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ON A CLASS OF FUNCTIONS ASSOCIATED WITH SĂLĂGEAN INTEGRAL OPERATOR

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Abstract. In this paper we define a new class of analytic functions using the Sălăgean integral operator. We also discuss the geometric significance, radius problem, coefficient estimates and convolution properties for functions belonging to this class. The geometric significance, radius problem, coefficient estimates and convolution properties have been discussed for functions belonging to this class.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of functions for which the domain $f(U)$ is starlike with respect to 0, is denoted by S^*. An analytic description of S^* is

$$S^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > 0, z \in U \right\}.$$

Denote by

$$C = \left\{ f \in \mathcal{A} : \Re \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

the class of normalized convex functions in U.

Denote by $P(\alpha)$ the class of functions with real part greater than α. For $\alpha = 0$, the class $P(\alpha)$ reduces to the class of functions with positive real part.

Definition 1. Let f and g be analytic functions in U.
We say that the function f is subordinate to the function g, if there exists a function w, which is analytic in U and $w(0) = 0, |w(z)| < 1, z \in U$, such that $f(z) = g(w(z))$,$\forall z \in U$. We denote by "$\prec$" the subordination relation.

Let $g \in \mathcal{A}$ where $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. The Hadamard product (convolution) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z)$$

.

Definition 2. [5]
For $f \in \mathcal{A}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}$, the operator I^n is defined by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t)\, t^{-1} dt, \ldots$$

$$I^n f(z) = I\left(I^{n-1} f(z)\right), z \in U$$

Remark 1. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k,$$

$z \in U$, $(n \in \mathbb{N}_0)$ *and* $z\left(I^n f(z)\right)' = I^{n-1} f(z)$.

Thulasiram et. al. [6] defined and studied a new class of functions $G(\lambda, \beta)$, for $0 \leq \lambda < 1$ and $0 \leq \beta < 1$ as follows:

$$f \in G(\lambda, \beta) \text{ if and only if } \Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right) > \beta, \quad \forall z \in U,$$

which can also be written as

$$f \in G(\lambda, \beta) \text{ if and only if } \frac{zf'(z) + \lambda z^2 f''(z)}{f(z)} \prec \frac{1 + (1 - 2\beta) z}{1 - z}, \quad \forall z \in U.$$

Hussain et. al. [1] defined and studied a new class of functions
$G_\lambda(n, A, B) : \mathcal{A} \to \mathcal{A}$ using the Sălăgean differential operator. Motivated by their work we define a new class of functions $R_\lambda(n, A, B) : \mathcal{A} \to \mathcal{A}$ using the Salagean integral operator.

Definition 3. Let $f \in \mathcal{A}$. Then for $0 \leq \lambda < 1, n \in \mathbb{N}_0$ and $-1 \leq B < A \leq 1$, we introduce a new class of analytic functions $R_\lambda(n, A, B)$.

$$f \in R_\lambda(n, A, B) \text{ if and only if}$$

$$\frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)} \prec \frac{1+Az}{1+Bz}, \ z \in U. \tag{2}$$

Remark 2. *[2] If we use the definition of subordination and properties for the Schwarz function, we can define the class $R_\lambda(n, A, B)$ the same way, for $f \in \mathcal{A}$, $f \in R_\lambda(n, A, B)$ if and only if $\left|\frac{p(z)-1}{A-Bp(z)}\right| < 1$ or equivalently $\left|p(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}$, where $p(z) = \frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)}$.*
Note that $f \in R_\lambda(n, A, B)$ if the image of each $z \in U$, under $p(z)$ lies inside the disc centred at $\left(\frac{1-AB}{1-B^2}, 0\right)$ and radius $\frac{A-B}{1-B^2}$.

Lemma 1. *[3] Let $u = u_1 + u_2$ and $v = v_1 + v_2$ and $\Psi(u, v)$ be a complex valued function satisfying the conditions:*

i) $\Psi(u, v)$ *is continous in* $D \subset \mathbb{C}^2$,

ii) $(1, 0) \in D$ *and* $\Re\Psi(1, 0) > 0$,

iii) $\Re\Psi(iu_2, v_1) \leq 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{1}{2}\left(1 + u_2^2\right)$.

If $h(z)$ is an analytic function in U such that $(h(z), zh'(z)) \in D$ and $\Re\Psi(h(z), zh'(z)) > 0$ for $z \in U$ then $\Re h(z) > 0$ in U.

Lemma 2. *[4] Let f be uniformly complex and g be a starlike function. Then for every F analytic in U, we have*

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{co}\,[F(U)]$$

where $\overline{co}\,[F(U)]$ denotes the closed convex hull of the set $F(U)$.

2. Main results

Theorem 1. Let $f \in \mathcal{A}$ be of the form (1). We say that f is in the class $R_\lambda(n, A, B)$ if

$$\sum_{k=2}^{\infty} \left\{ \left| \left[(1-\lambda)\frac{1}{k} + \lambda\right] \frac{1-B^2}{A-B} - \frac{1}{k^2} \cdot \frac{1-AB}{A-B} \right| + \frac{1}{k^2} \right\} \frac{1}{k^n} |a_k| \leq 1 - B, \ B \geq 0. \tag{3}$$

Proof. It is enough to show that $p(z) = \dfrac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)}$ lies in a circle with centre at $\left(\dfrac{1-AB}{1-B^2}, 0 \right)$ and radius $\dfrac{A-B}{1-B^2}$. This means that

$$\left| p(z) - \frac{1-AB}{1-B^2} \right| \le \frac{A-B}{1-B^2}. \tag{4}$$

We consider

$$\left| p(z) - \frac{1-AB}{1-B^2} \right| = \left| \frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)} - \frac{1-AB}{1-B^2} \right| =$$

$$= \left| \frac{z + \sum_{k=2}^{\infty} \left[(1-\lambda) \frac{1}{k} + \lambda \right] \frac{a_k}{k^n} z^k}{z + \sum_{k=2}^{\infty} \frac{a_k}{k^{n+2}} z^k} - \frac{1-AB}{1-B^2} \right| =$$

$$= \frac{\left| \frac{B(A-B)}{1-B^2} z + \sum_{k=2}^{\infty} \left\{ \left[(1-\lambda) \frac{1}{k} + \lambda \right] \frac{1}{k^n} \cdot \frac{1-B^2}{1-B^2} - \frac{1-AB}{1-B^2} \cdot \frac{1}{k^{n+2}} \right\} a_k z^k \right|}{\left| z + \sum_{k=2}^{\infty} \frac{a_k}{k^{n+2}} z^k \right|} =$$

$$\le \frac{\frac{B(A-B)}{1-B^2} + \sum_{k=2}^{\infty} \left| \left[(1-\lambda) \frac{1}{k} + \lambda \right] \frac{1}{k^n} \cdot \frac{1-B^2}{1-B^2} - \frac{1-AB}{1-B^2} \cdot \frac{1}{k^{n+2}} \right| |a_k|}{1 - \sum_{k=2}^{\infty} \frac{|a_k|}{k^{n+2}}}$$

$$= \frac{\frac{B(A-B)}{1-B^2} + \sum_{k=2}^{\infty} \left| \left[(1-\lambda) \frac{1}{k} + \lambda \right] \cdot \frac{1-B^2}{1-B^2} - \frac{1-AB}{1-B^2} \cdot \frac{1}{k^2} \right| |a_k| \frac{1}{k^n}}{1 - \sum_{k=2}^{\infty} \frac{|a_k|}{k^{n+2}}}. \tag{5}$$

From (4) and (5) we get (3).

Theorem 2. Let $f \in \mathcal{A}$ of the form (1). If $f \in R_\lambda (n, 1-2\alpha, -1)$ then $I^{n+2} f(z) \in S^*$.

Proof. If $f \in R_\lambda(n, 1-2\alpha, -1)$ then by (2) we have

$$\frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)} \prec \frac{1+(1-2\alpha) z}{1-z}, \ \forall z \in U$$

which means

$$\frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)} \in P(\alpha). \tag{6}$$

Let $p_1(z) = \frac{z\left(I^{n+2} f(z)\right)'}{I^{n+2} f(z)} = \frac{I^{n+1} f(z)}{I^{n+2} f(z)}, \quad p_1(z) \in P(\alpha)$, differentiating logarithmically, we obtain

$$\frac{z\left(I^{n+2} f(z)\right)'}{I^{n+2} f(z)} + \frac{z p_1(z)'}{p_1(z)} = \frac{z\left(I^{n+1} f(z)\right)'}{I^{n+1} f(z)}$$

or equivalently

$$p_1(z) + \frac{z p_1(z)'}{p_1(z)} = \frac{I^n f(z)}{I^{n+1} f(z)}.$$

Now

$$\frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)} = (1-\lambda) \frac{I^{n+1} f(z)}{I^{n+2} f(z)} + \lambda \frac{I^n f(z)}{I^{n+2} f(z)} =$$

$$= (1-\lambda) p_1(z) + \lambda \frac{I^n f(z)}{I^{n+1} f(z)} \frac{I^{n+1} f(z)}{I^{n+2} f(z)} =$$

$$= (1-\lambda) p_1(z) + \lambda \left(p_1(z) + \frac{z p_1(z)'}{p_1(z)}\right) p_1(z) =$$

$$= (1-\lambda) p_1(z) + \lambda z p_1'(z) + \lambda p_1^2(z). \tag{7}$$

Using (6) and (7) we have

$$(1-\lambda) p_1(z) + \lambda z p_1'(z) + \lambda p_1^2(z) \in P(\alpha)$$

which implies that,

$$\Re\left\{(1-\lambda) p_1(z) + \lambda z p_1'(z) + \lambda p_1^2(z)\right\} > \alpha,$$

or equivalently

$$\Re\left\{(1-\lambda) p_1(z) + \lambda z p_1'(z) + \lambda p_1^2(z) - \alpha\right\} > 0.$$

Now we define the function $\phi(u, v)$ by taking $u = p_1(z)$ and $v = z p_1'(z)$ as

$$\phi(u, v) = (i-\lambda) u \lambda u^2 + \lambda v - \alpha.$$

i) $\phi(u,v)$ is continous in $\mathbb{C}\times\mathbb{C}$,

ii) $\phi(1,0)=1-\lambda+\lambda-\alpha=1-\alpha>0$,

iii) $\Re\phi(iu_2,v_1)=-\lambda u_2^2+\lambda v_1-\alpha$.

If we put $v_1\leq -\frac{1}{2}\left(1+u_2^2\right)$, we get

$$\Re\phi(iu_2,v_1)\leq -\left[\lambda u_2^2+\frac{1}{2}\lambda\left(1+u_2^2\right)+\alpha\right],$$

Hence $\Re\phi(iu_2,v_1)\leq 0$. By Lemma 1 we have $\Re p_1(z)>0,\ z\in U$. Therefore

$$\Re\frac{z\left(I^{n+2}f(z)\right)'}{I^{n+2}f(z)}>0,$$

which implies

$$I^{n+2}f(z)\in S^*.$$

Theorem 3. If $I^{n+2}f(z)\in S^*$ then $f\in R_\lambda(n,1-2\alpha,-1)$, for $\lambda<\dfrac{1-\alpha}{2}$ and $|z|<r$, where r is unique root of

$$(1-\lambda)(1-r)^3-\lambda(1+r)^3-2\lambda r(1+r)-\alpha(1+r)(1-r)^2=0$$

in $(0,1)$.

Proof. Let $f\in S^*$ and $p_1(z)=\dfrac{z\left(I^{n+2}f(z)\right)'}{I^{n+2}f(z)}\in P(\alpha)$. By Theorem 2 we get

$$\frac{(1-\lambda)I^{n+1}f(z)+\lambda I^n f(z)}{I^{n+2}f(z)}=(1-\lambda)p_1(z)+\lambda z p_1'(z)+\lambda p_1^2(z)-\alpha.$$

We need to find r such that

$$\Re\left\{(1-\lambda)p_1(z)+\lambda z p_1'(z)+\lambda p_1^2(z)-\alpha\right\}>0.$$

Consider that

$$\begin{aligned}\Re\left\{(1-\lambda)p_1(z)+\lambda z p_1'(z)+\lambda p_1^2(z)-\alpha\right\}&\geq(1-\lambda)\Re p_1(z)-\lambda\left|zp_1'(z)\right|+\lambda\Re p_1^2(z)-\alpha,\\&\geq(1-\lambda)\left|p_1(z)\right|-\lambda\left|p_1(z)\right|^2-\lambda r\left|p_1'(z)\right|-\alpha.\end{aligned}$$

If we use growth and distortion theorems for class P, we get

$$\Re\left\{(1-\lambda)\,p_1(z)+\lambda z p_1'(z)+\lambda p_1^2(z)-\alpha\right\}\geq(1-\lambda)\left(\frac{1-r}{1+r}\right)-\lambda\left(\frac{1+r}{1-r}\right)^2-\frac{2\lambda r}{(1-r)^2}-\alpha$$

or equivalently

$$\Re\left\{(1-\lambda)\,p_1(z)+\lambda z p_1'(z)+\lambda p_1^2(z)-\alpha\right\}\geq$$

$$\geq\frac{(1-\lambda)\,(1-r)^3-\lambda\,(1+r)^3-2\lambda r\,(1+r)-\alpha\,(1+r)\,(1-r)^2}{(1-r)^2\,(1+r)}.$$

Thus

$$\Re\left\{(1-\lambda)\,p_1(z)+\lambda z p_1'(z)+\lambda p_1^2(z)-\alpha\right\}>0$$

if

$$(1-\lambda)\,(1-r)^3-\lambda\,(1+r)^3-2\lambda r\,(1+r)-\alpha\,(1+r)\,(1-r)^2=0.$$

Let

$$f_1(r)=(1-\lambda)\,(1-r)^3-\lambda\,(1+r)^3-2\lambda r\,(1+r)-\alpha\,(1+r)\,(1-r)^2 \tag{8}$$

$f_1(1)=-12\lambda$ and $f_1(0)=1-\alpha-2\lambda>0$ if $\lambda<\dfrac{1-\alpha}{2}$. Hence (8) has a unique real root in $(0,1)$.

Theorem 4. Let f of the form (1) be in $R_\lambda\,(n,A,B)$, then

$$|a_2|\leq\frac{(A-B)\,2^{n+2}}{2\lambda+1},\quad |a_3|\leq 3^{n+2}\,(A-B)\,\frac{2\lambda+1+(A-B)}{2\,(3\lambda+1)},$$

$$|a_{k+2}|\leq\frac{(k+2)^{n+2}\,(A-B)\left[1+\dfrac{1}{(k+1)^{n+2}}\,|a_{k+1}|+\displaystyle\sum_{j=2}^{\infty}\frac{1}{j^{n+2}}\,|a_j|\right]}{(1-\lambda)\,(k+2)+\lambda\,(k+2)^2-1}$$

Proof. Let

$$\frac{(1-\lambda)\,I^{n+1}f\,(z)+\lambda I^n f\,(z)}{I^{n+2}f\,(z)}=p(z)$$

or equivalently

$$(1-\lambda)\,I^{n+1}f\,(z)+\lambda I^n f\,(z)=p(z)I^{n+2}f\,(z)\,. \tag{9}$$

If $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$, then (9) becomes

$$z + \sum_{k=2}^{\infty} \left[(1-\lambda) \frac{1}{k^{n+1}} + \lambda \frac{1}{k^n} \right] a_k z^k = \left(1 + \sum_{k=1}^{\infty} p_k z^k \right) \left(z + \sum_{k=2}^{\infty} \frac{a_k}{k^{n+2}} z^k \right)$$

or equivalently

$$\sum_{k=2}^{\infty} \left[(1-\lambda) \frac{1}{k^{n+1}} + \lambda \frac{1}{k^n} - \frac{1}{k^{n+2}} \right] a_k z^k = \sum_{k=1}^{\infty} p_k z^{k+1} + \left(\sum_{k=1}^{\infty} p_k z^k \right) \left(\sum_{k=2}^{\infty} \frac{a_k}{k^{n+2}} z^k \right).$$

Fom this follows that

$$\sum_{k=2}^{\infty} \left[(1-\lambda) \frac{1}{k^{n+1}} + \lambda \frac{1}{k^n} - \frac{1}{k^{n+2}} \right] a_k z^k =$$

$$= \sum_{k=1}^{\infty} p_k z^{k+1} + \sum_{k=2}^{\infty} p_1 \frac{1}{k^{n+2}} a_k z^{k+1} + \sum_{k=2}^{\infty} \left[\sum_{j=2}^{k} \frac{1}{j^{n+2}} a_j p_{k+2-j} \right] z^{k+2}.$$

Equating coefficients of z^2, z^3:

$$|a_2| \le \frac{(A-B)\, 2^{n+2}}{2\lambda + 1}, \quad |a_3| \le 3^{n+2} (A-B) \frac{2\lambda + 1 + (A-B)}{2\,(3\lambda+1)},$$

where $|p_n| \le A - B$.

Equating coefficients of z^{k+2}:

$$|a_{k+2}| \le \frac{(k+2)^{n+2} (A-B) \left[1 + \frac{1}{(k+1)^{n+2}} |a_{k+1}| + \sum_{j=2}^{\infty} \frac{1}{j^{n+2}} |a_j| \right]}{(1-\lambda)(k+2) + \lambda (k+2)^2 - 1}, \; k \ge 2.$$

Remark 3. *The* (2) *inequality is sharp for* $f_0 \in \mathcal{A}$ *such that*

$$\frac{(1-\lambda)\, I^{n+1} f_0(z) + \lambda I^n f_0(z)}{I^{n+2} f_0(z)} = \frac{1 + Az}{1 + Bz},$$

where

$$f_0(z) = z + \frac{(A-B)\, 2^{n+2}}{2\lambda + 1} z^2 + 3^{n+2} (A-B) \frac{2\lambda + 1 + (A-B)}{2\,(3\lambda+1)} z^3 + \cdots.$$

3. Convolution properties

Theorem 5. Let $\Psi \in C$. If $f \in R_\lambda(n, 1-2\alpha, -1)$ then $\Psi * f \in R_\lambda(n, 1-2\alpha, -1)$.

Proof. Note that $I^n(\Psi * f) = \Psi * I^n f$. Let $E = \Psi * f$, then

$$\frac{(1-\lambda) I^{n+1} E + \lambda I^n E}{I^{n+2} E} = \frac{\Psi * \left[(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)\right]}{\Psi * I^{n+2} f(z)} =$$

$$= \frac{\Psi * p(z) I^{n+2} f(z)}{\Psi * I^{n+2} f(z)}, \quad \text{where} \quad p(z) = \frac{(1-\lambda) I^{n+1} f(z) + \lambda I^n f(z)}{I^{n+2} f(z)} \prec \frac{1+Az}{1+Bz}.$$

If $f \in R_\lambda(n, 1-2\alpha, -1)$ then by Theorem 2 , $I^{n+2} f \in S^*$ and by Lemma 2

$$\frac{\Psi * E(z) I^{n+2} f(z)}{\Psi * I^{n+2} f(z)} \in \overline{co}\,[p(U)].$$

Moreover $\overline{co}\,[p(U)] \prec \dfrac{1+Az}{1+Bz}$, hence $E \in R_\lambda(n, 1-2\alpha, -1)$.

Corollary 1. Let $F_c f(z) = \dfrac{c+1}{z^c} \displaystyle\int_0^z f(t) t^{c-1} dt$, $c > -1$ be the well-known Bernardi integral operator. It is known that

$$F_c f(z) = f(z) * \sum_{n=1}^{\infty} \frac{c+1}{c+n} z^n = f(z) * h(z),$$

where $h \in C$, $\forall\ \Re\, c > 0$.
So, $\forall\ \Re\, c > 0$ if $f \in R_\lambda(n, 1-2\alpha, -1)$ then $F_c f(z) \in R_\lambda(n, 1-2\alpha, -1)$.

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