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FUZZY-SET CATEGORIES WITH DYNAMICAL MEMBERSHIP DEGREES

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Abstract. In general, the scientific literature handles those categories of crisp and fuzzy sets that have static structures. We investigate a generalization of them such that the membership degrees are time dependant; therefore they have dynamical structures. We show that, to this aim, an adequate algebraic formalism can consist of double Kleisli constructions. We also show how to build equivalent categories from a pair of monads with related units. Finally, we investigate the existence of crisp abstractions of fuzzy-set categories.

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1. Introduction

To interpret many Real-World phenomena, we need the help of time dependant algebraic structures. In particular, set families such that some membership degrees change through time. For instance, the set of *healthy* people who live in a given town, in general, has a fuzzy subset. Body defenses, the doctor's actions, and diseases make that some membership degrees change.

In section 3, we introduce crisp dynamical sets. These are collections such that the membership degrees can only take values in $\{0, 1\}$. A crisp dynamical set A consists of pairs (x, I), where x denotes a member of A, and $I \subseteq \mathbb{R}$ is the range of time values at each of them the relation $x \in A$ is true. Accordingly, the membership degree of x at some time t is 1, whenever $t \in I$; otherwise the relation $x \in A$ is false.

In Theorem 1, we show sufficient conditions to build categories of crisp dynamical sets. Some of them are Kleisli constructions, and so are the fuzzy-set ones [2]. Since the second coordinate I of a dynamical-set member (x, I) is a subset of $\mathbb{R}$, we define the morphism composition through a natural transformation based on set intersections instead of T-norms [3]. We also show that, under this time-dependant algebraic structures, we cannot state the well known Russell's paradox: Real-World is a contradiction-free universe [7].

To build dynamical structures, we need an external parameter denoting the physical time. To this end, the method exposed in [5], consists of comonads and dual constructions of Kleisli categories. We extend this method in Section 4. Theorem 3 shows that we can build dynamical fuzzy-set categories as double Kleisli constructions [1]. In Theorem 6, we also show that an equivalent algebraic structure arises from a pair of monads, and morphism composition becomes simpler than in the former one.

A dynamical fuzzy-set A, is a collection of pairs $(x, \phi(x,t))$, where x is a member of A, and the value of $\phi(x,t) \in [0,1]$ is the membership degree of x at a time-value $t \in \mathbb{R}$. When the map $\phi(x,t)$ is not time dependant, A is equivalent to a static fuzzy set. Thus, dynamical fuzzy-set categories are extensions of static ones. The nature of the map $\phi(x,t)$ depends on the application field. They can be constrained by Real-World laws, like those consisting of differential equations. The constrain research is beyond the scope of this article. We only aim to state the foundational definitions and results.

Finally, in Section 4.1, we investigate whether there are crisp images of fuzzy set categories. In Theorem 7 we show that we can build them through abstractions. For instance, consider the set A of healthy people who were born in a town T. We can also write this definition as the conjunction of two predicates $p_{A,1}(x) \wedge p_{A,2}(x)$; where $p_{A,1}(x)$ denotes the predicate *"x was born in T"* , and $p_{A,2}(x)$ means *"x is a healthy human being"*. The set $A_1 = \{x \mid p_{A,1}(x)\}$ is an abstraction of A because we define it disregarding the property denoted by the predicate $p_{A,2}(x)$. Nevertheless, the property denoted by $p_{A,1}(x)$ either is true or false; hence A_1 is a crisp set.

We define a functor $\mathcal{H}$ sending A into the crisp-set A_1, under the condition of being every member of A a discernible object [6]. In other words, for every $a \in A$, there is a finitely definable predicate $p_a(x)$, such that the conjunction $p_A(x) \wedge p_a(x)$ specifies a, that is to say, $\{a\} = \{x \mid p_A(x) \wedge p_a(x)\}$. Thus, there is a finite symbol sequence $S = w_1 w_2 \dots w_n$, in some language $\mathcal{L}$, that denotes $p_A(x) \wedge p_a(x)$. This requirement leads to the countability of A because any Gödel-like numbering function sends S into a positive integer [6].

If A satisfies this condition, the image of it under $\mathcal{H}$ is

$$\mathcal{H}(A) = \Big\{ \{x \mid p_{A,1}(x) \wedge p_a(x)\} \Big| a \in A \Big\}.$$

If a map $f : A \to B$ sends the object specified by $p_A(x) \wedge p_a(x)$ into the one specified by $p_B(x) \wedge p_{f(a)}(x)$, then $\mathcal{H}f$ sends the object that $p_{A,1}(x) \wedge p_a(x)$ defines into the one specified by $p_{B,1}(x) \wedge p_{f(a)}(x)$. In these expressions, we are assuming that the predicate $p_B(x)$ that specifies B satisfies the relation $p_B(x) = p_{B,1}(x) \wedge p_{B,2}(x)$, where the predicate $p_{B,1}(x)$ defines a crisp set. Likewise, we assume that, for each member b of B, there is a predicate $p_b(x)$ such that $p_B(x) \wedge p_b(x)$ specifies b.

When $\{x \mid p_{A,1}(x) \wedge p_a(x)\}$ is a set of cardinality greater than 1, the domain and codomain of $\mathcal{H}f$ are collections of sets. We can avoid this complexity, substituting these sets by the abstractions consisting of generic symbols. As in [5], we denote them by the superscript "$^\curlyvee$". Thus, the expression $\{2 \cdot n \mid n \in \mathbb{N}\}^\curlyvee$ denotes the positive even integer concept. With this formalism, we can define $\mathcal{H}f$ as the map that sends $\{x \mid p_{A,1}(x) \wedge p_a(x)\}^\curlyvee$ into $\{x \mid p_{B,1}(x) \wedge p_{f(a)}(x)\}^\curlyvee$. Accordingly, we must define the object-map of $\mathcal{H}$ as follows.

$$\mathcal{H}(A) = \left\{ \big\{x \mid p_{A,1}(x) \wedge p_a(x)\big\}^\curlyvee \big| a \in A \right\}.$$

Thus, the functor $\mathcal{H}$ sends each fuzzy set into a crisp one.

2. Preliminaries

Let $\mathcal{T}$ be the standard topology for $\mathbb{R}$. From now on, we consider every closed subset of $\mathbb{R}$, as a physical-time range.

Definition 1. *We term t-class each nonempty subset $\mathcal{C}$ of $\mathcal{T}$-closed subsets of $\mathbb{R}$, provided that it is stable under arbitrary intersections, and contains a $\subseteq$-maximum element $\breve{C}$.*

Given a t-class $\mathcal{C}$, we denote by $\mathcal{F}_\mathcal{C} : \mathbf{Set} \to \mathbf{Set}$ the endofunctor defined as follows. The object-map sends each set X into $X \times \mathcal{C}$ and, for every couple of sets X and Y, the arrow-map sends each $f \in \hom_{\mathbf{Set}}(X, Y)$ into

$$f \times \mathrm{id}_X : X \times \mathcal{C} \to Y \times \mathcal{C}.$$

Now, let $\omega : \mathrm{Id} \to \mathcal{F}_\mathcal{C}$ and $\tau : \mathcal{F}_\mathcal{C}^2 \to \mathcal{F}_\mathcal{C}$ be the natural transformations such that, for every object X, the map $\omega_X : X \to X \times \mathcal{F}_\mathcal{C}(X)$ sends each $x \in X$ into $(x, \breve{C})$. Likewise, $\tau_X : \mathcal{F}_\mathcal{C}^2(X) \to \mathcal{F}_\mathcal{C}(X)$ sends each $(x, A, B) \in X \times \mathcal{C} \times \mathcal{C}$ into $(x, A \cap B)$. It is a straightforward consequence of these definitions that the following diagrams commute.

$$\begin{array}{ccccc} \mathcal{F}_\mathcal{C} & \xrightarrow{\mathcal{F}_\mathcal{C}\omega} & \mathcal{F}_\mathcal{C}^2 & \xleftarrow{\omega\mathcal{F}_\mathcal{C}} & \mathcal{F}_\mathcal{C} \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow {\scriptstyle \tau} & \swarrow {\scriptstyle \mathrm{id}} & \\ & & \mathcal{F}_\mathcal{C} & & \end{array} \qquad \begin{array}{ccc} \mathcal{F}_\mathcal{C}^3 & \xrightarrow{\mathcal{F}_\mathcal{C}\tau} & \mathcal{F}_\mathcal{C}^2 \\ {\scriptstyle \tau\mathcal{F}_\mathcal{C}} \downarrow & & \downarrow {\scriptstyle \tau} \\ \mathcal{F}_\mathcal{C}^2 & \xrightarrow[\tau]{} & \mathcal{F}_\mathcal{C} \end{array} \qquad (1)$$

Accordingly, $\mathfrak{M}(\mathcal{C}) = (\mathcal{F}_\mathcal{C}, \omega, \tau)$ is a monad. Let $\mathbf{Set}_{\mathfrak{M}(\mathcal{C})}$ denote the associated Kleisli category [4]. Thus, for each couple of objects X and Y, $\mathbf{Set}_{\mathfrak{M}(\mathcal{C})}$-morphisms from X into Y are all maps in $\hom_{\mathbf{Set}}(X, \mathcal{F}_{\mathcal{C}}(Y))$. The composition of two morphisms $f \in \hom_{\mathbf{Set}_{\mathfrak{M}(\mathcal{C})}}(X, Y)$ and $g \in \hom_{\mathbf{Set}_{\mathfrak{M}(\mathcal{C})}}(Y, Z)$ is

$$g * f = \tau \circ \mathcal{F}_{\mathcal{C}} g \circ f \tag{2}$$

Since the morphisms in $\hom_{\mathbf{Set}_{\mathfrak{M}}}(X, Y)$ are maps from X into $\mathcal{F}_{\mathcal{C}}(Y)$, each of them consists of a couple of ordinary ones (f, ϕ) with the same domain X. The codomain of f is Y, while ϕ sends each $x \in X$ into a closed set in $\mathcal{C}$. For each morphism (g, γ) in $\hom_{\mathbf{Set}_{\mathfrak{M}(\mathcal{C})}}(Y, Z)$, we can write the composition $(g, \gamma) * (f, \phi)$ explicitly, as follows.

$$\forall x \in X : \quad ((g, \gamma) * (f, \phi))(x) = (g(f(x)), \gamma(f(x)) \cap \phi(x)). \tag{3}$$

3. Dynamical Crisp-Set Categories

Roughly speaking, we term *dynamical crisp-set* any collection X the members of which can be dropped and added without modifying its definition. In our formalism, every closed subset of $\mathbb{R}$ is a physical time range.

Theorem 1. *Let $\mathfrak{C}$ be a family of t-classes. For each* $\mathrm{Ob}(\mathbf{Set})$*-object X, let Σ_X be a map from X into $\mathfrak{C}$, and let $\|\Sigma_X\|$ denote the class collection $\cup_{x \in X} \Sigma_X(x)$. If, for every couple of nonempty sets X and Y, $\mathbf{M}_\Sigma(X, Y)$ is a subset of*

$$\hom_{\mathbf{Set}}\left(X, Y \times \|\Sigma_X\|\right)$$

such that each member (f, ϕ) satisfies the relations

$$\forall x \in X : \quad \begin{cases} \phi(x) \in \Sigma_X(x) \\ \Sigma_Y(f(x)) \subseteq \Sigma_X(x), \end{cases} \tag{4}$$

there is a category $\mathbf{Set}(\Sigma)$*, with the same object-class as* $\mathbf{Set}$*, such that*

a *For every pair of sets (X, Y),*

$$\hom_{\mathbf{Set}(\Sigma)}(X, Y) = \mathbf{M}_\Sigma(X, Y).$$

b *For every set X, the associated identity is* $(\mathrm{id}_X, \delta_X)$*; where* $\mathrm{id}_X : X \to X$ *is the identity map, and for every $x \in X$, $\delta_X(x) \in \|\Sigma_X\|$ is the maximum element of the t-class $\Sigma_X(x)$.*

c *The composition of two morphisms*

$$(f,\phi) \in \hom_{\mathbf{Set}(\Sigma)}(X,Y)\imath \textit{ and } (g,\gamma) \in \hom_{\mathbf{Set}(\Sigma)}(Y,Z)$$

is

$$((g,\gamma) * (f,\phi))(x) = (g(f(x)), \phi(x) \cap \gamma(f(x))). \tag{5}$$

Proof. We show that $\mathbf{Set}(\Sigma)$ is stable under morphism composition. Let $(f,\phi) \in \hom_{\mathbf{Set}(\Sigma)}(X,Y)$ and $(g,\gamma) \in \hom_{\mathbf{Set}(\Sigma)}(Y,Z)$ be two morphisms and

$$(h,\eta) = (g,\gamma) * (f,\phi)$$

their composition. By Statement (**c**), for every $x \in X$,

$$(h(x), \eta(x)) = (g(f(x)), \phi(x) \cap \gamma(f(x))). \tag{6}$$

According to equation (4),

$$\gamma(f(x)) \in \Sigma_Y(f(x)) \subseteq \Sigma_X(x)$$

Since $\Sigma_X(x)$ is a t-class, it is stable under intersections; hence

$$\eta(x) = \phi(x) \cap \gamma(f(x)) \in \Sigma_X(x). \tag{7}$$

Likewise,

$$\Sigma_Z(g(f(x))) \subseteq \Sigma_Y(f(x)) \subseteq \Sigma_X(x); \tag{8}$$

hence (h,η) belongs to $\mathbf{M}_\Sigma(X,Z)$.

To see the morphism associativity, let (j,ϑ) be a member of $\hom_{\mathbf{Set}(\Sigma)}(Z,V)$. By straightforward computations, for every $x \in X$,

$$\begin{aligned} &((j,\varphi) * ((g,\gamma) * (f,\phi)))(x) = \\ &\qquad (j(g(f(x))), \phi(x) \cap \gamma(f(x)) \cap \varphi(g(f(x)))) = \\ &\qquad\qquad (((j,\varphi) * (g,\gamma)) * (f,\phi))(x) \end{aligned} \tag{9}$$

and the composition law is associative.

Finally, by hypothesis, for every object X, the map δ_X sends every $x \in X$ into the maximum of $\Sigma_X(x)$. Accordingly, for every morphism $(f,\phi) \in \hom_{\mathbf{Set}(\Sigma)}(X,Y)$,

$$((f,\phi) * (\mathrm{id}_X, \delta_X))(x) = (f(x), \phi(x) \cap \delta_X(x)) = (f(x), \phi(x)).$$

Thus, $(\mathrm{id}_X, \delta_X)$ is the identity associated with X.

Corollary 2. *If* $(f, \phi) \in \hom_{\mathbf{Set}(\Sigma)}(X, Y)$ *is a morphism such that, for every* x *in* X, $\phi(x) \neq \emptyset$, *then*

$$\forall x \in X : \quad \inf\{t \mid t \in \delta_X(x)\} \leq \inf\{t \mid t \in \phi(x)\}. \tag{10}$$

Proof. By definition, for each $x \in X$, $\delta_X(x)$ is the maximum of $\Sigma_X(x)$; hence, equation (4) leads to $\forall x \in X : \phi(x) \subseteq \delta_X(x)$. This relation leads to (10).

To be consistent, we only can construct the power set $\wp(X)$ of a set X after the time-value t_0 at which each of its members exists. Thus, if

$$E = \bigcap_{x \in X} \delta_X(x) \neq \emptyset,$$

then we can determine the real number t_0 as follows.

$$t_0 = \inf\left\{ t \mid t \in \bigcap_{x \in X} \delta_X(x) \right\}; \tag{11}$$

Taking into account that $X \in \wp(X)$, the powerset only can exist at a time-value greater than t_0. The following axiom guarantees this requirement.

Axiom 1. If X is a member of $\mathrm{Ob}(\mathbf{Set}(\Sigma))$ that satisfies the condition

$$\bigcap_{x \in X} \delta_X(x) \neq \emptyset,$$

then

$$\forall x \in X : \quad \inf\left\{ t \mid t \in \bigcap_{x \in X} \delta_X(x) \right\} < \inf\{t \mid t \in \delta_{\wp(X)}(X)\}.$$

Definition 2. *Given any* t*-class family* $\mathfrak{C}$, *we denote as* $\mathfrak{C}$-dynamical crisp-set category *any subcategory* $\mathbf{Dyn}(\mathfrak{C})$ *of* $\mathbf{Set}(\Sigma)$, *provided that their objects satisfy Axiom 1.*

A straightforward consequence of Axiom 1 is the impossibility of constructing self-contained dynamical sets. Notice that, if $X \in X$, then, by Axiom 1,

$$\inf\{t \mid t \in \delta_X(X)\} < \inf\{t \mid t \in \delta_{\wp(X)}(X)\} \tag{12}$$

which leads to a contradiction, because both sets, $\delta_X(X)$ and $\delta_{\wp(X)}(X)$, denote the existence range of X; therefore $\delta_X(X) = \delta_{\wp(X)}(X)$; which contradicts (12). As a consequence, in the Real World, where objects occur at a given time, we cannot state Russell's paradox [7].

3.1. Operators

Given a dynamical crisp set A, for every $x \in A$, let $\Lambda_A(x)$ denote the time-value range at each of which x belongs to A. We define unions, intersections, and complements of dynamical sets as follows.

$$\Lambda_{A\cup B}(x) = \Lambda_A(x) \cup \Lambda_B(x) \tag{13}$$

$$\Lambda_{A\cap B}(x) = \Lambda_A(x) \cap \Lambda_B(x) \tag{14}$$

$$\Lambda_{\complement A}(x) = \complement\Lambda_A(x) \tag{15}$$

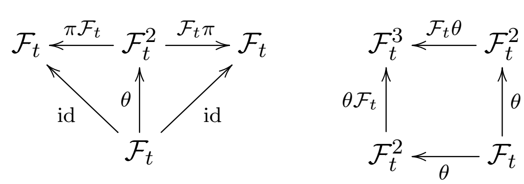
Thus, the operator Λ preserves unions, intersections, and complements.

4. Dynamical Fuzzy-Set Categories

We say a category $\mathbf{C}$ to be a dynamical fuzzy-set one, whenever the membership degrees are time dependant and can take values in the interior of $[0,1]$. Thus, these categories involve an external parameter denoting time. In [5], these parameters are added through comonads. This is why we state their algebraic structure as double Kleisli categories [1]. When a t-class family $\mathfrak{C}$ is a singleton $\{\mathcal{C}\}$, the category $\mathbf{Set}(\Sigma)$ defined in Theorem 1 coincides with the Kleisli construction $\mathbf{Set}_{\mathfrak{M}(\mathcal{C})}$ defined above. Given a time-value $t \in \mathbb{R}$, we denote by $\mathcal{C}_t$ the two-member t-class $\{\{t\}, \emptyset\}$. From now on, we say $\mathcal{C}_t$ to be a t-snapshot class; the corresponding functor $\mathcal{F}_{\mathcal{C}_t}$, a t-snapshot one; and $\left(\mathcal{F}_{\mathcal{C}_t}, \omega, \tau\right)$ a t-snapshot monad. Since there is no confusion, we denote the t-snapshot functor $\mathcal{F}_{\mathcal{C}_t}$, simply, by $\mathcal{F}_t$. For disambiguation, we denote the natural transformations ω and τ with the subscript t. Thus, for every set X, the map $\omega_{t,X} : X \to \mathcal{F}_t(X)$ sends each $x \in X$ into the pair $(x, \{t\}) \in X \times \{\{t\}, \emptyset\}$.

Let $\pi : \mathcal{F}_t \to \mathrm{Id}$ and $\theta : \mathcal{F}_t \to \mathcal{F}_t^2$ be the natural transformations defined as follows. For each set X, the map π_X is the projection that sends each $(x, A) \in X \times \mathcal{C}_t$ into x. Likewise, θ_X sends each (x, A) into (x, A, A).

For every t-snapshot endofunctor $\mathcal{F}_t$, $(\mathcal{F}_t, \pi, \theta)$ is a comonad [5], that we say to be a t-snapshot one. Indeed, the following diagrams commute.



(16)

Let $\mathcal{G} : \mathbf{Set} \to \mathbf{Set}$ be the endofunctor that sends each set X into $X \times [0,1]$ and every map $f : X \to Y$ into $f \times \mathrm{id}$. Let $\mathrm{Id} \xrightarrow{\eta} \mathcal{G}$ be the natural transformation that, for each set X, the map η_X sends each $x \in X$ into $(x, 1)$. Given a T-norm $* : [0,1] \times [0,1] \to [0,1]$ [3], let $\mathcal{G}^2 \xrightarrow{\mu} \mathcal{G}$ denote the natural transformation that, for every set X, the map μ_X sends each (x, α_0, α_1) into $(x, \alpha_0 * \alpha_1)$. With these assumptions, $(\mathcal{G}, \eta, \mu)$ is a monad. The associated Kleisli construction is a category of sets with fuzzy subsets [4].

Theorem 3. *Let $(\mathcal{F}_t, \pi, \theta)$ be any t-snapshot comonad. If $\mathcal{F} \circ \mathcal{G} \xrightarrow{\sigma} \mathcal{G} \circ \mathcal{F}$ is the natural transformation such, that for every set X, the map σ_X sends each $(x, \alpha, A) \in X \times [0,1] \times \mathcal{C}_t$ into $(x, A, \alpha) \in X \times \mathcal{C}_t \times [0,1]$, there is a category $\mathbf{Set}_{\mathcal{F}_\mathbf{t},\mathcal{G}}$ such that*

a *Both categories* **Set** *and* $\mathbf{Set}_{\mathcal{F}_\mathbf{t},\mathcal{G}}$ *have the same object-class.*

b *For every couple of objects X and Y,*

$$\hom_{\mathbf{Set}_{F,G}}(X, Y) = \hom_{\mathbf{Set}}\left(\mathcal{F}_t(X), \mathcal{G}(Y)\right).$$

c *For every object X, the identity is $\eta_X \circ \pi_X : \mathcal{F}_t(X) \to \mathcal{G}(X)$.*

d *The composition of each pair of morphisms $f : X \to Y$ and $g : Y \to Z$ is*

$$g \bullet f = \mu_Z \circ \mathcal{G}g \circ \sigma_Y \circ \mathcal{F}_t f \circ \theta_X. \tag{17}$$

Proof. It is sufficient to show that σ is a distributive law of $\mathcal{G}$ over $\mathcal{F}_t$ [1], that is to say, for each object X, the following statements hold.

1 $\sigma_X \circ \mathcal{F}_t\eta_X = \eta_{\mathcal{F}_t(X)}$.

2 $\mathcal{G}\pi_X \circ \sigma_X = \pi_{\mathcal{G}(X)}$.

3 $\sigma_X \circ \mathcal{F}_t\mu_X = \mu_{\mathcal{F}_t(X)} \circ \mathcal{G}\sigma_X \circ \sigma_{\mathcal{G}(X)}$.

4 $\mathcal{G}\theta_X \circ \sigma_X = \sigma_{\mathcal{F}_t(X)} \circ \mathcal{F}_t\sigma_X \circ \theta_{\mathcal{G}(X)}$.

On the one hand, for every $(x, A) \in \mathcal{F}_t(X)$,

$$\sigma_X \circ \mathcal{F}_t\eta_X(x, A) = \sigma_X(x, 1, A) = (x, A, 1).$$

On the other hand, $\eta_{\mathcal{F}_t(X)}(x, A) = (x, A, 1)$; therefore, statement (**1**) is true.

By straightforward computations,

$$\forall (x, \alpha, A) \in X \times [0,1] \times \mathcal{C}_t : \quad \mathcal{G}\pi_X \circ \sigma_X(x, \alpha, A) = (x, \alpha) = \pi_{\mathcal{G}(X)}(x, \alpha, A)$$

and Statement (**2**) holds.

Analogously,

$$\forall (x,\alpha,\beta,A) \in X \times [0,1] \times [0,1] \times \mathcal{C}_t :$$
$$\sigma_X \circ \mathcal{F}_t \mu_X(x,\alpha,\beta,A) = (x, A, \alpha * \beta) = \mu_{\mathcal{F}_t(X)} \circ \mathcal{G}\sigma_X \circ \sigma_{\mathcal{G}(X)}(x,\alpha,\beta,A);$$

hence Statement (**3**) is also true.

Finally, to show Statement (**4**),

$$\forall (x,\alpha,A) \in X \times [0,1] \times \mathcal{C}_t :$$
$$\mathcal{G}\theta_X \circ \sigma_X(x,\alpha,A) = (x,A,A,\alpha) = \sigma_{\mathcal{F}_t(X)} \circ \mathcal{F}_t\sigma_X \circ \theta_{\mathcal{G}(X)}(x,\alpha,A).$$

Thus, σ is a distributive law of $\mathcal{G}$ over $\mathcal{F}_t$, and there exists the double Kleisli category $\mathbf{Set}_{\mathcal{F}_t,\mathcal{G}}$ that satisfies the statements **a**, **b**, **c**, and **d** [1].

Definition 3. *Given a monad $(\mathcal{G},\eta,\mu)$ and a t-snapshot comonad $(\mathcal{F}_t,\pi,\theta)$, that satisfy the conditions of Theorem 3, we denote as* dynamical fuzzy-set category *each subcategory $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ of $\mathbf{Set}_{\mathcal{F}_t,\mathcal{G}}$, such that, every morphism*

$$\left(\mathcal{F}_t(X) \xrightarrow{(f,\phi)} \mathcal{G}(Y)\right),$$

satisfies the following conditions.

a *The map f factors through the projection $\pi_X : X \times \mathcal{C}_t \to X$.*

b *For every morphism (f,ϕ), there is a map*

$$\breve{\phi} : X \times \mathcal{C}_t \to [0,1]$$

such, that $\forall (x,A) \in X \times \mathcal{C}_t$:

$$\phi(x,A) = \begin{cases} \breve{\phi}(x,t) \text{ if } A = \{t\} \\ 0 \text{ if } A = \emptyset \end{cases} \tag{18}$$

Remark 1. *For each $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$-object X, the unit is $\eta_X \circ \pi_X$. This map only satisfies (18) when $A = \{t\}$. In any case, when for some $x \in X$, $A = \emptyset$, then x cannot be a member of X. To avoid this inconvenient, in the next theorem we state a modified category $\mathbf{C}^*_{\mathcal{F}_t,\mathcal{G}}$, in which every identity satisfies Definition 3.*

Since, for every morphism (f,ϕ), the codomain of ϕ is $[0,1]$, as usual, we can define unions and intersections by T-norms and conorms.

According to the preceding definition, the object-class of $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ consists of all ordinary sets. For every couple of objects X and Y,

$$\hom_{\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}}(X,Y) = \hom_{\mathbf{Set}}\left(\mathcal{F}_t(X),\mathcal{G}(Y)\right).$$

Taking into account (17), the composition of two morphisms $\left(X \xrightarrow{(f,\phi)} Y\right)$ and $\left(Y \xrightarrow{(g,\gamma)} Z\right)$ is

$$\begin{aligned}\forall(x,\{t\})\in\mathcal{F}_t(X):\quad & ((g,\gamma)\bullet(f,\phi))(x,\{t\}) = \\ \Big(\mu_Z\circ\mathcal{G}(g,\gamma)\circ\sigma_Y\circ(f,\phi)\circ\theta_X\Big)(x,\{t\}) &= \left(\breve{g}\left(\breve{f}(x)\right),\breve{\phi}(x,t)*\breve{\gamma}(\breve{f}(x),t)\right). \end{aligned} \tag{19}$$

Likewise,

$$((g,\gamma)\bullet(f,\phi))(x,\emptyset) = \left(\breve{g}\left(\breve{f}(x)\right),0\right); \tag{20}$$

where, according to condition (**a**) in Definition 3,

$$\begin{aligned}\forall(x,\{t\})\in X\times\mathcal{C}_t:\quad & f(x,\{t\}) = \breve{f}\circ\pi_X(x,\{t\})\\ \forall(y,\{t\})\in Y\times\mathcal{C}_t:\quad & g(y,\{t\}) = \breve{g}\circ\pi_Y(y,\{t\})\end{aligned}$$

Lemma 4. *For each fuzzy dynamical-set category $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ and every morphism (f,ϕ) the following statements hold. $\forall t\in\mathbb{R}$:*

a *If $\phi(x,\{t\}) = \breve{\phi}(x,t) = 1$, then*

$$\forall\delta>0:\quad \breve{\phi}(x,t+\delta)-\breve{\phi}(x,t)\le 0$$

b *If $\phi(x,\{t\}) = \breve{\phi}(x,t) = 0$, then*

$$\forall\delta>0:\quad \breve{\phi}(x,t+\delta)-\breve{\phi}(x,t)\ge 0$$

Proof. For every fixed $x\in X$, the image of $\breve{\phi}(x,t)$ is the unit interval $[0,1]$.

Corollary 5. *With the same conditions as in the preceding lemma, if $\breve{\phi}(x,t)$ is differentiable and satisfies some differential equation*

$$F\left(\frac{\partial^n\breve{\phi}(x,t)}{\partial t^n},\frac{\partial^{n-1}\breve{\phi}(x,t)}{\partial t^{n-1}}\ldots\breve{\phi}(x,t)\right)=0$$

the following relations are true. If there are two positive real numbers t_0 *and* t_1 *such that* $\breve{\phi}(x,t_0)=0$ *and* $\breve{\phi}(x,t_1)=1$*, then*

$$\left.\frac{\partial\breve{\phi}(x,t)}{\partial t}\right|_{t=t_0}\geq 0 \tag{21}$$

$$\left.\frac{\partial\breve{\phi}(x,t)}{\partial t}\right|_{t=t_1}\leq 0 \tag{22}$$

Proof. It is a straightforward consequence of both equations (4) and (4).

In the next theorem, we show that we can also build categories of dynamical fuzzy sets from two monads $\mathfrak{M}=(\mathcal{F}_t,\omega_t,\tau_t)$ and $\mathfrak{G}=(\mathcal{G},\eta,\mu)$. It is sufficient that the unit η of $\mathfrak{G}$ factors through ω_t.

Theorem 6. *Let* $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ *be the category of dynamical fuzzy sets associated with a* t*-snapshot comonad* $(\mathcal{F}_t,\pi,\theta)$ *and a monad* $(\mathcal{G},\eta,\mu)$*. Let* $(\mathcal{F}_t,\omega_t,\tau_t)$ *be a* t*-snapshot monad, and* $\mathcal{F}_t\xrightarrow{\xi}\mathcal{G}$ *the natural transformation that, for each object* X*, the map* ξ_X *sends every* $(x,A)\in X\times\mathcal{C}_t$ *into* $(x,1)\in X\times[0,1]$*. With these assumptions the following statements hold.*

a *The unit* η *of the monad* $(\mathcal{G},\eta,\mu)$ *is equal to the composition* $\xi\circ\omega_t$*.*

b *There is a category* $\mathbf{C}^*_{\mathcal{F}_t,\mathcal{G}}$*, with the same object-class as* $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ *such that, every non-identity morphism in* $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ *belongs to* $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$*, and vice versa. The composition* $\mathbf{g}\Diamond\mathbf{f}$ *of two morphisms*

$$\mathbf{f}\in\hom_{\mathbf{C}^*_{\mathcal{F}_t,\mathcal{G}}}(X,Y)\ \textit{and}\ \mathbf{g}\in\hom_{\mathbf{C}^*_{\mathcal{F}_t,\mathcal{G}}}(Y,Z)$$

is defined as follows.

$$\mathbf{g}\Diamond\mathbf{f}=\mu_Z\circ\mathcal{G}(\mathbf{g}\circ\omega_{t,Y})\circ\mathbf{f}. \tag{23}$$

c *For each object* X*, the associated identity is*

$$(\pi_X,\delta_X)\in\hom_{\mathbf{C}^*_{\mathcal{F}_t,\mathcal{G}}}(X,X)=\hom_{\mathbf{Set}}(\mathcal{F}_t(X),\mathcal{G}(X)); \tag{24}$$

where, for every $x\in X$*,*

$$\delta_X(x,A)=\begin{cases}1 & \textit{if } A\neq\emptyset;\\ 0 & \textit{otherwise.}\end{cases} \tag{25}$$

Proof.

a For every object X, $\eta_X(x) = (x, 1) = \xi_X(\omega_{t,X}(x))\ (1)$, and Statement (**a**) is true.

b It is sufficient to show that (17) and (23) are equivalent. By straightforward computations we can see that, if $\mathbf{f} = (f, \phi)$ and $\mathbf{g} = (g, \gamma)$, for every $(x, A) \in X \times \mathcal{C}_t$, if $A = \{t\}$,

$$\begin{aligned}(\mathbf{g} \bullet \mathbf{f})\,(x, \{t\}) = ((g, \gamma) \bullet (f, \phi))\,(x, \{t\}) = \\ (g\,(f(x, \{t\}), \{t\})\,, \phi(x, \{t\}) * \gamma\,(f(x, \{t\}), \{t\})) = \\ ((g, \gamma) \diamondsuit (f, \phi))\,(x, \{t\}) = (\mathbf{g} \diamondsuit \mathbf{f})\,(x, \{t\}) \quad (26)\end{aligned}$$

If $A = \emptyset$, taking into account Definition 3,

$$\begin{aligned}(\mathbf{g} \bullet \mathbf{f})\,(x, \emptyset) = ((g, \gamma) \bullet (f, \phi))\,(x, \emptyset) = \\ \left(\breve{g}\left(\breve{f}(x)\right), 0\right) = ((g, \gamma) \diamondsuit (f, \phi))\,(x, \{t\}) = (\mathbf{g} \diamondsuit \mathbf{f})\,(x, \emptyset); \quad (27)\end{aligned}$$

where $g = \breve{g} \circ \pi_Y$ and $f = \breve{f} \circ \pi_X$.

c Every morphism $(f, \phi) \in \hom_{\mathbf{C}^*_{\mathcal{F}_t, \mathcal{G}}}(X, Y)$, satisfies the relation

$$\begin{aligned}\forall (x, A) \in X \times \mathcal{C}_t : \\ ((f, \phi) \bullet (\eta_X \circ \pi_X))\,(x, A) = (f(x, A), \phi(x, A)) = \\ \left(\breve{f}(x), \delta_X(x, A)\right) = ((f, \phi) \diamondsuit (\pi_X, \delta_X))\,(x, A) \quad (28)\end{aligned}$$

Thus, (π_X, δ_X) is the identity associated with X.

Remark 2. *As in [5], we can extend dynamical fuzzy set categories with arbitrary hidden parameters. Let H be a parameter set, and $\mathcal{F}_{t,H} : \mathbf{Set} \to \mathbf{Set}$ the endofunctor that sends each X into $X \times \mathcal{C}_t \times H$ and every map $f : X \to Y$ into $f \times \mathrm{id}_{\mathcal{C}_t} \times \mathrm{id}_H$. This endofunctor together with the natural transformations $\mathcal{F}_{t,H} \xrightarrow{\pi} \mathrm{Id}$ and $\mathcal{F} \xrightarrow{\theta} \mathcal{F}^2$ form a comonad, whenever π_X sends each (x, A, h) into x, and θ_X sends each (x, A, h) into (x, A, h, A, h) If $\mathcal{F}_{t,H} \circ \mathcal{G} \xrightarrow{\sigma} \mathcal{G} \circ \mathcal{F}_{t,H}$ is the natural transformation such, that for each object X, the map σ_X sends (x, α, A, h) into (x, A, h, α), then we can construct the extension $\mathbf{C}_{\mathcal{F}_{t,H}, \mathcal{G}}$ as in Theorem 3.*

The members of H can denote frequencies, events, states and so on. For instance, consider the set of data in a memory. Membership degrees vanish or increases depending on events, occurrence frequencies among others parameters.

4.1. Functors from Fuzzy to Crisp Set Categories

Let $\mathbf{D}_{\mathcal{F}_t,\mathcal{G}}$ be the full subcategory of $\mathbf{C}_{\mathcal{F}_t,\mathcal{G}}$ of all discernible sets with discernible members [6]. Thus, for each set $A \in \mathrm{Ob}\,(\mathbf{D}_{\mathcal{F}_t,\mathcal{G}})$ there is a finitely definable predicate $p_A(x)$ such that $A = \{x \mid p_A(x)\}$. Accordingly, we are assuming that, for each set $A \in \mathrm{Ob}\,(\mathbf{D}_{\mathcal{F}_t,\mathcal{G}})$ and every $a \in A$, there is a finitely definable predicate $p_a(x)$ determining it, that is to say, $\{a\} = \{x \mid p_A(x) \wedge p_a(x)\}$. As a consequence, A is countable [6], because there is some language $\mathfrak{L}$ together with a finite symbol sequence $s = w_1 w_2 \dots w_n$ in $\mathfrak{L}$ that denotes $p_A(x) \wedge p_a(x)$. Take into account that, each Gödel-like numbering function sends every finite symbol sequence into a positive integer [6].

Let λ be the map that sends each predicate into its truth-value. If the object $A \in \mathrm{Ob}\,(\mathbf{D}_{\mathcal{F}_t,\mathcal{G}})$ is a set with fuzzy subsets, there is, at least, one $a \in A$ such that $\lambda(p_A(a)) < 1$.

Axiom 2. If for some x_0, the predicate $p_A(x)$ satisfies the relation $\lambda(p_A(x_0)) < 1.0$, there are two predicates $p_{A,1}(x)$ and $p_{A,2}(x)$ such that

$$p_A(x) = p_{A,1}(x) \wedge p_{A,2}(x) \tag{29}$$

$$\forall a \in A : \quad \lambda\,(p_{A,1}(a)) = 1.0 \tag{30}$$

As in [5], if an expression E denotes a class, we denote by $E^{\curlyvee}$ the generic member of E. Thus, $\{2n \mid n \in \mathbb{N}\}^{\curlyvee}$ denotes even-positive integer concept. If E is a singleton $\{a\}$, then $E^{\curlyvee} = \{a\}^{\curlyvee} = a$.

Definition 4. *Given a finitely definable object A by a predicate $p_A(x)$, if $p_A(x)$ is equivalent to a conjunction of predicates $\bigwedge_{1\le i\le m} p_{A,i}(x)$, for each positive integer $n \le m$, we say the generic object $O^{\curlyvee}$ of the class*

$$O = \{x \mid p_{A,1} \wedge p_{A,2}(x) \dots p_{A,n-1}(x) \wedge p_{A,n+1}(x) \dots p_{A,m}(x)\}\,.$$

to be an abstraction of A.

Theorem 7. *If every object in a category $\mathbf{D}_{\mathcal{F}_t,\mathcal{G}}$ of dynamical sets satisfies Axiom 2, there is a functor $\mathcal{H} : \mathbf{C}_{\mathcal{F}_t,\mathcal{G}} \to$ **Set** that sends each dynamical set into a crisp abstraction of it.*

Proof. By the definition of $\mathbf{D}_{\mathcal{F}_t,\mathcal{G}}$, for each $a \in A = \{x \mid p_A(x)\}$, there is a predicate $p_a(x)$ such that $\{a\} = \{x \mid p_A(x) \wedge p_a(x)\}$. For every object A and every predicate $p_A(x)$, there are $p_{A,1}(x)$ and $p_{A,2}(x)$ that satisfy both relations (29) and (30).

We define the object-map of $\mathcal{H}$ by

$$\mathcal{H}(A) = \bigcup_{a\in A} \left\{ \{x \mid p_{A,1}(x) \wedge p_a(x)\}^{\curlyvee} \right\} \tag{31}$$

According to (30), the membership degree of every member of $\mathcal{H}(A)$ is 1.0; hence it is a crisp set and belongs to **Set**.

Now, If $(f,\phi) = (\breve{f} \circ \pi_A, \phi)$ is a $\mathbf{D}_{\mathcal{F}_t,\mathcal{G}}$-morphism with domain A, let $\mathcal{H}(f,\phi)$ be the map that sends each $\{x \mid p_{A,1} \wedge p_a(x)\}^{\Upsilon}$ in $\mathcal{H}(A)$ into

$$\left\{x \mid p_{\breve{f}[A],1}(x) \wedge p_{\breve{f}(a)}(x)\right\}^{\Upsilon}.$$

If $(g,\gamma) = (\breve{g} \circ \pi_B, \gamma)$ is a morphism such that its domain B contains the codomain of (f,ϕ), then $\mathcal{H}(g \bullet f)$ sends $\{x \mid p_{A,1}(x) \wedge p_a(x)\}^{\Upsilon}$ into

$$\left\{x \mid p_{\breve{g}[B],1}(x) \wedge p_{\breve{g}(\breve{f}(a))}(x)\right\}^{\Upsilon};$$

therefore $\mathcal{H}$ preserves morphism compositions.

To see that $\mathcal{H}$ preserves identities, equation (24) leads to

$$\mathcal{H}(\mathrm{id}_A \circ \pi_A, \delta_X) = \mathrm{id}_A.$$

The image under $\mathcal{H}$ of every object A is a crisp set. Likewise, the image of every $\mathbf{D}_{\mathcal{F}_t,\mathcal{G}}$-morphism is an ordinary map. Thus, $\mathcal{H}$ is a functor from $\mathbf{D}_{\mathcal{F}_t,\mathcal{G}}$ into a subcategory of **Set**.

5. Conclusion

The fuzzy-set theory is an important source of scientific literature. Since subsets with static membership degrees are particular cases of the dynamical ones, our generalization gives rise to a large research field. The aim of this article is, simply, to start up this subject. The algebraic structures stated in both Theorem 1 and Theorem 6, satisfy this aim. Now, there is a large class of open problems namely, to find out the laws that constrain or determine the dynamical membership functions. This task depends on the nature of the application fields. Finally, Theorem 1 shows that there are crisp-abstractions for some fuzzy-set categories.

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