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FIXED POINT THEOREMS FOR A PAIR OF GENERALIZED CONTRACTIVE MAPPINGS OVER A METRIC SPACE WITH AN APPLICATION TO HOMOTOPY

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Abstract. In this paper, two generalized contractive type mappings, namely generalized weak contraction mapping and special-type weak contraction mapping have been introduced. Existence of fixed points, coincidence points and common fixed points for these type of mappings have been shown here. Examples and counter examples have been cited in support of our theorems. Moreover, one of our fixed point theorems is applied to homotopy theory.

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1. Introduction

The theory of fixed points has now been an emerging area in mathematics specially in nonlinear functional analysis because of its wide applicability in various fields of mathematics as well as physical science, life sciences, mathematical economics, approximation theory, optimal control problems and the like. The field of the fixed point theory as on today is vast and open to lots of techniques and ideas. Historically after the initial jerk on the existence of fixed point of mappings by Brouwer [5] and Schauder [10], a polish mathematician S. Banach [1] proved one of his first remarkable theorem commonly known as Banach Contraction Principle Theorem. Following this discovery a literature in fixed point theory, specially in a setting of metric spaces had flooded by many workers who had successfully extended Banach contraction principle theorem in many directions. Of late, V. Berinde has introduced a generalized contractive mapping, namely weak contraction mapping (See [2]). This mapping is so strong in nature that it generalizes not only just contraction map but also Kannan maps, Chatterjee maps, Zamifirescu maps and also partially Ćirić quasi contraction maps. Subsequently in 2008 V. Berinde has established a more generalized mapping, Ćirić almost contraction mapping [3], which generalizes the aforesaid weak contraction maps and Ćirić quasi contraction maps. Also a special contractive type mapping was introduced by him, namely $\phi-$contraction or (ϕ, L) weak contraction enabled to establish fixed point theorems using $(c)-$comparison function (See [4]). Motivated by the background of these literatures and with one of interesting works done by Saha and Baisnab [9], we now introduce here a class of mappings namely (i) generalized weak contraction mappings and (ii) special-type weak contraction mappings. It is to be noted that any weak contraction map is a generalized weak contraction but the reverse implication is not necessarily true.

In the last century due to prolonged investigations on homotopy theory it is revealed that recent trends towards research works promote contributions for providing connections between homotopy theory and a higher dimensional category theory. Homotopy theory is a main part of algebraic topology which is used to study topological objects upto homotopy equivalence. It is a fact that a fixed point theory might be considered as perturbation stable whenever the theorems do hold with respect to a mapping under consideration also hold with respect to small perturbation of data dependence mapping T. As a matter of fact that researchers are paying more interests in establishing the validation of theorems for small perturbation as well as for any deformation of the mapping T. In other words we are very much interested to obtain theorems for a class of mappings T which remain valid for another class of mappings that are homotopic to T. For detail one can refer the literatures (See [6] and [11]).

In this paper we have been able to prove some fixed point theorems, coincidence point theorems and common fixed point theorems for a class of mappings as dealt by us. Examples and counter examples have been provided in strengthening of the hypothesis of our theorems. Finally we have been able to show an application of our fixed point theorem to homotopy theory.

2. Preliminaries

Before going to our main results we need some basic preliminaries.

Definition 1. *[2] Let (X, d) be a metric space. A map $T : X \to X$ is called weak contraction if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta\, d(x, y) + L\, d(y, Tx) \tag{1}$$

for all $x, y \in X$.

Theorem 1. *[2] Let (X, d) be a complete metric space and $T : X \to X$ a weak contraction i.e. a map satisfying (1) with $\delta \in (0, 1)$ and some $L \geq 0$. Then*

1) $F(T) = \{x \in X : Tx = x\} \neq \phi$,

2) *For any* $x_0 \in X$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *is given by* $x_n = Tx_{n-1} = T^n x_0$ *converges to some* $x^* \in F(T)$,

3) *The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, ... \tag{2}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, ... \tag{3}$$

hold, where δ *is the constant appearing in* (1).

Theorem 2. *[2] Let* (X, d) *be a complete metric space and* $T : X \to X$ *a weak contraction for which there exists* $\theta \in (0, 1)$ *and some* $L_1 \geq 0$ *such that*

$$d(Tx, Ty) \leq \theta d(x, y) + L_1\, d(x, Tx) \tag{4}$$

for all $x \in X$*. Then*

1) T *has a unique fixed point i.e.* $F(T) = \{x^*\}$,

2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *is given by* $x_n = Tx_{n-1} = T^n x_0$ *converges to* x^**, for any* $x_0 \in X$,

3) *Then a priori and a posteriori error estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \quad n = 0, 1, 2, ... \tag{5}$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \quad n = 1, 2, ... \tag{6}$$

hold.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta d(x_{n-1}, x^*), \quad n = 1, 2, ... \tag{7}$$

Example 1. *[2] Let* $T : [0, 1] \to [0, 1]$ *be the identity map, i.e.,* $Tx = x$*, for all* $x \in [0, 1]$*. Then* T *is not a Ćirić quasi-contraction map but it is a weak contraction map with* $\delta \in (0, 1)$ *arbitrary and* $L \geq 1 - \delta$*. The set of fixed points of* T *is the entire interval* $[0, 1]$ *that is* $F(T) = [0, 1]$.

Example 2. *[2] Let* $T : [0, 1] \to [0, 1]$ *be given by* $Tx = \frac{2}{3}$*, for* $x \in [0, 1)$ *and* $T1 = 0$*. Then* T *is a weak contraction map with* $\delta \geq \frac{2}{3}$ *and* $L \geq \delta$*. Also* T *is a Ćirić quasi-contraction map for* $h \in [\frac{2}{3}, 1)$.

Definition 2. *[7] A mapping* $T : (M, d) \to (M, d)$ *is said to be orbitally continuous if* $u \in M$ *and such that* $u = \lim_{i\to\infty} T^{n_i}x$ *for some* $x \in M$*, then* $Tu = \lim_{i\to\infty} TT^{n_i}x$.

3. Main Results

Now as a generalization of Berinde's weak contraction map here we introduce a class of generalized contractive mappings.

3.1. Generalized weak contraction map and fixed point theorems

Definition 3. *Let (X, d) be a metric space. Then a mapping $T : X \to X$ is said to be a generalized weak contraction mapping if there exists three functions $a, b, \delta : X \times X \to [0, \infty)$ satisfying $0 < a(x, y) < 1$ for all $x, y \in X$ such that*

$$d(T^n x, T^n y) \leq a(x, y)^n \delta(x, y) + b(x, y) \sum_{m=1}^{n} a(x, y)^{n-m} d(T^{m-1} y, T^m x) \qquad (8)$$

$\forall x, y \in X$ and $\forall n \in \mathbb{N}$.

Due to the symmetry of distance function we also have the following

$$d(T^n x, T^n y) \leq a(y, x)^n \delta(y, x) + b(y, x) \sum_{m=1}^{n} a(y, x)^{n-m} d(T^{m-1} x, T^m y) \qquad (9)$$

$\forall x, y \in X$ and $\forall n \in \mathbb{N}$.

Theorem 3. *Let (X, d) be a complete metric space and suppose that $T : X \to X$ be a generalized weak contraction mapping. If T is orbitally continuous then it has atleast one fixed point in X. Furthermore if $0 < a(x, y) + b(x, y) < 1$ for all $x, y \in X$ then the fixed point of T is also unique.*

Proof. Let $x_0 \in X$. We construct the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Then we get,

$$\begin{aligned} d(T^n x_0, T^{n+1} x_0) &\leq a(x_0, Tx_0)^n \delta(x_0, Tx_0) \\ &\quad + b(x_0, Tx_0) \sum_{m=1}^{n} a(x_0, Tx_0)^{n-m} d(T^{m-1}(Tx_0), T^m x_0) \\ &= a(x_0, Tx_0)^n \delta(x_0, Tx_0) \quad \forall n \in \mathbb{N}. \qquad (10) \end{aligned}$$

For $n \in \mathbb{N}$ and for $p \geq 1$ we get,

$$\begin{aligned}
& d(T^n x_0, T^{n+p} x_0) \\
\leq\ & d(T^n x_0, T^{n+1} x_0) + d(T^{n+1} x_0, T^{n+2} x_0) + ... \\
& + d(T^{n+p-1} x_0, T^{n+p} x_0) \\
\leq\ & [a(x_0, Tx_0)^n + a(x_0, Tx_0)^{n+1} + ... + a(x_0, Tx_0)^{n+p-1}]\delta(x_0, Tx_0) \\
=\ & a(x_0, Tx_0)^n \frac{1 - a(x_0, Tx_0)^p}{1 - a(x_0, Tx_0)} \delta(x_0, Tx_0) \\
\leq\ & \frac{a(x_0, Tx_0)^n}{1 - a(x_0, Tx_0)} \delta(x_0, Tx_0). \qquad (11)
\end{aligned}$$

Hence for any $p \geq 1$, $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$ and therefore $\{x_n = T^n x_0\}$ is Cauchy in (X, d). Since X is complete then there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is orbitally continuous in X and $T^n x_0 \to z$ as $n \to \infty$ so we have $T^{n+1} x_0 \to Tz$ as $n \to \infty$. It follows that $Tz = z$. Hence z is a fixed point of T.

Now if $0 < a(x, y) + b(x, y) < 1$ for all $x, y \in X$ then we show that this fixed point is unique. Let z_1 and z_2 be two fixed points of T then $Tz_1 = z_1$ and also $Tz_2 = z_2$, which in turn implies that $T^n z_1 = z_1$ and $T^n z_2 = z_2$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
d(z_1, z_2) &= d(T^n z_1, T^n z_2) \\
&\leq a(z_1, z_2)^n \delta(z_1, z_2) \\
&\quad + b(z_1, z_2) \sum_{m=1}^{n} a(z_1, z_2)^{n-m} d(T^{m-1} z_2, T^m z_1) \\
&= a(z_1, z_2)^n \delta(z_1, z_2) + b(z_1, z_2)[\sum_{m=1}^{n} a(z_1, z_2)^{n-m}] d(z_1, z_2) \\
&= a(z_1, z_2)^n \delta(z_1, z_2) + b(z_1, z_2) \frac{1 - a(z_1, z_2)^n}{1 - a(z_1, z_2)} d(z_1, z_2) \\
&< a(z_1, z_2)^n \delta(z_1, z_2) + \frac{b(z_1, z_2)}{1 - a(z_1, z_2)} d(z_1, z_2). \qquad (12)
\end{aligned}$$

Now since $0 < a(z_1, z_2) + b(z_1, z_2) < 1$ then $[1 - \frac{b(z_1, z_2)}{1 - a(z_1, z_2)}] d(z_1, z_2) \leq a(z_1, z_2)^n \delta(z_1, z_2)$, implying that $d(z_1, z_2) \leq \frac{1 - a(z_1, z_2)}{1 - (a(z_1, z_2) + b(z_1, z_2))} a(z_1, z_2)^n \delta(z_1, z_2) \to 0$ as $n \to \infty$. Therefore $d(z_1, z_2) = 0$ i.e. $z_1 = z_2$ and hence T has a unique fixed point in X.

Remark 1. *If we consider $a(x, y) + b(x, y) \geq 1$ for a mapping $T : X \to X$ that satisfies* (8)*, then we see that T does not admit a unique fixed point in X. The following is the example of justification to our statement.*

Example 3. *Let (X, d) be a metric space with more than one point and $T : X \to X$ be the identity mapping on X. Let us choose $a, b : X \times X \to X$ by $a(x, y) = \frac{1}{2}$ and $b(x, y) = \frac{1}{2}$ for all $x, y \in X$ and $\delta : X \times X \to X$ by $\delta(x, y) = d(x, y)$ $\forall x, y \in X$ then*

$$\begin{aligned} d(T^n x, T^n y) = d(x, y) &\leq a(x, y)^n d(x, y) + b(x, y) \sum_{m=1}^{n} a(x, y)^{n-m} d(x, y) \\ &= a(x, y)^n \delta(x, y) + b(x, y) \sum_{m=1}^{n} a(x, y)^{n-m} d(T^{m-1} y, T^m x) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So T is a generalized weak contraction mapping. Here $a(x, y) + b(x, y) = 1$ and we see that T has more than one fixed points in X.

It is to be noted that the conditions used in Theorem 3 are just sufficient. For its justification we cite the following examples.

Example 4. *Let $X = \mathbb{R}$ be the metric space with usual metric. Let $Y = \{0, \frac{1}{2}, \frac{1}{2^2}, ...\} \subset X$. Also let $T : Y \to Y$ be defined by*

$$T(x) = \begin{cases} \frac{1}{2}, & x \in \{0, \frac{1}{2}\} \\ \frac{1}{2^{k+1}}, & x \in \{\frac{1}{2^k} : k \geq 2\} \end{cases} \tag{13}$$

Now let us choose $x_0 = \frac{1}{4}$. Then the sequence $\{T^n x_0\}$ converges to 0 as $n \to \infty$ but $T(T^n x_0) \to 0 \neq \frac{1}{2} = T0$ as $n \to \infty$. So T is not orbitally continuous in Y. Let us take $\delta : Y \times Y \to Y$ by $\delta(x, y) = 1 + x + y$ for all $x, y \in Y$, $a(x, y) = \frac{1}{2}$ and $b(x, y) = 1$ $\forall x, y \in Y$. By routine verification it can be seen that T is a generalized weak contraction mapping. Also we see that $\frac{1}{2}$ is the unique fixed point of T in Y. Here $a(x, y) + b(x, y) = \frac{1}{2} + 1 = \frac{3}{2} > 1$ for all $x, y \in Y$.

Remark 2. *If in Example 3, X is assumed to be an incomplete metric space then T also admit a fixed point in X.*

Remark 3. *Any Berinde weak contraction (contractive condition (1)) is orbitally continuous in X.*

In fact if T is a weak contraction map such that for some $y \in X$, $T^{n_i} y \to u$ as $i \to \infty$, where $u \in X$ and $\{n_i\}_{i \in \mathbb{N}}$ is a subsequence in X, then

$$\begin{aligned} d(T^{n_i+1} y, Tu) &\leq \delta d(T^{n_i} y, u) + L d(u, T^{n_i+1} y) \\ &\leq \delta d(T^{n_i} y, u) + L[d(u, T^{n_i} y) + d(T^{n_i} y, T^{n_i+1} y)] \end{aligned} \tag{14}$$

Now for any $n \in \mathbb{N}$ and for any $x \in X$ we have $d(T^n x, T^{n+1} x) \leq \delta^n d(x, Tx)$. Therfore from (14) we get, $d(T^{n_i+1} y, Tu) \leq d(T^{n_i} y, u) + L d(u, T^{n_i} y) + L \delta^{n_i} d(y, Ty) \to 0$ as $n \to \infty$. So $TT^{n_i} y \to Tu$ as $i \to \infty$. Hence T is orbitally continuous in X.

Corollary 4. *Let (X,d) be a complete metric space and $T : X \to X$ be a weak contraction i.e. a map satisfying*

$$d(Tx,Ty) \leq \delta\, d(x,y) + L\, d(y,Tx) \tag{15}$$

for all $x,y \in X$, where $\delta \in (0,1)$ and $L \geq 0$. Then T has atleast one fixed point in X.

Proof. Given that T satisfies the relation (15). Then for any $x,y \in X$ we have,

$$\begin{aligned} d(T^2x,T^2y) &\leq \delta\, d(Tx,Ty) + L\, d(Ty,T^2x) \\ &\leq \delta[\delta\, d(x,y) + L\, d(y,Tx)] + L\, d(Ty,T^2x) \\ &= \delta^2\, d(x,y) + L[d(Ty,T^2x) + \delta\, d(y,Tx)] \end{aligned} \tag{16}$$

and also

$$\begin{aligned} d(T^3x,T^3y) &\leq \delta\, d(T^2x,T^2y) + L\, d(T^2y,T^3x) \\ &\leq \delta[\delta^2\, d(x,y) + L\{d(Ty,T^2x) + \delta\, d(y,Tx)\}] + L\, d(T^2y,T^3x) \\ &= \delta^3\, d(x,y) + L[d(T^2y,T^3x) + \delta\, d(Ty,T^2x) + \delta^2\, d(y,Tx)] \end{aligned} \tag{17}$$

Proceeding in this way we get,

$$d(T^nx,T^ny) \leq \delta^n d(x,y) + L\sum_{m=1}^{n} \delta^{n-m} d(T^{m-1}y,T^mx) \quad \forall x,y \in X, \forall n \in \mathbb{N}. \tag{18}$$

Therefore T is a generalized weak contraction mapping for $\delta(x,y) = d(x,y)$, $a(x,y) = \delta$ and $b(x,y) = L$ $\forall x,y \in X$. Also T is orbitally continuous in X. By Theorem 3 T has atleast one fixed point in X.

Remark 4. *Any Banach contraction map, Kannan contractive map, Chatterjee type map and Zamfirescu mapping are of Berinde type weak contraction (See [2]) so they are also generalized weak contraction mapping and therefore they have atleast one fixed point in a complete metric space X.*

Also a Ćirić contraction map or quasi contraction map is a weak contraction for Lipschitz constant $\alpha \in (0, \frac{1}{2})$ (See [2]). So any quasi contraction map (with the constant $0 < \alpha < \frac{1}{2}$) is also a generalized weak contraction mapping and thus it has atleast one fixed point in a complete metric space X.

Note 1. *Any generalized weak contraction map may not be a weak contraction map. The following example is given for its justification.*

Remark 5. *The mapping cited in Example 4 is a generalized weak contraction map but not a weak contraction mapping since it is not orbitally continuous in X.*

Theorem 5. *Let (X,d) be a complete metric space. Suppose that $T,S : X \to X$ are two mappings satisfying the following conditions:*

$$\begin{aligned} (i) \qquad & d(T^n x, S^n y) \\ \leq \; & a(x,y)^{n-1}\delta(x,y) \\ & + b(x,y)\min\Big(\sum_{m=1}^{n-1} a(x,y)^{n-m-1} d(S^m y, T^{m+1}x), \sum_{m=1}^{n-1} a(x,y)^{n-m-1} d(T^m x, S^{m+1}y)\Big) \end{aligned} \tag{19}$$

for all $x,y \in X$ and for all $n \geq 2$, where $\delta(x,y), b(x,y)$ are non-negative reals and $0 < a(x,y) < 1 \;\forall\; x,y \in X$,

(ii) $T^n x_0 = S^n x_0$ for all $n \geq 2$, for some $x_0 \in X$ and

(iii) T and S both are orbitally continuous in X.

Then T and S have atleast one common fixed point in X. Moreover if $a(x,y) + b(x,y) < 1$ for all $x,y \in X$ then the fixed point is also unique.

Proof. Let us consider the sequence $\{x_n\}$ in X defined by $x_n = T^n x_0$ if n is even and $x_n = S^n x_0$ if n is odd. Then for $n \in \mathbb{N}$ we have,

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(T^{2n}x_0, S^{2n+1}x_0) \\ &\leq a(x_0, Sx_0)^{2n-1}\delta(x_0, Sx_0) \\ &\quad + b(x_0, Sx_0)\min\Big(\sum_{m=1}^{2n-1} a(x_0, Sx_0)^{2n-m-1} d(S^{m+1}x_0, T^{m+1}x_0), \\ &\quad \sum_{m=1}^{2n-1} a(x_0, Sx_0)^{2n-m-1} d(S^{m+2}x_0, T^m x_0)\Big) \\ &= a(x_0, Sx_0)^{2n-1}\delta(x_0, Sx_0) \end{aligned} \tag{20}$$

and also for $n \geq 2$,

$$\begin{aligned} d(x_{2n-1}, x_{2n}) &= d(T^{2n}x_0, S^{2n-1}x_0) \\ &\leq a(Tx_0, x_0)^{2n-2}\delta(Tx_0, x_0) \\ &\quad + b(Tx_0, x_0)\min\Big(\sum_{m=1}^{2n-2} a(Tx_0, x_0)^{2n-m-2} d(S^m x_0, T^{m+2}x_0), \\ &\quad \sum_{m=1}^{2n-2} a(Tx_0, x_0)^{2n-m-2} d(S^{m+1}x_0, T^{m+1}x_0)\Big) \\ &= a(Tx_0, x_0)^{2n-2}\delta(Tx_0, x_0) \end{aligned} \tag{21}$$

So from (20) and (21) we have, for all $n \geq 2$

$$d(x_n, x_{n+1}) \leq a(x_0, Sx_0)^{n-1}\delta(x_0, Sx_0) + a(Tx_0, x_0)^{n-1}\delta(Tx_0, x_0) \tag{22}$$

Then by a routine verification we see that $\{x_n\}$ is a Cauchy sequence in X, since X is complete so there exists $z \in X$ such that $\{x_n\}$ converges to z. Therefore $T^{2n}x_0 \to z$ and $S^{2n-1}x_0 \to z$ as $n \to \infty$. Now since T is orbitally continuous in X so $S^{2n+1}x_0 = T^{2n+1}x_0 \to Tz$ as $n \to \infty$. Therefore $Tz = z$. Similarly we get $Sz = z$. Hence z is a common fixed point of T and S in X.
Moreover if $a(x, y) + b(x, y) < 1$ for all $x, y \in X$ then we prove that this fixed point is unique.

Let, w be another common fixed point of T and S in X. Then $Tw = Sw = w$. Therefore for all $n \geq 2$,

$$\begin{aligned}
d(z, w) &= d(T^n z, S^n w) \\
&\leq a(z, w)^{n-1}\delta(z, w) + b(z, w)\min\Big(\sum_{m=1}^{n-1} a(z, w)^{n-m-1} d(S^m w, T^{m+1}z), \\
&\quad \sum_{m=1}^{n-1} a(z, w)^{n-m-1} d(S^{m+1}w, T^m z)\Big) \\
&= a(z, w)^{n-1}\delta(z, w) + b(z, w)\sum_{m=1}^{n-1} a(z, w)^{n-m-1} d(z, w) \\
&= a(z, w)^{n-1}\delta(z, w) + b(z, w)\frac{1 - a(z, w)^n}{1 - a(z, w)} d(z, w) \\
&\leq a(z, w)^{n-1}\delta(z, w) + \frac{b(z, w)}{1 - a(z, w)} d(z, w)
\end{aligned} \tag{23}$$

Thus $d(z, w) \leq \frac{1-a(z,w)}{1-a(z,w)-b(z,w)} a(z, w)^{n-1}\delta(z, w) \to 0$ as $n \to \infty$, which implies that $d(z, w) = 0$ that is $z = w$. Therefore T and S have a unique common fixed point in X.

3.2. Special-type weak contraction map and fixed point theorems

Definition 4. *A self mapping T over a metric space (X, d) is said to be a special-type weak contraction if it satisfies the condition*

$$d(Tx, Ty) \leq a\, d(x, y) + L\, d(y, Tx) d(x, Ty) \ \ \forall\, x, y \in X \tag{24}$$

where $a \in (0, 1)$ and $L \geq 0$.

Theorem 6. *Let (X, d) be a complete metric space and $T : X \to X$ be a special-type weak contraction. Then T has atleast one fixed point in X. Moreover if $L > 0$ and $diam(X) < \frac{1-a}{L}$ then T has exactly one fixed point in X.*

Proof. Let $x_0 \in X$ and we construct the Picard iterative sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$.

Then, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a\ d(x_{n-1}, x_n) + L\ d(x_n, Tx_{n-1})d(x_{n-1}, Tx_n) = a\ d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. So $\{x_n\}$ is Cauchy sequence in X, since X is complete so there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Now,

$$\begin{aligned}
& d(T^{n+1}x_0, Tz) \\
\leq\ & a\ d(T^n x_0, z) + L\ d(z, T^{n+1}x_0)d(T^n x_0, Tz) \\
\leq\ & a\ d(T^n x_0, z) + L\ d(z, T^{n+1}x_0)[d(T^n x_0, z) + d(z, Tz)] \\
=\ & a\ d(T^n x_0, z) + L\ d(z, T^{n+1}x_0)d(T^n x_0, z) + Ld(z, Tz)d(z, T^{n+1}x_0)
\end{aligned}$$

Then $d(T^{n+1}x_0, Tz) \to 0$ as $n \to \infty$. Therefore by taking $n \to \infty$ we have $d(z, Tz) \leq d(z, T^{n+1}x_0) + d(T^{n+1}x_0, Tz) \to 0$. Thus $d(z, Tz) = 0$ implying that $Tz = z$. Hence T has a fixed point in X.

Now let us assume that $L > 0$ and $diam(X) < \frac{1-a}{L}$. Suppose z_1, z_2 be two distinct fixed points of T then we get

$$\begin{aligned}
d(z_1, z_2) = d(Tz_1, Tz_2) &\leq a\ d(z_1, z_2) + L\ d(z_2, Tz_1)d(z_1, Tz_2) \\
&= a\ d(z_1, z_2) + L\ d(z_1, z_2)^2
\end{aligned}$$

Hence $d(z_1, z_2) \geq \frac{1-a}{L}$, a contradiction that $diam(X) < \frac{1-a}{L}$. Therefore T has a unique fixed point in X.

Remark 6. *Let (X, d) be a metric space and $T : X \to X$ be a contraction map, then T is also a special-type weak contraction map. Now any contraction map is continuous but there are discontinuous maps which are special-type weak contraction. See the following example.*

Example 5. *Let $X = [0, 1]$ with usual metric on $\mathbb{R}$. Let $T : X \to X$ be given by*

$$Tx = \begin{cases} \frac{x}{3}, & x \in [0, \frac{2}{3}] \\ \frac{1}{3}, & x \in (\frac{2}{3}, 1] \end{cases} \tag{25}$$

Also let us take $a = \frac{2}{3}$ and $L = \frac{3}{4}$. Then T is a special-type weak contraction. We see that 0 is the fixed point of T in X.

Note 2. *There are mappings which satisfy neither Ćirić's contractive condition, nor Banach, Kannan, Chatterjea and Zamifiresu contractive conditions but is a special-type weak contraction mapping. Example 6 supports our proposition.*

Example 6. *Let $X = [0,1]$ be the usual metric space and $T : X \to X$ be given by*

$$Tx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{2}{3}) \\ 1, & x \in [\frac{2}{3}, 1] \end{cases} \tag{26}$$

Then T is a special-type contraction for any constant $a \in (0,1)$ and for the constant $L \geq 9$ but it satisfies neither Ćirić's contractive condition, nor Banach, Kannan, Chatterjea and Zamifiresu contractive conditions.

Example 7. *There are mappings which are Berinde weak contraction but not a special-type weak contraction. Let $X = \mathbb{R}$ be the metric space with usual metric and $T : X \to X$ be the identity mapping, then it is a Berinde weak contraction. If T is a special type weak contraction then there exists $a \in (0,1)$ and $L > 0$ such that*

$$d(Tx, Ty) \leq a\, d(x,y) + L\, d(y, Tx) d(x, Ty) \ \ \forall\, x, y \in X$$

Now if $x \neq y$ then $d(x,y) \leq a\, d(x,y) + L\, d(x,y)^2$ implying that $d(x,y) \geq \frac{1-a}{L}$, which is not true. So T can not be a special-type weak contraction.

Note 3. *If (X,d) be a metric space such that $\sup\{d(x,y) : x \in X, y \in T(X)\} \leq 1$ then clearly any special-type weak contraction is also a weak contraction.*

Example 8. *The condition that X is complete in Theorem 6 is just sufficient. If we take $X = (0,1]$ with usual metric and $T : X \to X$ by*

$$Tx = \begin{cases} \frac{1}{2}, & x \in (0, \frac{2}{3}) \\ 1, & x \in [\frac{2}{3}, 1] \end{cases} \tag{27}$$

Then T is a special-type weak contraction for any constant $0 < a < 1$ and for the constant $L \geq 9$ and has two fixed points $\frac{1}{2}, 1$. Here we see that X is not complete.

Example 9. *The condition that diam$(X) < \frac{1-a}{L}$ is sufficient for the existence of unique fixed point of a special-type weak contraction mapping. In Example 5 we see that T has a unique fixed point in X, though diam$(X) \nless \frac{1-\frac{2}{3}}{\frac{3}{4}} = \frac{4}{9}$.*

Theorem 7. *Let (X,d) be a complete metric space and f be a continuous mapping on X. Let $g : X \to X$ be a mapping which commutes with f, satisfies $g(X) \subset f(X)$ and*

$$d(gx, gy) \leq a\, d(fx, fy) + L\, d(fx, gy) d(fy, gx) \ \ \forall x, y \in X \tag{28}$$

where $0 < a < 1$ and $L \geq 0$. Then f and g have atleast one coincidence point in X. Moreover if $L > 0$ and diam$(X) < \frac{1-a}{L}$ then f, g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be fixed. Then $gx_0 \in g(X) \subset f(X)$. So there exists $x_1 \in X$ such that $gx_0 = fx_1$. Proceeding in a similar way we can construct a sequence $\{y_n\}$ by $y_n = gx_{n-1} = fx_n$ for all $n \geq 1$. Now,

$$\begin{aligned} d(y_n, y_{n+1}) = d(gx_{n-1}, gx_n) &\leq a\, d(fx_{n-1}, fx_n) + L\, d(fx_{n-1}, gx_n) d(fx_n, gx_{n-1}) \\ &= a\, d(y_{n-1}, y_n) \;\; \forall n \in \mathbb{N} \end{aligned}$$

Therefore $\{y_n\}$ is a Cauchy sequence in X. Since X is complete so there exists $z \in X$ such that $\{y_n\}$ converges to z. As f is continuous in X so $fy_n \to fz$ as $n \to \infty$. Now, $gy_{n-1} = gfx_{n-1} = fgx_{n-1} = fy_n \to fz$ as $n \to \infty$. Then,

$$\begin{aligned} d(gy_n, gz) &\leq a\, d(fy_n, fz) + L\, d(fy_n, gz) d(fz, gy_n) \\ &\leq a\, d(fy_n, fz) + L\, [d(fy_n, fz) + d(fz, gz)] d(fz, gy_n) \\ &= a\, d(fy_n, fz) + L\, d(fy_n, fz) d(fz, gy_n) + L\, d(fz, gz)] d(fz, gy_n) \end{aligned}$$

Hence we get $gy_n \to gz$ as $n \to \infty$ and therefore we get $fz = gz$ showing that f and g have a coincidence point in X.
Now let, $L > 0$ and $diam(X) < \frac{1-a}{L}$. Then,

$$\begin{aligned} d(g^2z, gz) &\leq a\, d(fgz, fz) + L\, d(fgz, gz) d(fz, g^2 z) \\ &= a\, d(g^2 z, gz) + L\, d(g^2 z, gz)^2 \end{aligned}$$

Now if $g^2 z \neq gz$ then $d(g^2 z, gz) \geq \frac{1-a}{L}$, a contradiction to the fact that $diam(X) < \frac{1-a}{L}$. So $g^2 z = gz$. Therefore $f(gz) = g(gz) = gz$. Thus gz is a common fixed point of f, g. Now let z_1, z_2 be two distinct common fixed points of f and g. Then proceeding in the same way as above we get $d(z_1, z_2) \geq \frac{1-a}{L}$, which leads to a contradiction. Therefore f and g have a unique common fixed point in X.

Remark 7. *Any two mappings f and g satisfying Jungck's contrctive condition [8] also satisfy the contractive condition due to Theorem 7.*

The conditions used in Theorem 7 are just sufficient. For its justification we cite the following examples.

Example 10. *Let $X = [0, 1]$ with usual metric and $g, f : X \to X$ be defined by*

$$g(x) = \begin{cases} \frac{1}{4}, & x \in [0, \frac{2}{3}] \\ 1, & x \in (\frac{2}{3}, 1] \end{cases}, f(x) = \begin{cases} x, & x \in [0, \frac{2}{3}] \\ 1, & x \in (\frac{2}{3}, 1] \end{cases} \tag{29}$$

Then f and g commute with $g(X) \subset f(X)$. Also they satisfy the contractive condition of Theorem 7 for $a = \frac{1}{4}$ and $L = 3$. We also see that the set of coincidence points of f and g is $\{\frac{1}{4}\} \cup (\frac{2}{3}, 1]$. But here f is not continuous at $x = \frac{2}{3}$.

Example 11. *Let us take $X = [0,1]$ with usual metric on $\mathbb{R}$ and $f, g : X \to X$ be defined by*

$$g(x) = \begin{cases} \frac{1}{4}, & x \in [0, \frac{2}{3}] \\ \frac{1}{2}, & x \in (\frac{2}{3}, 1] \end{cases}, f(x) = \frac{x}{2} \ \ \forall x \in [0,1] \tag{30}$$

Then $g(X) \subset f(X)$ and satisfy the condition (28) for the constant $L = 18$ and for any constant $a \in (0,1)$. Also f is continuous. Here $\frac{1}{2}, 1$ are the coincidence points of f and g in X though f and g do not commute.

Example 12. *Let $X = [0,1]$ with usual metric of $\mathbb{R}$ and $g, f : X \to X$ be defined by*

$$g(x) = \begin{cases} \frac{1}{4}, & x \in [0, \frac{2}{3}] \\ 1, & x \in (\frac{2}{3}, 1] \end{cases}, f(x) = \begin{cases} \frac{x}{2}, & x \in [0, \frac{2}{3}] \\ x, & x \in (\frac{2}{3}, 1] \end{cases} \tag{31}$$

Then $g(X) \subset f(X)$ and f and g satisfy the contractive condition (28) for the constant $L = 9$ and for any constant $a \in (0,1)$, also $\frac{1}{2}, 1$ are the coincidence points of f and g. But here neither f and g commute nor f is continuous.

Example 13. *Let us take $X = [0,1)$ with usual metric and $f, g : X \to X$ are defined by $fx = \frac{x}{2}$ and $gx = \frac{x}{4}$ for all $x \in X$. Then f and g satisfy all the conditions of Theorem 7 without being X is complete but 0 is the unique coincidence point of f and g in X.*

Example 14. *Let $X = [0,1]$ be the metric space with usual metric in $\mathbb{R}$ and $f, g : X \to X$ be defined by $fx = \frac{x}{2}$ and $gx = \frac{x}{6}$ $\forall x \in X$. Then f and g satisfy the contractive condition due to Theorem 7 for $a = \frac{1}{3}$ and for $L = 1$. Here f, g have a unique common fixed point in X but $diam(X) \not< \frac{1}{3}$.*

Theorem 8. *Let (X,d) be a complete metric space and $T, S : X \to X$ be two mappings satisfying*

$$d(Tx, Sy) \leq a\, d(x,y) + L\, d(x, Sy) d(y, Tx) \ \ \forall x, y \in X \tag{32}$$

where $a \in (0,1)$ and $L \geq 0$. Then T and S have atleast one common fixed point in X. Moreover if $diam(X) < \frac{1-a}{L}$ then they have a unique common fixed point in X.

Proof. let $x_0 \in X$ and let us construct the sequence $\{x_n\}$ in X by $x_{2n} = Sx_{2n-1}$ for all $n \in \mathbb{N}$ and $x_{2n+1} = Tx_{2n}$ $\forall\, n \geq 0$. Then

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Tx_{2n}, Sx_{2n-1}) \\ &\leq a\, d(x_{2n}, x_{2n-1}) + L\, d(x_{2n}, Sx_{2n-1}) d(x_{2n-1}, Tx_{2n}) \\ &= a\, d(x_{2n}, x_{2n-1}) \ \ \forall\, n \geq 1 \end{aligned}$$

and also,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Tx_{2n}, Sx_{2n+1}) \\ &\leq a\, d(x_{2n}, x_{2n+1}) + L\, d(x_{2n}, Sx_{2n+1}) d(x_{2n+1}, Tx_{2n}) \\ &= a\, d(x_{2n}, x_{2n+1}) \ \ \forall\, n \geq 1 \end{aligned}$$

Therefore $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. So $\{x_n\}$ is Cauchy sequence in X. Since X is complete then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Therefore $Tx_{2n} \to z$ and $Sx_{2n-1} \to z$ as $n \to \infty$. Now,

$$\begin{aligned} d(Tx_{2n}, Sz) &\leq a\, d(x_{2n}, z) + L\, d(x_{2n}, Sz) d(z, Tx_{2n}) \\ &\leq a\, d(x_{2n}, z) + L\, d(z, Tx_{2n})[d(x_{2n}, z) + d(z, Sz)] \\ &= a\, d(x_{2n}, z) + L\, d(z, x_{2n+1}) d(x_{2n}, z) + L\, d(z, x_{2n+1}) d(z, Sz) \end{aligned}$$

So $Tx_{2n} \to Sz$ as $n \to \infty$ and therefore $Sz = z$. Similarly we have $Tz = z$. Therefore z is a common fixed point of T, S.

Now let $diam(X) < \frac{1-a}{L}$ and u, v be two distinct common fixed points of T and S. By a routine calculation we get $d(u, v) \geq \frac{1-a}{L}$, a contradiction. Thus T and S have a unique common fixed point in X.

4. Application of fixed point theorem to homotopy theory

In this section, we obtain a homotopy result as an application of Theorem 6. First we give the definition of homotopy between two functions.

Definition 5. *[11] Let X, Y be two topological spaces, and let $G, S : X \to Y$ be two continuous mappings. Then, a homotopy from G to S is a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = Gx$ and $H(x, 1) = Sx$, for all $x \in X$. Also, G and S are called homotopic mappings.*

Theorem 9. *Let (X, d) be a complete metric space and U be an open and V be a closed subset of X with $U \subset V$.*

Let the operator $H : V \times [0, 1] \to X$ satisfies the following conditions:

a) $x \neq H(x, t)$ for every $x \in V \backslash U$ and for any $t \in [0, 1]$,

b) $d(H(x, t), H(y, t)) \leq ad(x, y) + Ld(y, H(x, t))d(x, H(y, t))$ for all $x, y \in V$ and for any $t \in [0, 1]$, where $a \in (0, 1)$ and $L \geq 0$,

c) There exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that $d(H(x, t), H(x, s)) \leq |g(t) - g(s)|$ for all $t, s \in [0, 1]$ and for every $x \in V$,

d) $\sup_{x,y \in V, t \in [0,1]} d(x, H(y, t)) < \frac{1-a}{L}$.

Then $H(., 0)$ has a fixed point if and only if $H(., 1)$ has a fixed point.

Proof. Let us define the set $G = \{t \in [0,1] : H(.,t)$ has a fixed point in U $\}$. Also let us assume that $H(.,0)$ has a fixed point. Since the condition (a) holds then there exists $x \in U$ such that $H(x,0) = x$. Therefore $0 \in G$ and hence G is a non-empty subset of $[0,1]$. We just want to show that G is a clopen subset of $[0,1]$. Hence from the connectedness of $[0,1]$ it will readily follow that $G = [0,1]$.

First we show that G is open. let $t_0 \in G$. Then there exists $x_0 \in U$ such that $H(x_0,t_0) = x_0$. Now since U is open so there exists $r > 0$ such that $B(x_0,r) \subset U$. Let us take $0 < \epsilon \leq rL[\frac{1-a}{L} - \sup_{x \in \overline{B(x_0,r)}} d(x_0, H(x,t_0))]$. Since g is continuous therefore there exists $\delta(\epsilon) > 0$ such that $|g(t) - g(t_0)| < \epsilon$ whenever $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)) \subset [0,1]$. Now let $x \in \overline{B(x_0,r)}$ then

$$
\begin{aligned}
d(H(x,t),x_0) &= d(H(x,t),H(x_0,t_0)) \\
&\leq d(H(x,t),H(x,t_0)) + d(H(x,t_0),H(x_0,t_0)) \\
&\leq |g(t) - g(t_0)| + ad(x,x_0) + Ld(x,H(x_0,t_0))d(x_0,H(x,t_0)) \\
&= |g(t) - g(t_0)| + [a + Ld(x_0,H(x,t_0))]d(x,x_0) \\
&\leq \epsilon + (1 - \frac{\epsilon}{r})r = r, \qquad (33)
\end{aligned}
$$

whenever $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)) \subset [0,1]$. Therefore for every fixed $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon))$ $H(.,t)$ is a self mapping on $\overline{B(x_0,r)}$. Now since $H(.,t)$ satisfies all the conditions of Theorem 6, we have $H(.,t)$ has a fixed point in $\overline{B(x_0,r)} \subset V$, but it must be in U as condition (a) holds. Therefore $t \in G$ for every $t \in (t_0 - \delta(\epsilon), t_0 + \delta(\epsilon))$. Hence $(t_0 - \delta(\epsilon), t_0 + \delta(\epsilon)) \subset G$. So G is open in $[0,1]$.

Now we will show that G is closed also. Let $\{t_n\} \subset G$ be such that $t_n \to t^* \in [0,1]$ as $n \to \infty$. Then there exists $x_n \in U$ such that $x_n = H(x_n,t_n)$ for all $n \in \mathbb{N}$. Moreover we have,

$$
\begin{aligned}
d(x_n,x_m) &= d(H(x_n,t_n),H(x_m,t_m)) \\
&\leq d(H(x_n,t_n),H(x_n,t_m)) + d(H(x_n,t_m),H(x_m,t_m)) \\
&\leq |g(t_n) - g(t_m)| + ad(x_n,x_m) + Ld(x_n,H(x_m,t_m))d(x_m,H(x_n,t_m)) \\
&= |g(t_n) - g(t_m)| + d(x_n,x_m)[Ld(x_m,H(x_n,t_m)) + a], \qquad (34)
\end{aligned}
$$

which implies that $d(x_n,x_m) \leq [1-a-L\sup_{x,y \in V, t \in [0,1]} d(x,H(y,t))]^{-1}|g(t_n) - g(t_m)|$ for any $n,m \in \mathbb{N}$. So taking $n,m \to \infty$ we see that $\{x_n\}$ is Cauchy in X. Since X is complete then there exists $x^* \in V$ such that $x_n \to x^*$ as $n \to \infty$. We just show that $H(x^*,t^*) = x^*$. Now,

$$
\begin{aligned}
d(x_n,H(x^*,t^*)) &= d(H(x_n,t_n),H(x^*,t^*)) \\
&\leq d(H(x_n,t_n),H(x_n,t^*)) + d(H(x_n,t^*),H(x^*,t^*)) \\
&\leq |g(t_n) - g(t^*)| + ad(x_n,x^*) + Ld(x_n,H(x^*,t^*))d(x^*,H(x_n,t^*)), \qquad (35)
\end{aligned}
$$

implying that $d(x_n, H(x^*, t^*)) \leq [1 - a - L \sup_{x,y \in V} d(x, H(y, t^*))]^{-1} |g(t_n) - g(t^*)|$ for all $n \geq 1$. Taking $n \to \infty$ we get $x_n \to H(x^*, t^*)$ and thus $H(x^*, t^*) = x^*$. Hence $x^* \in U$ and we have $t^* \in G$. Thus G is closed and hence $G = [0, 1]$. So from the construction of G we can say that $H(., 1)$ has also a fixed point in X.

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