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ON MEROMORPHICALLY $\lambda-$ STARLIKE FUNCTIONS WITH RESPECT TO $K-$ SYMMETRIC POINTS OF COMPLEX ORDER

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Abstract. In this present investigation, we introduce the subclass $\mathcal{N}^{k}_{\gamma,\lambda}(\phi)$, of meromorphically starlike functions with respect to $k-$symmetric points of complex order $\gamma(\gamma \neq 0)$ Punctured open unit disk Δ^{*}. Some interesting subordination criteria, inclusion relations and the integral representation for functions belonging to this class are provided. The results obtained generalize some known results, and some other new results are obtained.

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1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, $\mathbb{C}$ being, as, usual, the set of complex numbers.

A function $f \in \mathcal{A}$ subordinate to an univalent function $g \in \mathcal{A}$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$. Let Σ be the family of analytic functions $w(z)$ in the unit disc Δ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$, for z in Δ. Note that $f(z) \prec g(z)$ if there is function $w(z)$ in $\wp$ such that $f(z) = g(w(z))$. Further, let $\wp$ be the class of analytic functions $\phi(z)$ which are regular in Δ and satisfy the conditions $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(\Delta)$ is symmetric with respect to the the real axis, such a function has a Taylor series of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \,(B_1 > 0)\,.$$

Let $\mathcal{N}$ denote the class of all meromorphic functions in the punctured open unit disk $\Delta^{*} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ of the form

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \qquad (a_k \geq 0, \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\}). \tag{1}$$

For a given positive integer k, let $\varepsilon = \exp(2\pi i/k)$ and

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k=1} \varepsilon^{\nu} f(\varepsilon^{\nu} z), \ (z \in \Delta^*). \tag{2}$$

We denote by $\mathcal{S}^k(\phi)$, $\mathcal{C}^k(\phi)$ and $\mathcal{N}^k_{\gamma,\lambda}(\phi)$ the familiar subclasses of $\mathcal{N}$ consisting of meromorphically starlike, convex and $\lambda-$starlike functions with respect to $k-$symmetric points in Δ^*. That is

$$\mathcal{S}^k(\phi) = \left\{ f \in \mathcal{N} : -\frac{zf'(z)}{f_k(z)} \prec \phi(z) \ \right\},$$

$$\mathcal{C}^k(\phi) = \left\{ f \in \mathcal{N} : -\frac{(zf'(z))'}{f_k(z)} \prec \phi(z) \right\},$$

and

$$\mathcal{N}^k_{\gamma,\lambda}(\phi) = \left\{ f \in \mathcal{N} : -\frac{\lambda z(zf'(z))' + (1+\lambda)zf'(z)}{\lambda z f_k'(z) + (1+\lambda) f_k(z)} \prec \phi(z) \right\}, \tag{3}$$

where $\phi(z) \in \wp$ and $z \in \Delta^*$.

Let $f(z) \in \mathcal{N}$ be given by (1) the class $\mathcal{S}_\gamma(\phi)$ is defined by

$$1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} + 1\right) \prec \phi(z), \ (z \in \Delta^*, \ \gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\} \text{ and } \phi(z) \in \wp).$$

Furthermore, a function $f(z) \in \mathcal{N}$, the class $\mathcal{C}_\gamma(\phi)$ is defined by

$$1 - \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)} + 2\right) \prec \phi(z), \ (z \in \Delta^*, \ \gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\} \text{ and } \phi(z) \in \wp).)$$

Motivated by the classes $\mathcal{N}^k_\lambda(\phi)$, $\mathcal{S}_\gamma(\phi)$ and $\mathcal{C}_\gamma(\phi)$, we now introduce and investigate the following subclasses of $\mathcal{N}$, and obtain some interesting results.

Moreover, for some non-zero complex number γ, we consider the subclasses $\mathcal{N}^k_{\gamma,\lambda}(\phi)$, $\mathcal{M}^k_{\gamma,\lambda}(\phi)$ of $\mathcal{N}$ as follows:

Definition 1. *A function $f \in \mathcal{N}$ is belongs to the class $\mathcal{N}^k_{\gamma,\lambda}(\phi)$ if satisfies the following subordination condition*

$$1 - \frac{1}{\gamma}\left(\frac{\lambda z(zf'(z))' + (1+\lambda)zf'(z)}{\lambda z f_k'(z) + (1+\lambda) f_k(z)} + 1\right) \prec \phi(z), \tag{4}$$

where $\phi(z) \in \wp$and $\lambda \geq 0$.

Definition 2. *A function $f \in \mathcal{N}$ is belongs to the class $\mathcal{M}^k_{\gamma,\lambda}(\phi)$ if satisfies the following subordination condition*

$$1 - \frac{1}{\gamma}\left(\frac{\lambda z(zf'(z))' + (1+\lambda)zf'(z)}{\lambda z\zeta_k'(z) + (1+\lambda)\zeta_k(z)} + 1\right) \prec \phi(z), \tag{5}$$

where $\zeta_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^\nu \zeta(\varepsilon^\nu z)$, $\zeta(z) \in \mathcal{N}^k_{\gamma,\lambda}(\phi)$, $\phi(z) \in \wp$ *and* $\lambda \geq 0$.

By giving specific values to the parameters k, γ and λ in the class $\mathcal{N}^k_{\gamma,\lambda}(\phi)$, we get the following new subclasses of meromorphically functions.

Remark 1. *Putting $k = 1$, we obtain the following definition*

Definition 3. *A function $f \in \mathcal{N}$ is belongs to the class $\mathcal{N}_{\gamma,\lambda}(\phi)$ if satisfies the following subordination condition*

$$1 - \frac{1}{\gamma}\left(\frac{\lambda z(zf'(z))' + (1+\lambda)zf'(z)}{\lambda zf'(z) + (1+\lambda)f(z)} + 1\right) \prec \phi(z), \tag{6}$$

where $\phi(z) \in \wp$*and* $\lambda \geq 0$.

Remark 2. *If we set $\lambda = 0$, we have following definition*

Definition 4. *A function $f \in \mathcal{N}$ is belongs to the class $\mathcal{S}^k_\gamma(\phi)$ if satisfies the following subordination condition*

$$1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{f_k(z)} + 1\right) \prec \phi(z),$$

where $\phi(z) \in \wp$.

Remark 3.

$(i)\mathcal{N}_{\gamma,0}(\frac{1+Az}{1+Bz}) = \mathcal{S}_\gamma$ with $\gamma \in \mathbb{C}^*$ and $1 \leq B < A \leq -1$, (see Bulboca et al [2]);

$(ii)\mathcal{N}_{\gamma,-1}(\frac{1+Az}{1+Bz}) = \mathcal{C}_\gamma$ with $\gamma \in \mathbb{C}^*$ and $1 \leq B < A \leq -1$, (see Bulboca et al [2]);
$(iii)\mathcal{N}_{\gamma,0}(\frac{1+z}{1-z}) = \mathcal{S}_\gamma$ with $\gamma \in \mathbb{C}^*$,(see Aouf [1]);
$(iv)\mathcal{N}_{\gamma,-1}(\frac{1+z}{1-z}) = \mathcal{C}_\gamma$ with $\gamma \in \mathbb{C}^*$,(see Aouf [1]);
$(v)\mathcal{N}_{1,0}(\frac{1+(1-2\alpha)\beta z}{1-\beta z}) = \mathcal{S}(\alpha,\beta)$ with $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, (see El-Ashwah and Aouf [3]);
$(vii)\mathcal{N}_{1,-1}(\frac{1+(1-2\alpha)\beta z}{1-\beta z}) = \mathcal{C}(\alpha,\beta)$ with $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, (see El-Ashwah and Aouf [3]);
$(viii)\mathcal{N}_{(1-\alpha)e^{i\mu}\cos\mu,0}(\frac{1+z}{1-z}) = \mathcal{S}^\mu(\alpha)$ with $\mu \in \mathbb{R}$, $|\mu| \leq \pi/2$ and $0 \leq \alpha < 1$ (see [7] for $p = 1$);

$(viiii)\mathcal{N}_{(1-\alpha)e^{i\mu}\cos\mu,-1}(\frac{1+z}{1-z}) = \mathcal{C}^{\mu}(\alpha)$ with $\mu \in \mathbb{R}$, $|\mu| \leq \pi/2$ and $0 \leq \alpha < 1$ (see [7] for $p = 1$);

Considering $\mu \in \mathbb{R}$, $|\mu| \leq \pi/2$ and $0 \leq \alpha < 1$, for the special cases $\gamma = e^{i\gamma}\cos\mu$, $\phi(z) = \frac{1+(1-2\alpha)\beta z}{1-\beta z}$ we will get the notations

$$\begin{aligned} \mathcal{S}^{\mu}[\alpha,\beta] &= \mathcal{N}_{e^{i\mu}\cos\mu,0}(\frac{1+(1-2\alpha)\beta z}{1-\beta z}), \\ \mathcal{S}^{\mu}[\alpha,\beta] &= \mathcal{N}_{e^{i\mu}\cos\mu,-1}(\frac{1+(1-2\alpha)\beta z}{1-\beta z}). \end{aligned}$$

Lemma 1. *[6] Let $\alpha, \beta \in \mathbb{C}$. Suppose that $h(z)$ is convex and univalent in Δ with*

$$h(0) = 1 \text{ and } \Re[\alpha h(z) + \beta] > 0 \ (z \in \Delta), \tag{7}$$

and let $q(z)$ is analytic in Δ with $q(0) = 1$ and $q(z) \prec h(z)$.

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \wp$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\alpha q(z) + \beta} \prec h(z)$$

implies that $p(z) \prec h(z)$.

Lemma 2. *(see [4],[5]) Let $\alpha, \beta \in \mathbb{C}$. Suppose that $h(z)$ is convex and univalent in satisfies (7). If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \wp$ and satisfies the subordination*

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \beta} \prec h(z)$$

implies that $p(z) \prec h(z)$.

2. Main Result

Unless otherwise mentioned, we assume throughout this article that $f \in \mathcal{N}$, $\lambda > 0$, $\phi \in \wp$ and $\gamma \in \mathbb{C}^*$

Proposition 1. *Let $\Re\frac{1}{\lambda}[1-\lambda\gamma(\phi(z)-1)] > 0$ then $f \in \mathcal{S}_{\gamma}(\phi)$, whenever $f \in \mathcal{N}_{\gamma,\lambda}(\phi)$.*

Proof. Set

$$p(z) = 1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} + 1\right), \tag{8}$$

then p is analytic function with $p(0) = 1$. By simple computation, we have

$$\begin{aligned} & 1 - \frac{1}{\gamma}\left(\frac{\lambda z(zf'(z))' + (1+\lambda)zf'(z)}{\lambda zf'(z) + (1+\lambda)f(z)} + 1\right) \\ = \quad & p(z) + \frac{\lambda zp'(z)}{1 - \lambda\gamma(p(z)-1)} \prec \phi(z), \ (z \in \Delta). \end{aligned}$$

The result of proposition 1 yields from Lemma 2, by taking $\alpha = -\gamma$ and $\beta = \frac{\lambda\gamma+1}{\lambda}$.

Proposition 2. *Let* $\Re\frac{1}{\lambda}[1 - \lambda\gamma(\phi(z) - 1)] > 0,$ *then*

$$F(z) = I_\lambda = \frac{1}{\lambda z^{\frac{1}{\lambda}+1}} \int_0^z t^{\frac{1}{\lambda}} f(t)dt \in \mathcal{S}_\gamma(\phi) \tag{9}$$

whenever $f(z) \in \mathcal{S}_\gamma(\phi)$.

Proof. Let

$$p(z) = 1 - \frac{1}{\gamma}\left(\frac{zF'(z)}{F(z)} + 1\right),$$

it is easy to obtain that

$$p(z) + \frac{\lambda zp'(z)}{1 - \lambda\gamma(p(z)-1)} = 1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} + 1\right) \prec \phi(z).$$

Since $f \in \mathcal{S}_\gamma(\phi)$. From Lemma 2, we get $p(z) = 1 - \frac{1}{\gamma}\left(\frac{zF'(z)}{F(z)} + 1\right) \prec \phi(z)$.

Theorem 3. *Let* $\Re\frac{1}{\lambda}[1 - \lambda\gamma(\phi(z) - 1)] > 0$. *Then* $f_k \in \mathcal{N}_{\gamma,\lambda}(\phi)$ *and* $f_k \in \mathcal{S}_\gamma(\phi)$, *whenever* $f \in \mathcal{N}^k_{\gamma,\lambda}(\phi)$.

Proof. Let $f \in \mathcal{N}^k_{\gamma,\lambda}(\phi)$. Replacing z by $\varepsilon^\mu z$ ($\mu = 0, 1, \dots, k-1$; $\varepsilon^k = 1$) in (4), then

$$1 - \frac{1}{\gamma}\left(\frac{(1+\lambda)z\varepsilon^\mu f'(\varepsilon^\mu z) + \lambda z\varepsilon^\mu[f'(\varepsilon^\mu z) + \varepsilon^\mu zf''(\varepsilon^\mu z)]}{(1+\lambda)f_k(\varepsilon^\mu z) + \lambda z\varepsilon^\mu f_k'(\varepsilon^\mu z)} + 1\right) \prec \phi(z). \tag{10}$$

According to the definition of f_k and $\varepsilon^k = 1$, we know that

$$f_k(\varepsilon^\mu z) = \varepsilon^{-\mu} f_k(z) \text{ and } f_k'(\varepsilon^\mu z) = \varepsilon^{-2\mu} f_k'(z) = \frac{1}{k}\sum_{\mu=0}^{k-1} \varepsilon^{2\mu} f_k'(\varepsilon^\mu z) \tag{11}$$

For $\mu = 0, 1, \dots, k-1$,and summing up, we can get

$$\begin{aligned} & \frac{1}{k}\sum_{\mu=0}^{k-1}\left[1 - \frac{1}{\gamma}\left(\frac{(1+\lambda)\varepsilon^{2\mu} zf'(\varepsilon^\mu z) + \lambda z\varepsilon^{2\mu}(zf'(\varepsilon^\mu z))'}{\lambda zf_k'(z) + (1+\lambda)f_k(z)} + 1\right)\right] \\ = \quad & 1 - \frac{1}{\gamma}\left(\frac{(1+\lambda)zf_k'(z) + \lambda z(zf_k'(z))'}{\lambda zf_k'(z) + (1+\lambda)f_k(z)} + 1\right). \end{aligned}$$

Hence there exist ζ_μ in Δ^* such that

$$1-\frac{1}{\gamma}\left(\frac{(1+\lambda)zf_k'(z)+\lambda z(zf_k'(z))'}{(1+\lambda)f_k(z)+\lambda zf_k'(z)}+1\right)\prec\frac{1}{k}\sum_{\mu=0}^{k-1}\phi(\zeta_\mu)=\phi(\zeta_\mu),$$

for $\zeta_\mu\in\Delta^*$, since $\phi(\Delta^*)$ is convex. That $f_k\in\mathcal{N}_{\gamma,\lambda}(\phi)$. Since $\Re\frac{1}{\lambda}[1-\lambda\gamma(\phi(z)-1)]>0$. Thus by using proposition 2 we obtain $f_k\in\mathcal{S}_\gamma(\phi)$.

Theorem 4. *Let* $\Re\frac{1}{\lambda}[1-\lambda\gamma(\phi(z)-1)]>0$. *Then* $f\in\mathcal{S}^k_\gamma(\phi)$, *whenever* $f\in\mathcal{N}^k_{\gamma,\lambda}(\phi)$.

Proof. Let $f\in\mathcal{N}^k_{\gamma,\lambda}(\phi)$. Then by Definition 1, we have

$$1-\frac{1}{\gamma}\left(\frac{(1+\lambda)zf'(z)+\lambda z(zf'(z))'}{(1+\lambda)f_k(z)+\lambda zf_k'(z)}+1\right)\prec\phi(z).$$

Putting $p(z)=1-\frac{1}{\gamma}\left(\frac{zf'(z)}{f_k(z)}+1\right)$ and $q(z)=1-\frac{1}{\gamma}\left(\frac{zf_k'(z)}{f_k(z)}+1\right)$, it is easy to obtain that

$$\begin{aligned}&1-\frac{1}{\gamma}\left(\frac{(1+\lambda)zf'(z)+\lambda z(zf'(z))'}{(1+\lambda)f_k(z)+\lambda zf_k'(z)}+1\right)\\=\;&p(z)+\frac{\lambda zp'(z)}{1-\lambda\gamma(q(z)-1)}\prec\phi(z),\ (z\in\Delta).\end{aligned}$$

Since $f\in\mathcal{N}^k_{\gamma,\lambda}(\phi)$, then by using Theorem 3, we can see that $q(z)\prec\phi(z)$. Now an application of lemma 1, yield

$$p(z)=1-\frac{1}{\gamma}\left(\frac{zf'(z)}{f_k(z)}+1\right)\prec\phi(z).$$

That is $f\in\mathcal{S}^k_\gamma(\phi)$. We thus complete the proof of Theorem 4.

Theorem 5. *Let* $\Re\frac{1}{\lambda}[1-\lambda\gamma(\phi(z)-1)]>0$ *in* Δ, $f\in\mathcal{S}^k_\gamma(\phi)$ *and let* F *be the integral operator defined by* (9), *then* $F\in\mathcal{S}^k_\gamma(\phi)$.

Proof. Let a function $f_k(z)$ of the form (2) with $F(z)$ in the place of $f(z)$.That is $F_k(z)=\frac{1}{k}\sum_{\mu=0}^{k-1}\varepsilon^\mu F(\varepsilon^\mu z)$. We can see that $F_k(z)=\frac{1}{\lambda z^{\frac{1}{\lambda}+1}}\int_0^z(t^{\frac{1}{\lambda}}f_k(t)dt)$ and then differentiating with respect to z, we get

$$(1+\lambda)F_k(z)+\lambda zF_k'(z)=f_k(z). \tag{12}$$

From (9), we have

$$(1+\lambda)F(z)+\lambda zF'(z)=f(z). \tag{13}$$

Since $f \in \mathcal{S}_{\gamma}^{k}(\phi)$, we can apply Theorem 3 with $\lambda = 0$ to deduce $f_k \in \mathcal{S}_{\gamma}(\phi)$. Now an application of proposition 2, we have $F_k \in \mathcal{S}_{\gamma}(\phi)$, that is

$$1 - \frac{1}{\gamma}\left(\frac{zF_k'(z)}{F_k(z)} + 1\right) \prec \phi(z). \tag{14}$$

If we denote

$$p(z) = 1 - \frac{1}{\gamma}\left(\frac{zF'(z)}{F_k(z)} + 1\right), \tag{15}$$

and

$$q(z) = 1 - \frac{1}{\gamma}\left(\frac{zF_k'(z)}{F_k(z)} + 1\right), \tag{16}$$

then $p(z)$ is analytic in Δ with $p(0) = 1$ and $q(z)$ is analytic in Δ with $q(0) = 1$, $q(z) \prec \phi(z)$. Differentiating in (13) and using (15), we have

$$\gamma p(z) - \gamma - 1 + \frac{\lambda z p'(z)}{\left(1 + \lambda + \lambda \frac{zF_k'(z)}{F_k(z)}\right)} = \frac{zf'(z)}{(1+\lambda)F_k(z) + \lambda z F_k'(z)}. \tag{17}$$

Using (12) and (16), (17) gives

$$p(z) + \frac{\lambda z p'(z)}{1 - \lambda\gamma(q(z) - 1)} = 1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{f_k(z)} + 1\right) \prec \phi(z).$$

Now an application of Lemma 1, we get $p(z) \prec \phi(z)$, which proves the theorem.

Theorem 6. *Let* $\Re\frac{1}{\lambda}[1 - \lambda\gamma(\phi(z) - 1)] > 0$. *Then* $\mathcal{M}_{\gamma,\lambda}^{k}(\phi) \subset \mathcal{M}_{\gamma,0}^{k}(\phi)$.

Proof. Let $f \in \mathcal{M}_{\gamma,\lambda}^{k}(\phi)$. Setting

$$p(z) = 1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{\varsigma_k(z)} + 1\right), q(z) = 1 - \frac{1}{\gamma}\left(\frac{zf_k'(z)}{\varsigma_k(z)} + 1\right),$$

we have

$$\begin{aligned} & 1 - \frac{1}{\gamma}\left(\frac{(1+\lambda)zf'(z) + \lambda z(zf'(z))'}{(1+\lambda)f_k(z) + \lambda z f_k'(z)} + 1\right) \\ = \quad & p(z) + \frac{\lambda z p'(z)}{1 - \lambda\gamma(q(z) - 1)} \prec \phi(z). \end{aligned}$$

Since $\varsigma(z) \in \mathcal{N}_{\gamma,\lambda}^{k}(\phi)$, from Theorem 3 we have $q(z) \prec \phi(z)$. Again an application of Lemma 1 yields $p(z) = 1 - \frac{1}{\gamma}\left(\frac{zf'(z)}{\varsigma_k(z)} + 1\right) \prec \phi(z)$ which establishes the theorem.

Theorem 7. *Let $f \in \mathcal{N}^k_{\gamma,\lambda}(\phi)$, then we have*

$$f_k(z) = \frac{1}{\lambda(z^{\frac{1}{\lambda}+1})} \int_0^z \exp\left\{ \frac{-\gamma}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu t} \frac{\phi(w(\zeta)) - 1}{\zeta} d\zeta \right\} t^{\frac{1}{\lambda}-1} dt, \tag{18}$$

where $f_k(z)$ is given by equality (2) and $w(z) \in \wp$.

Proof. Suppose that $f \in \mathcal{N}^k_{\gamma,\lambda}(\phi)$. We know that the condition (5) can be written as follows:

$$1 - \frac{1}{\gamma}\left(\frac{(1+\lambda)zf'(z) + \lambda z(zf'(z))'}{(1+\lambda)f_k(z) + \lambda z f_k'(z)} + 1 \right) = \phi(w(z)). \tag{19}$$

By similarly applying the arguments given in the proof for Theorem 3 to (19), we obtain:

$$1 - \frac{1}{\gamma}\left(\frac{(1+\lambda)zf_k'(z) + \lambda z(zf_k'(z))'}{(1+\lambda)f_k(z) + \lambda z f_k'(z)} + 1 \right) = \frac{1}{k} \sum_{\mu=0}^{k-1} \phi(w(\varepsilon^\mu z)). \tag{20}$$

From (20), we have

$$\left(\frac{(1+\lambda)f_k'(z) + \lambda(zf_k'(z))'}{(1+\lambda)f_k(z) + \lambda z f_k'(z)} + \frac{1}{z} \right) = \frac{-\gamma}{k} \sum_{\mu=0}^{k-1} \frac{\phi(w(\varepsilon^\mu z)) - 1}{z}. \tag{21}$$

Integrating this equality, we get

$$\log z \left[(1+\lambda) f_k(z) + \lambda z f_k'(z) \right] = \frac{-\gamma}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{\phi(w(\varepsilon^\mu \rho)) - 1}{\rho} d\rho, \tag{22}$$

$$(1+\lambda) f_k(z) + \lambda z f_k'(z) = \frac{1}{z} \exp \frac{-\gamma}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{\phi(w(\zeta)) - 1}{\zeta} d\zeta. \tag{23}$$

From (23), we can get equality (18) easily. This completes the proof of Theorem 7.

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