 =: (1-\rho; k_1, \ldots, k_r)$, $A_3 =: \left(\frac{1+a+b}{2} - \rho; k_1, \ldots, k_r\right)$, $B_1 =: \left(\frac{\upsilon-\sigma}{2}; \frac{h_1}{2}, \ldots, \frac{h_r}{2}\right)$, $B_2 =: \left(-\frac{\upsilon+\sigma+1}{2}; \frac{h_1}{2}, \ldots, \frac{h_r}{2}\right)$, $B_3 =: \left(-\rho + \frac{1+a}{2}; k_1, \ldots, k_r\right)$, *and* $B_4 =: \left(-\rho + \frac{1+b}{2}; k_1, \ldots, k_r\right)$.

Proof. We denote the left side of (21) by $\mathcal{I}_1$. Then expressing the I-function of several variables in terms of Mellin-Barnes contour integral with the help of (1), and changing the order of integrations, which is permissible under the stated conditions, we obtain

$$\mathcal{I}_1 = \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_r} \phi(s_1, \cdots, s_r) \prod_{k=1}^{r} \phi_k(s_k)\ z_k^{s_k} \left[\int_0^1 x^{\sigma + \sum_{i=1}^r h_i s_i} P_\upsilon(x)\, \mathrm{d}x\right]$$
$$\times \left[\int_0^1 y^{\rho-1+\sum_{i=1}^r k_i s_i} (1-y)^{\rho-1+\sum_{i=1}^r k_i s_i}\, {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\ x\right] \mathrm{d}y\right] \mathrm{d}s_1 \cdots \mathrm{d}s_r. \tag{22}$$

We evaluate the inner x- and y-integrals in (22) with the help of the integral formulas in Lemma 1 and obtain

$$\int_0^1 x^{\sigma+\sum_{i=1}^r h_i s_i}\, P_\upsilon(x)\, \mathrm{d}x = \frac{\sqrt{\pi}\, 2^{-\sigma-1-\sum_{i=1}^r h_i s_i} \Gamma\left(\sigma + 1 + \sum_{i=1}^r h_i s_i\right)}{\Gamma\left(1 + \frac{\sigma-\upsilon}{2} + \frac{\sum_{i=1}^r h_i s_i}{2}\right) \Gamma\left(\frac{\sigma+\upsilon+3}{2} + \frac{\sum_{i=1}^r h_i s_i}{2}\right)} \tag{23}$$

and

$$\int_0^1 y^{\rho-1+\sum_{i=1}^r k_i s_i} (1-y)^{\rho-1+\sum_{i=1}^r k_i s_i}\ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\ x\right] \mathrm{d}y$$
$$= \frac{\pi \Gamma\left(\frac{a+b+1}{2}\right)}{2^{2\rho+\sigma-1} \Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)}\ \frac{\Gamma\left(\rho + \sum_{i=1}^r k_i s_i\right) \Gamma\left(\rho - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2} + \sum_{i=1}^r k_i s_i\right)}{2^{2\sum_{i=1}^r k_i s_i} \Gamma\left(\rho - \frac{1}{2}a + \frac{1}{2} + \sum_{i=1}^r k_i s_i\right) \Gamma\left(\rho - \frac{1}{2}b + \frac{1}{2} + \sum_{i=1}^r k_i s_i\right)}. \tag{24}$$

Finally, substituting the equations (23) and (24) in (22) and reinterpreting the resulting multiple Mellin-Barnes contour integral in terms of I-function of r-variables, we obtain the desired result (21).

Theorem 3. *Let* $h_i, k_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$, $\lambda \in \mathbb{R}^+$, $\Re(\upsilon) > 0$, $\Re(\rho) > 0$, *and* $\Re(\sigma) > -1$. *Also let*

$$\Re(1+\sigma) + \sum_{i=1}^r h_i \min_{1 \le j \le m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > 0,\ \Re(\tau) + \sum_{i=1}^r k_i \min_{1 \le j \le m^{(i)}} Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > 0,$$

and

$$\Re\left(\frac{\tau}{\lambda}\right)+\sum_{i=1}^{r}\frac{k_i}{\lambda}\min_{1\le j\le m^{(i)}}\Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right)>0.$$

Further let

$$\left|\arg\left[z_i\left(1-x\right)^{h_i}\ y^{k_i}\left(1-y^{\lambda}\right)^{k_i}\right]\right|<\frac{1}{2}\Omega_i\pi,$$

where Ω_i $(i=1,\ldots,r)$ *are the same in* (6). *Then*

$$\int_{-1}^{1}\int_{0}^{1}\left(x+1\right)^{\sigma}y^{\tau-1}\left(1-y^{\lambda}\right)^{\tau-1}P_{\upsilon}\left(x\right)\ I\begin{pmatrix} z_1\left(1-x\right)^{h_1}\ y^{k_1}\left(1-y^{\lambda}\right)^{k_1}\\ \vdots\\ z_r\left(1-x\right)^{h_r}\ y^{k_r}\left(1-y^{\lambda}\right)^{k_r}\end{pmatrix}\mathrm{d}x\,\mathrm{d}y$$

$$=\frac{2^{\sigma+1}}{\lambda}I_{U:p_r+4,q_r+3;Y}^{V;0,n_r+4;X}\left(\begin{array}{c|c} z_1 2^{h_1} & \mathbb{A}:\ C_1,C_1,C_2,C_3,A:\ \mathbf{A}\\ \vdots & \vdots\\ z_r 2^{h_r} & \mathbb{B}:\ B,D_1,D_2,D_3:\ \mathbf{B}\end{array}\right),\tag{25}$$

where $C_1:=(-\sigma;h_1,\ldots,h_r)$, $C_2=:\left(1-\frac{\tau}{\lambda};\frac{k_1}{\lambda},\ldots,\frac{k_r}{\lambda}\right)$, $C_3=:(1-\tau;k_1,\ldots,k_r)$, $D_1:=(-1-\sigma-\upsilon;h_1,\ldots,h_r)$, $D_2=:(\upsilon-\sigma;h_1,\ldots,h_r)$, $D_3=:\left(1-\tau-\frac{\tau}{\lambda};\frac{k_1}{\lambda},\ldots,\frac{k_r}{\lambda}\right)$.

Proof. A similar argument as in the proof of Theorem 3 will establish the result here. We omit the details.

4. Special cases and remarks

The integral formulas in Theorems 2 and 3, being very general, can be specialized to yield a number of relatively simple integral formulas involving certain special functions reducible from the multivariable I-function (1). Here we demonstrate only two integral formulas in the following corollaries.

To do this, let $U=V=\mathbb{A}=\mathbb{B}=0$. Then the multivariable I-function reduces to the multivariable H-function (see, e.g., [2, 3, 4, 9, 11]). Also let

$$\Omega_i'=:\sum_{k=1}^{n^{(i)}}\alpha_k^{(i)}-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)}+\sum_{k=1}^{m^{(i)}}\beta_k^{(i)}-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)}+\sum_{k=1}^{n_s}\alpha_{sk}^{(i)}-\sum_{k=n_s+1}^{p_s}\alpha_{sk}^{(i)}-\sum_{k=1}^{q_s}\beta_{sk}^{(i)}\tag{26}$$

$$(i=1,\ldots,r).$$

We consider only two special cases of the main results, given in Corollaries 4 and 5.

Corollary 4. *Let* $\left|\arg\left(z_i x^{h_i} y^{k_i} (1-y)^{k_i}\right)\right| < \frac{1}{2}\Omega_i' \pi$ $(i = 1, \ldots, r)$ *with* Ω_i' *the same as in* (26)*. Also let the other conditions and notations be the same as in* Theorem 2*. Then*

$$\int_{-1}^{1}\int_{0}^{1} (x+1)^{\sigma}\; y^{\tau-1}\left(1-y^{\lambda}\right)^{\tau-1}\; P_{\upsilon}(x)\; {}_2F_1\left[\begin{matrix} a,\; b\,; \\ \frac{1}{2}(a+b+1)\,; \end{matrix}\; x\right] H\left[\begin{matrix} z_1\, x^{h_1}\, y^{k_1}\,(1-y)^{k_1} \\ \vdots \\ z_r\, x^{h_r}\, y^{k_r}\,(1-y)^{k_r} \end{matrix}\right] \mathrm{d}x\,\mathrm{d}y$$

$$= \frac{\sqrt{\pi^3}\,\Gamma\left(\frac{a+b+1}{2}\right)}{2^{2\rho+\sigma-1}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\; H^{0,n_r+3;X}_{p_r+3,q_r+4;Y}\left[\begin{matrix} z_1\, 2^{-2k_1-h_1} \\ \vdots \\ z_r\, 2^{-2k_r-h_r} \end{matrix}\;\middle|\; \begin{matrix} A_1, A_2, A_3, A : \mathbf{A} \\ \vdots \\ B, B_1, B_2, B_3, B_4 : \mathbf{B} \end{matrix}\right]. \tag{27}$$

Corollary 5. *Let*

$$\left|\arg\left[z_i\,(1-x)^{h_i}\; y^{k_i}\left(1-y^{\lambda}\right)^{k_i}\right]\right| < \frac{1}{2}\Omega_i'\pi,$$

where Ω_i' $(i = 1, \ldots, r)$ *are the same in* (26)*. Also let the other conditions and notations be the same as in* Theorem 3*. Then*

$$\int_{-1}^{1}\int_{0}^{1} (x+1)^{\sigma}\; y^{\tau-1}\left(1-y^{\lambda}\right)^{\tau-1}\; P_{\upsilon}(x)\; H\left[\begin{matrix} z_1\;(1-x)^{h_1}\; y^{k_1}\;\left(1-y^{\lambda}\right)^{k_1} \\ \vdots \\ z_r\;(1-x)^{h_r}\; y^{k_r}\;\left(1-y^{\lambda}\right)^{k_r} \end{matrix}\right] \mathrm{d}x\,\mathrm{d}y$$

$$= \frac{2^{\sigma+1}}{\lambda}\; H^{0,n_r+4;X}_{p_r+4,q_r+3;Y}\left[\begin{matrix} z_1\, 2^{h_1} \\ \vdots \\ z_r\, 2^{h_r} \end{matrix}\;\middle|\; \begin{matrix} C_1, C_1, C_2, C_3, A : \mathbf{A} \\ \vdots \\ B, D_1, D_2, D_3 : \mathbf{B} \end{matrix}\right]. \tag{28}$$

A similar argument given above will establish a large number of integral formulas whose integrand include diverse special functions such as the multivariable A-function (see [6]), the multivariable Aleph-function (see [1]), the Aleph-function of two variables (see [8]; see also [15]), the I-function of two variables (see [16]), the H-function of two variables (see [18]; see also [13]), the Aleph-function of one variable (see [21, 22]), the I-function of one variable (see [14]), the A-function of one variable (see [5]).

5. Conclusion

We established two double definite integral formulas whose integrands include the multivariable I-function. The integral formulas presented here, being very general, are found to reduce in a large number of relatively simple integral formulas whose integrands contain various special functions deducible from the multivariable I-function. The multivariable I-function can also be suitably specialized to yield a large number of special functions of one or several variables or product of several such special functions which are expressible in terms of E, G and H-functions of one, two and more variables.

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