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SOME INTERPOLATION OPERATORS ON A CURVED DOMAIN

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Abstract. One of the quite simple procedures for constructing multidimensional approximation operators consist in the composition of univariate approximation operators, using tensor product and boolean sum operators. In this paper, using these type of interpolation operators, constructed for functions defined on a triangle and a square with on curved side, we plot the graphs of the interpolation errors of the corresponding interpolation formulas. Also, we give the maximum interpolation errors for the two test functions.

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Keywords: Approximation operators, triangle, square, curved side, graphs, interpolation errors, test functions.

1. Introduction

Approximation operators on polygonal domains with some curved sides have important applications especially in finite element method for differential equations with given boundary conditions and in computer aided geometric design. Such operators were considered in the papers [8], [7], [6]. Lately, such problems were studied in [2], [1], [3], [11], [9],[10], using interpolation operators.

The aim of this paper is to plot the graphs and to find the maximum of the interpolation errors of the interpolation formula of the Lagrange and Hermite type operators on a triangle with one curved side and, respectively, on a square with one curved side.

2. Interpolation operators on a triangle

Let $\tilde{T_h} = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \geq h\}$ be the standard triangle. In [11] the authors consider a standard triangle,$\tilde{T_h}$, having the vertices
$V_1 = (h, 0), V_2 = (0, h)$ and $V_3 = (0, 0)$, two straight sides Γ_1, Γ_2, along the coordinate

axes, and the third side Γ_3 (opposite to the vertex V_3), which is defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e. $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$ and F a real-valued function defined on $\tilde{T}_h$.

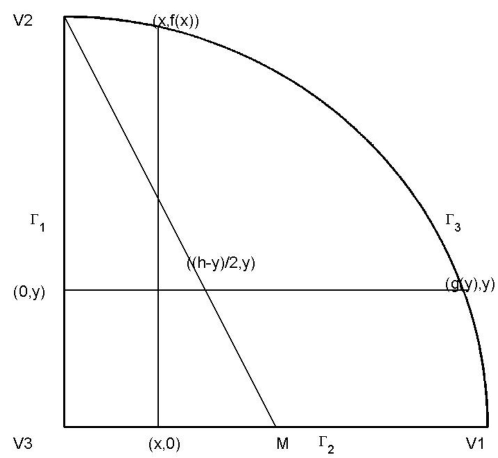

Figure 1: Triangle $\tilde{T}_h$

In [11] the authors construct certain Lagrange, Hermite and Birkhoff type operators, which interpolate a given function and some of its derivatives on the border of a triangle with one curved side, as well as some of their product and Boolean sum operators.

In [2] we introduce an Lagrange operator L_2^x which interpolate the function, F, and some interior line of triangle $\tilde{T}_h$ and we consider the case when the interior line is a median (Figure 1). L_2^x interpolate the function F with respect to x in the points $(0, y)$, $\left(\frac{h-y}{2}, y\right)$, $(g(y), y)$:

$$\begin{aligned}(L_2^x F)(x,y) &= \frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)} F(0,y) \\ &+ \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]} F\Big(\frac{h-y}{2}, y\Big) \\ &+ \frac{x(2x-h+y)}{g(y)[2g(y)-h+y]} F(g(y),y). \qquad (1)\end{aligned}$$

Using this operator and the Hermite type operator from [11]:

$$\begin{aligned}(H_2^yF)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}F(x,0)+\frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))\\ &+ \frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)),\end{aligned} \tag{2}$$

we constructe in [1] new interpolation operators.

Let P be

$$P:=H_2^yL_2^x$$

and

$$F=PF+R_1F \tag{3}$$

approximation formula generated by P, with:

$$\begin{aligned}(PF)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}\Big[\frac{(2x-h)(x-h)}{h^2}F(0,0)-\frac{4x(x-h)}{h^2}F\Big(\frac{h}{2},0\Big)+\\ &+ \frac{x(2x-h)}{h^2}F(h,0)\Big]\\ &+ \frac{y[2f(x)-y]}{f^2(x)}F(x,f(x))+\frac{y[y-f(x)]}{f(x)}F^{(0,1)}(x,f(x)).\end{aligned} \tag{4}$$

If there exist $F^{(0,1)}$ on the side Γ_3 then P verifies the interpolation properties:

$$\begin{aligned}PF &= F,\, on\, \Gamma_2\cup\Gamma_3\\ (PF)^{(0,1)} &= F^{(1,0)},\, on\, \Gamma_3\end{aligned}$$

and $dex(P)=2$.

Let S be

$$S:=H_2^y\oplus L_2^x$$

and

$$F=SF+R_2F \tag{5}$$

approximation formula generated by S, with:

$$\begin{aligned}(SF)(x,y) &= \frac{[y-f(x)]^2}{f^2(x)}F(x,0)+\frac{(2x-h+y)[x-g(y)]}{(h-y)g(y)}F(0,y)\\ &+ \frac{4x[x-g(y)]}{(h-y)[h-y-2g(y)]}F\Big(\frac{h-y}{2},y\Big)+\frac{x(2x-h+y)}{g(y)[2g(y)-h+y]}F(g(y),y)\\ &- \frac{[y-f(x)]^2}{f^2(x)}\Big[\frac{(2x-h)(x-h)}{h^2}F(0,0)-\frac{4x(x-h)}{h^2}F\Big(\frac{h}{2},0\Big)+\frac{x(2x-h)}{h^2}F(h,0)\Big]\end{aligned} \tag{6}$$

Let consider $F:\tilde{T}_h\to\mathbb{R}$ then :

1. $SF = F$, on $\partial\tilde{T}_h$.

2. $dex(S) = 2$.

We consider the following test functions, generally used in literature, (see, e.g., [12]):

$$\begin{aligned} \text{Gentle} : F_1(x,y) &= exp\big[-\frac{81}{16}((x-0.5)^2+(y-0.5)^2)\big]/3, \\ \text{Saddle} : F_2(x,y) &= \frac{(1.25+cos5.4y)}{6+6(3x-1)^2}. \end{aligned} \tag{7}$$

In Figure 2- Figure 9 we plot the graphs of the interpolation errors for L_2F_i, H_2F_i, PF_i, and SF_i, $i=\overline{1,2}$, on $\tilde{T}_1$, $(h=1)$ considering $f:[0,1]\to[0,1]$, $f(x)=\sqrt{1-x^2}$.

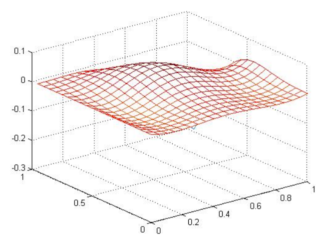

Figure 2: The interpolation error for L_2F_1

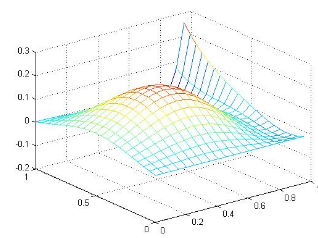

Figure 3: The interpolation error for H_2F_1

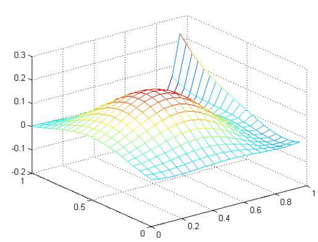

Figure 4: The interpolation error for PF_1

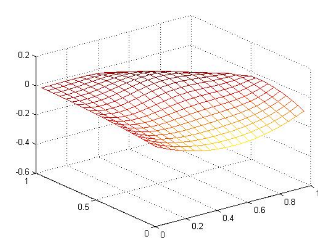

Figure 5: The interpolation error for SF_1

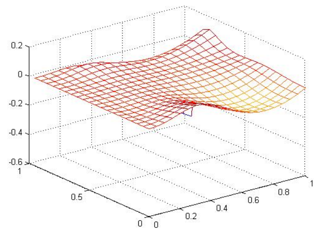

Figure 6: The interpolation error for L_2F_2

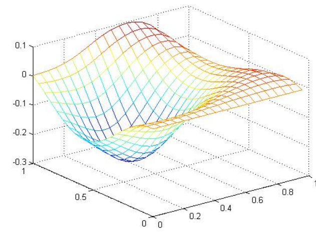

Figure 7: The interpolation error for H_2F_2

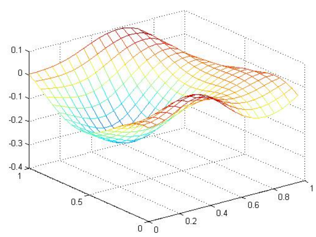

Figure 8: The interpolation error for PF_2

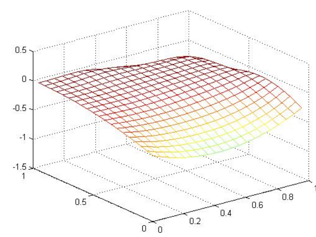

Figure 9: The interpolation error for SF_2

The following table contains the maximum interpolation errors for the functions given in 7, defined on $\tilde{T}_1$, $(h = 1)$.

Table 1: The interpolation error

Max error	F_1	F_2
L_2F	0.0532	0.1070
H_2F	0.2554	0.0946
PF	0.2281	0.0913
SF	0.0406	0.0297

3. Interpolation operators on a square

In [10] the authors consider D_h the square with one curved side having the vertices $V_1 = (0, 0), V_2 = (h, 0), V_3 = (h, h)$ and $V_4 = (0, h)$, three straight sides Γ_1, Γ_2, along the coordinate axes, Γ_3 parallel to axis Ox, and the curved side Γ_4 which is defined by the function g, such that $g(h) = g(0) = h$.

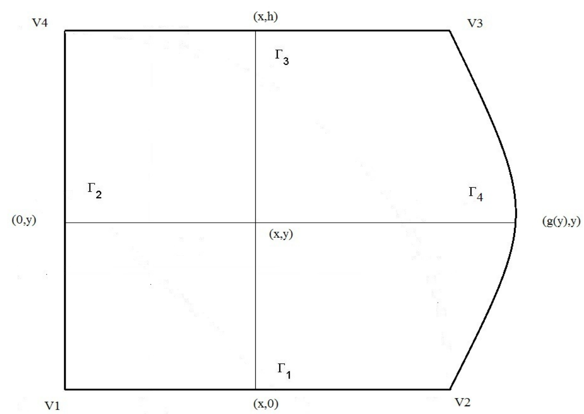

Figure 10: The square D_h

Let F be a real-valued function defined on D_h and $(0, y), (g(y), y)$, respectively, $(x, 0), (x, h)$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in D_h$, intersect the sides Γ_2, Γ_4, respectively Γ_1 and Γ_3.

They construct and analyze Bernstein-type operators on the square with one and two curved side.

In [3] we construct the Lagrange operators L_1 and L_2 defined by

$$\begin{aligned}(L_1F)(x,y) &= \frac{g(y)-x}{g(y)}F(0,y) + \frac{x}{g(y)}F(g(y),y),\\ (L_2F)(x,y) &= \frac{h-y}{h}F(x,0) + \frac{y}{h}F(x,h).\end{aligned}$$

1) Both operators L_1 and L_2 interpolates the function F along two sides of the square D_h:

$$\begin{aligned}(L_1F)(0,y) &= F(0,y) \quad (L_1F)(g(y),y) = F(g(y),y),\ \ y \in [0,h]\\ (L_2F)(x,0) &= F(x,0) \quad (L_2F)(x,h) = F(x,h) \quad x \in [0,h]\end{aligned}$$

2) The degree of exactness $\text{dex}(L_i) = 1, \quad i = 1, 2.$

Let P_{21}^L be the product of the operators L_2 and L_1, i.e., $P_{21} = L_2 L_1$.
We have

$$\begin{aligned}(P_{21^L}F)(x,y) &= \frac{(h-x)(h-y)}{h^2}F(0,0) + \frac{x(h-y)}{h^2}F(h,0)\\ &+ \frac{y(h-y)}{h^2}F(0,h) + \frac{xy}{h^2}F(h,h)\end{aligned}$$

1) The interpolation properties: $P_{21}^L F = F$, on the four vertices of the square V_1, V_2, V_3 and V_4.

2) The degree of exactness $\mathrm{dex}(P_{21}^L) = 1$.

Let S_{21}^L be the Boolean sum of the operators L_2 and L_1, i.e., $S_{21}^L = L_2 \oplus L_1 = L_2 + L_1 - L_2 L_1$.

We have

$$\begin{aligned}(S_{21}^L)(x,y) &= \frac{h-y}{h}F(x,0) + \frac{y}{h}F(x,h) + \frac{g(y)-x}{g(y)}F(0,y) +\\ &+ \frac{x}{g(y)}F(g(y),y) - \frac{(h-x)(h-y)}{h^2}F(0,0)\\ &- \frac{x(h-y)}{h^2}F(h,0) - \frac{y(h-x)}{h^2}F(0,h) - \frac{xy}{h^2}F(h,h)\end{aligned}$$

1) The interpolation properties:$S_{21}^L F = F$ on ∂D_h.

2) The degree of exactness $\mathrm{dex}(S_{21}^L) = 1$.

Suppose that the real valued function F is defined on the square D_h and it possesses the partial derivatives $F^{(1,0)}$ on the side Γ_4 and $F^{(0,1)}$ on Γ_3.

We consider the operators H_1 and H_2 defined by

$$\begin{aligned}(H_1 F)(x,y) &= \frac{[x-g(y)]^2}{g^2(y)}F(0,y) + \frac{x[2g(y)-x]}{g^2(y)}F(g(y),y)\\ &+ \frac{x[x-g(y)]}{g(y)}F^{(1,0)}(g(y),y),\\ (H_2 F)(x,y) &= \frac{(y-h)^2}{h^2}F(x,0) + \frac{y(2h-y)}{h^2}F(x,h)\\ &+ \frac{y(y-h)}{h}F^{(0,1)}(x,h)\end{aligned}$$

1) The interpolation properties:

$$\begin{aligned}(H_1 F) &= F, \mathrm{on}\, \Gamma_2 \cup \Gamma_4 \quad (H_1 F)^{(1,0)} = F^{(1,0)}, \mathrm{on}\, \Gamma_4\\ (H_2 F) &= F, \mathrm{on}\, \Gamma_1 \cup \Gamma_3 \quad (H_2 F)^{(0,1)} = F^{(0,1)}, \mathrm{on}\, \Gamma_3.\end{aligned}$$

2) The degree of exactness $dex(H_1) = dex(H_2) = 2$
Using the Gentle test function:

$$\text{Gentle} : F_1(x, y) \quad = \quad exp\big[-\frac{81}{16}((x - 0.5)^2 + (y - 0.5)^2)\big]/3,$$

in Figure 10- Figure 13 we plot the graphs of the interpolation errors for L_2F_1, $P_{21}F_1$,$S_{21}F_i$ and H_2F_1, on D_1, $(h = 1)$ considering $f : [0, 1] \to [0, 1]$, $f(x) = \sqrt{1 - x^2}$.

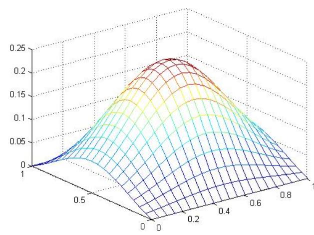

Figure 11: The interpolation error for L_2F_1

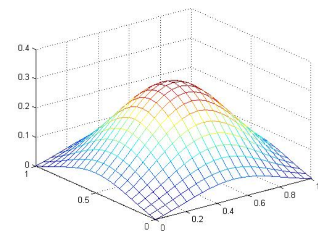

Figure 12: The interpolation error for $P_{21}F_1$

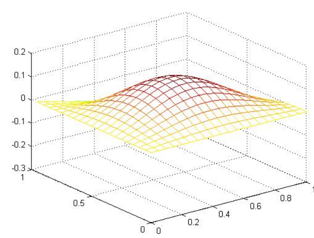

Figure 13: The interpolation error for $S_{21}F_1$

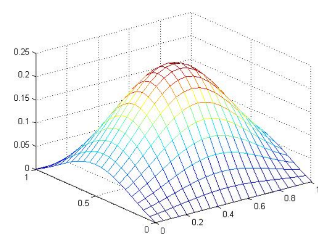

Figure 14: The interpolation error for H_2F_1

The following table contains the maximum interpolation errors for the function given in 8, defined on D_1, $(h = 1)$.

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Table 2: The interpolation error

Max error	F_1
L_2F	0.2393
$P_{21}F$	0.3068
$S_{21}F$	0.1369
H_2	0.2393

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