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NEW CRITERIA FOR UNIVALENCE OF A GENERAL INTEGRAL OPERATOR

C. Bărbatu, D. Breaz

Abstract. For some classes of analytic functions f, g,, h and k in the open unit disk $\mathbb{U}$, we consider the general integral operator $\mathcal{G}_n$, that was introduced in a recent work [1] and we obtain new conditions of univalence for this family of integral general operators. The key tools in the proofs of our results are the Becker's and the Pascu's univalence criteria. Also, in order to prove these results we use Nehari's well-known inequality. Some corollaries of the main results are also considered. Relevant connections of the results presented here with various other known results are briefly indicated.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} :| z |< 1\}$$

and satisfy the following usual normalization conditions:

$$f(0) = f^{'}(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

Becker's gave the following univalence criterion:

Theorem 1. *(Becker[2]) If the function f is regular in unit disk $\mathbb{U}$ and $f(z) = z + a_2z^2 + ...$ and*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad for\,all\ z \in \mathbb{U},$$

then the function f is univalent in $\mathbb{U}$.

The following univalence condition was derived by Pascu.

Theorem 2. *(Pascu [12]) Let $\delta \in \mathbb{C}$ with $Re\delta > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2Re\delta}}{Re\delta}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then for any complex γ with $Re\gamma \geq Re\delta$, the integral operator

$$F_\gamma(z) = \left(\gamma\int_0^z t^{\gamma-1}f'(t)\,dt\right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Nehari, on the other hand, proved another important results.

Theorem 3. *(Nehari[11]) If the function g is regular in unit disk $\mathbb{U}$ and $|g(z)| < 1$ in $\mathbb{U}$, then for all $\xi \in \mathbb{U}$ the following inequalities hold*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1-\bar{z}\xi}\right| \tag{2}$$

and

$$\left|g'(z)\right| \leq \frac{1-|g(z)|^2}{1-|z|^2},$$

the equalities hold in case $g(z) = \varepsilon\frac{z+u}{1+\bar{u}z}$ where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 1. *(Nehari[11]) For $z = 0$, from inequality (2) we obtain for every $\xi \in \mathbb{U}$*

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1+|g(0)|\,|g(\xi)|}.$$

Considering $g(0) = a$ *and* $\xi = z$, *then*

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|},$$

for all $z \in \mathbb{U}$.

The problem of univalence for some generalized integral operators have discussed by many authors such as: (see Breaz et al.[3]-[4], Oprea et al. [8]-[9], Pescar et al. [14]-[17] and Ularu et al. [19]) .

We consider the integral operator:

$$\mathcal{G}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(f_i{}'(t) e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)} \right)^{\beta_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)} \right)^{\gamma_i} \right] dt \right\}^{\frac{1}{\delta}}, \qquad (3)$$

where f_i, g_i, h_i, k_i are analytic in $\mathbb{U}$ and $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, for all $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$.

Remark 2. *The integral operator* $\mathcal{G}_n$ *defined by (3), introduced by Bărbatu and Breaz in the paper [1], is a general integral operator of Pfaltzgraff, Kim-Merkes and Oversea types which extends also the other operators as follows:*

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \gamma_1 = 0$ *and* $k_1(z) = z$ *we obtain the integral operator which was studied by Kim-Merkes [6]*

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt.$$

ii) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$, $\beta_1 = \gamma_1 = 0$ *and* $g_1(z) = 0$ *we obtain the integral operator which was studied by Pfaltzgraff [18]*

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t) \right)^\alpha dt.$$

iii) For $\alpha_i - 1 = \gamma_i = 0$ *and* $k_i(z) = z$ *we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [3]*

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt \right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [13].

iv) For $\alpha_i - 1 = \alpha_i$, $\beta_i = \gamma_i = 0$ and $g_i(z) = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [4]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[f_i'(t)\right]^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [17] .

v) For $\alpha_i - 1 = 0$, $k_i(z) = z$ and $k_i'(z) = 1$ we obtain the integral operator which was defined and studied by Pescar [15]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(f_i{}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Frasin in [5] and by Oversea in [10].

vi) For $\alpha_i - 1 = \alpha_i$ și $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by A. Oprea and D. Breaz in [9]

$$\mathcal{G}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(f_i{}'(t) e^{g_i(t)}\right)^{\alpha_i}\right] dt,$$

this integral operator is a generalization of the integral operator introduced by Ularu and Breaz in [19].

vii) For $\alpha_i - 1 = 0$ we obtain the integral operator which was defined and studied by Pescar in [15]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} \left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i} dt\right]^{\frac{1}{\delta}}.$$

Thus, the integral operator $\mathcal{G}_n$, introduced here by the formula (3), can be considered as an extension and a generalization of these operators above mentioned.

2. Main results

Our main results give sufficient conditions for the general integral operator $\mathcal{G}_n$ defined by (3) to be univalent in the open disk $\mathbb{U}$.

Theorem 4. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$, $f_i(z) = z + a_{2i}z^2 + ...$, $g_i(z) = z + b_{2i}z^2 + ...$, $h_i(z) = z + c_{2i}z^2 + ...$, $k_i(z) = z + d_{2i}z^2 + ...$, $i = \overline{1,n}$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq 1, \quad \left|g_i'(z)\right| \leq 1, \quad \left|\frac{zh_i'(z) - h_i(z)}{zh_i(z)}\right| \leq 1,$$

$$\left|\frac{zk_i'(z) - k_i(z)}{zh_i(z)}\right| \leq 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \leq 1, \tag{4}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{\sum_{i=1}^{n} \left(|\alpha_i - 1| + |\beta_i| + |\gamma_i|\right)}{\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|\right)} < 1, \tag{5}$$

$$\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|\right) \leq \frac{1}{\underset{|z|\leq 1}{max}\left[2\left(1 - |z|^2\right)|z| \, \frac{|z|+|k|}{1+|k||z|}\right]}, \tag{6}$$

where

$$|k| = \frac{\left|\sum_{i=1}^{n} \left[(\alpha_i - 1)(2a_{2i} + 1) + \beta_i (c_{2i} + d_{2i}) + 2\gamma_i (c_{2i} + d_{2i})\right]\right|}{2\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|\right)}, \tag{7}$$

then for every δ, $Re\delta \geq Re\gamma$, the function $\mathcal{G}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. We have $h_i, k_i \in \mathcal{S}$, $\frac{h_i(z)}{z} \neq 0, \frac{k_i(z)}{z} \neq 0$ for all $i = \overline{1,n}$.

Let us define the function

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(f_i{}'(t)e^{g_i(t)}\right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)}\right)^{\beta_i} \left(\frac{h_i{}'(t)}{k_i{}'(t)}\right)^{\gamma_i}\right] \mathrm{d}t,$$

for all $f_i, g_i, h_i, k_i \in \mathcal{S}$, $i = \overline{1,n}$.

We have $\prod_{i=1}^{n} \left[\left(f_i{}'(z)e^{g_i(z)}\right)^{\alpha_i - 1} \left(\frac{h_i(z)}{k_i(z)}\right)^{\beta_i} \left(\frac{h_i{}'(z)}{k_i{}'(z)}\right)^{\gamma_i}\right] = 1$, when $z = 0$.

Consider the function

$$h_n(z) = \frac{1}{2\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|\right)} \frac{G_n''(z)}{G_n'(z)}.$$

The function h_n has the form

$$h_n(z) = \frac{1}{2\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\beta_i| \, |\gamma_i|\right)} \sum_{i=1}^{n} (\alpha_i - 1)\left(\frac{f_i''(z)}{f_i'(z)} + g_i'(z)\right) +$$

$$+\frac{1}{2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\sum_{i=1}^{n}\beta_i\left(\frac{h_i'(z)-h_i(z)}{zh_i(z)}-\frac{k_i'(z)-k_i(z)}{zk_i(z)}\right)+$$
$$+\frac{1}{2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\sum_{i=1}^{n}\gamma_i\left(\frac{h_i''(z)}{h_i'(z)}-\frac{k_i''(z)}{k_i'(z)}\right). \tag{8}$$

By using the relations (8), (4) and (5), we obtain

$$|h_n(z)|\le\frac{1}{2\prod_{i=1}^{n}\left(||\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\sum_{i=1}^{n}|\alpha_i-1|\left(\left|\frac{f_i''(z)}{f_i'(z)}\right|+\left|g_i'(z)\right|\right)+$$
$$+\frac{1}{2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\sum_{i=1}^{n}|\gamma_i|\left(\left|\frac{h_i'(z)-h_i(z)}{zh_i(z)}\right|+\left|\frac{k_i'(z)-k_i(z)}{zk_i(z)}\right|\right)+$$
$$+\frac{1}{2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\sum_{i=1}^{n}|\beta_i|\left(\left|\frac{h_i''(z)}{h_i'(z)}\right|+\left|\frac{k_i''(z)}{k_i'(z)}\right|\right)\le$$
$$\le\frac{1}{2\prod_{i=1}^{n}\left(||\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\sum_{i=1}^{n}\left[2|\alpha_i-1|+2\,|\beta_i|+2\,|\gamma_i|\right]\le 1,$$
$$|h_n(0)|=\frac{\left|\sum_{i=1}^{n}\left[(\alpha_i-1)\left(2a_{2i}+1\right)+\beta_i\left(c_{2i}+d_{2i}\right)+\gamma_i\left(2c_{2i}+2d_{2i}\right)\right]\right|}{2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}=|k|\,.$$

Applying Remark 1.4 for the function h_n, we obtain

$$|h_n(z)|=\frac{1}{2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)}\left|\frac{G_n''(z)}{G_n'(z)}\right|\le\frac{|z|+|h_n(0)|}{1+|h_n(0)|\,|z|}\le\frac{|z|+|k|}{1+|k|\,|z|}. \tag{9}$$

From (9), we get

$$\left|\left(1-|z|^2\right)\frac{zG_n''(z)}{G_n'(z)}\right|\le 2\prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)\left(1-|z|^2\right)|z|\,\frac{|z|+|k|}{1+|k|\,|z|}, \tag{10}$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$.

Let us define function $\varphi:[0,1]\to\mathbb{R}$,

$$\varphi(x)=2\left(1-x^2\right)x\frac{x+|k|}{1+|k|\,x},\ x=|z|\,.$$

Since $\varphi(0)=0$, $\varphi(1)=0$ and $\varphi(\frac{1}{2})=\frac{3}{4}\cdot\frac{1+2|k|}{2+|k|}$, it results

$$\max_{x\in[0,1]}\varphi(x)>0.$$

Using this result and the form (10), we have

$$\left|\left(1-|z|^2\right)\frac{zG_n''(z)}{G_n'(z)}\right| \le \prod_{i=1}^{n}\left(|\alpha_i-1|\,|\beta_i|\,|\gamma_i|\right)\max_{|z|<1}\left[2\left(1-|z|^2\right)|z|\,\frac{|z|+|k|}{1+|k|\,|z|}\right], \quad (11)$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$.

Applying the condition (6) in relation (11), we obtain

$$\left|\left(1-|z|^2\right)\frac{zG_n''(z)}{G_n'(z)}\right| \le 1.$$

Finally, we apply Theorem 1.1, we conclude that, the general integral operator $\mathcal{G}_n$ given by (3) is in the class $\mathcal{S}$.

Theorem 5. *Let $\delta,\gamma,\alpha_i,\beta_i,\gamma_i\in\mathbb{C}$, $c=Re\gamma>0$ and $f_i.g_i,h_i,k_i\in\mathcal{S}$, $f_i(z)=z+a_{2i}z^2+...$, $g_i(z)=z+b_{2i}z^2+...$, $h_i(z)=z+c_{2i}z^2+...$, $k_i(z)=z+d_{2i}z^2+...$, $i=\overline{1,n}$ and M_i,N_i,P_i positive real numbers. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right|<1,\quad \left|\frac{zg_i'(z)}{g_i(z)}\right|<M_i,\quad |g_i(z)|\le 1,$$

$$\left|\frac{zh_i'(z)}{h_i(z)}\right|<N_i,\quad \left|\frac{zk_i'(z)}{k_i(z)}\right|<P_i,\quad \left|\frac{zh_i''(z)}{h_i'(z)}\right|\le 1,\quad \left|\frac{zk_i''(z)}{k_i'(z)}\right|\le 1, \quad (12)$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$,

$$\sum_{i=1}^{n}\left[|\alpha_i-1|\left(M_i+1\right)+|\beta_i|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]\le\frac{1}{\underset{|z|\le 1}{max}\left[\frac{1-|z|^{2c}}{c}\frac{|z|+|k|}{1+|k||z|}\right]}, \quad (13)$$

where

$$|k|=\frac{\left|\sum_{i=1}^{n}\left[\left(\alpha_i-1\right)\left(2a_{2i}+b_{2i}\right)+\beta_i\left(c_{2i}+d_{2i}+2\right)+2\gamma_i\left(c_{2i}+d_{2i}\right)\right]\right|}{\sum_{i=1}^{n}\left[|\alpha_i-1|\left(M_i+1\right)+|\beta_i|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}, \quad (14)$$

then for every δ, $Re\delta\ge Re\gamma$, the function $\mathcal{G}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. Either now the function

$$h_n(z)=\frac{1}{\sum_{i=1}^{n}\left[|\alpha_i-1|\left(M_i+1\right)+|\beta_i|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}\frac{zG_n''(z)}{G_n'(z)} \quad (15)$$

$$h_n(z)=\frac{\sum_{i=1}^{n}(\alpha_i-1)\left(\frac{zf_i''(z)}{f_i'(z)}+zg_i{}'(z)\right)}{\sum_{i=1}^{n}\left[|\alpha_i-1|\left(M_i+1\right)+|\beta_i|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}+$$

$$+\frac{\sum_{i=1}^{n}\left\{\beta_i\left[\left(\frac{zh_i'(z)}{h_i(z)}-1\right)-\left(\frac{zk_i'(z)}{k_i(z)}-1\right)\right]+\gamma_i\left(\frac{zh_i''(z)}{h_i'(z)}-\frac{zk_i''(z)}{k_i'(z)}\right)\right\}}{\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}, \qquad (16)$$

$$|h_n(z)|\leq\frac{\sum_{i=1}^{n}\left|\alpha_i-1\right|\left(\left|\frac{zf_i''(z)}{f_i'(z)}\right|+\left|\frac{zg_i'(z)}{g_i(z)}\right|\left|g_i(z)\right|\right)}{\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}+$$

$$+\frac{\sum_{i=1}^{n}\left\{\left|\beta_i\right|\left[\left(\left|\frac{zh_i'(z)}{h_i(z)}\right|+1\right)+\left(\left|\frac{zk_i'(z)}{k_i(z)}\right|+1\right)\right]+\left|\gamma_i\right|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right\}}{\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]},$$

$$|h_n(z)|<\frac{\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(1+M_i\cdot 1\right)+\left|\beta_i\right|\left(N_i+1+P_i+1\right)+\left|\gamma_i\right|\left(1+1\right)\right]}{\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}, \qquad (17)$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$.

So $|h_n(z)|<1$.

We have

$$|h_n(0)|=$$

$$=\frac{\left|\sum_{i=1}^{n}\left[\left(\alpha_i-1\right)\left(2a_{2i}+b_{2i}\right)+\beta_i\left(c_{2i}+d_{2i}+2\right)+\gamma_i\left(2c_{2i}+2d_{2i}\right)\right]\right|}{\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]}=|k|$$

and by applying Remark 1.4, we get

$$|h_n(z)|\leq\frac{|z|+|k|}{1+|k|\,|z|}. \qquad (18)$$

From (15) and (18) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zG_n''(z)}{G_n'(z)}\right|\leq$$

$$\leq\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]\max_{|z|\leq 1}\left[\frac{1-|z|^{2c}}{c}\frac{|z|+|k|}{1+|k|\,|z|}\right], \qquad (19)$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$.

From (13) and (19) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zG_n''(z)}{G_n'(z)}\right|\leq 1.$$

Finally, by applying Theorem 1.2 to the function G_n, we deduce that function $\mathcal{G}_n$ given by (3) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.1, we have:

Corollary 6. *Let $\alpha \in \mathbb{C}$, $Re\alpha > 0$ and $f, g, h, k \in \mathcal{S}$, $f(z) = z + a_2z^2 + ...$, $g(z) = z + b_2z^2 + ...$, $h(z) = z + c_2z^2 + ...$, $k(z) = z + d_2z^2 + ...$ If*

$$\left|\frac{f^{''}(z)}{f^{'}(z)}\right| < 1, \quad \left|g^{'}(z)\right| < 1, \quad \left|\frac{zh^{'}(z) - h(z)}{zh(z)}\right| < 1,$$

$$\left|\frac{zk^{'}(z) - k(z)}{zh(z)}\right| < 1, \quad \left|\frac{h^{''}(z)}{h^{'}(z)}\right| < 1, \quad \left|\frac{k^{''}(z)}{k^{'}(z)}\right| < 1, \tag{20}$$

for all $z \in \mathbb{U}$ and the constant $|\alpha|$ satisfies the condition

$$|\alpha| \leq \frac{1}{\underset{|z|\leq 1}{max}\left[2\left(1 - |z|^2\right)|z| \frac{2|z|+|2a_2+3c_2+3d_2+1|}{2+|2a_2+3c_2+3d_2+1||z|}\right]}, \tag{21}$$

then the function $\mathcal{G}$, defined by

$$\mathcal{G}(z) = \left[\alpha \int_0^z t^{\alpha-1}\left(f'(t)e^{g(t)}\frac{h(t)}{k(t)}\frac{h'(t))}{k'(t)}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}}, \tag{22}$$

is in the class $\mathcal{S}$.

Letting $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 7. *Let $\gamma, \alpha_i, \beta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$, $f_i(z) = z + a_{2i}z^2 + ...$, $g_i(z) = z + b_{2i}z^2 + ...$, $h_i(z) = z + c_{2i}z^2 + ...$, $k_i(z) = z + d_{2i}z^2 + ...$, $i = \overline{1, n}$. If*

$$\left|\frac{f_i^{''}(z)}{f_i^{'}(z)}\right| \leq 1, \quad \left|g_i^{'}(z)\right| \leq 1, \quad \left|\frac{zh_i^{'}(z) - h_i(z)}{zh_i(z)}\right| \leq 1, \quad \left|\frac{zk_i^{'}(z) - k_i(z)}{zh_i(z)}\right| \leq 1, \tag{23}$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\frac{\sum_{i=1}^n (|\alpha_i - 1| + |\beta_i|)}{\prod_{i=1}^n (|\alpha_i - 1| |\beta_i|)} < 1, \tag{24}$$

$$\prod_{i=1}^n (|\alpha_i - 1| |\beta_i|) \leq \frac{1}{\underset{|z|\leq 1}{max}\left[2\left(1 - |z|^2\right)|z| \frac{|z|+|k|}{1+|k||z|}\right]}, \tag{25}$$

where

$$|k| = \frac{\left|\sum_{i=1}^n \left[(\alpha_i - 1)(2a_{2i} + 1) + \beta_i (c_{2i} + d_{2i})\right]\right|}{2\prod_{i=1}^n (|\alpha_i - 1| |\beta_i|)}, \tag{26}$$

then the integral operator $\mathcal{V}_n$, defined by

$$\mathcal{V}_n(z) = \int_0^z \prod_{i-1}^{n} \left[\left(f_i{}'(t) e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)} \right)^{\beta_i} \right] dt, \tag{27}$$

is in the class $\mathcal{S}$.

Remark 3. *Taking in (27) $\beta_i = 0$, we obtain a result that was defined in [9].*

Letting $\delta = 1$ and $\beta_i = 0$ in Theorem 2.1, we have the following corollary:

Corollary 8. *Let $\gamma, \alpha_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $f_i, g_i, h_i, k_i \in \mathcal{S}$, $f_i(z) = z + a_{2i}z^2 + \dots$, $g_i(z) = z + b_{2i}z^2 + \dots$, $h_i(z) = z + c_{2i}z^2 + \dots$, $k_i(z) = z + d_{2i}z^2 + \dots$, $i = \overline{1,n}$. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \le 1, \quad \left| g_i'(z) \right| \le 1,$$

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \le 1, \quad \left| g_i'(z) \right| \le 1, \quad \left| \frac{h_i''(z)}{h_i'(z)} \right| \le 1, \quad \left| \frac{k_i''(z)}{k_i'(z)} \right| \le 1, \tag{28}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\frac{\sum_{i=1}^{n} \left(|\alpha_i - 1| + |\gamma_i| \right)}{\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\gamma_i| \right)} < 1, \tag{29}$$

$$\prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\gamma_i| \right) \le \frac{1}{\underset{|z| \le 1}{max} \left[2 \left(1 - |z|^2 \right) |z| \, \frac{|z| + |k|}{1 + |k||z|} \right]}, \tag{30}$$

where

$$|k| = \frac{\left| \sum_{i=1}^{n} \left[(\alpha_i - 1)(2a_{2i} + 1) + 2\gamma_i (c_{2i} + d_{2i}) \right] \right|}{2 \prod_{i=1}^{n} \left(|\alpha_i - 1| \, |\gamma_i| \right)}, \tag{31}$$

then the integral operator $\mathcal{W}_n$, defined by

$$\mathcal{W}_n(z) = \int_0^z \prod_{i-1}^{n} \left[\left(f_i{}'(t) e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i{}'(t))}{k_i{}'(t)} \right)^{\gamma_i} \right] dt, \tag{32}$$

is in the class $\mathcal{S}$.

Remark 4. *To the integral operator given by (32) if we take $\gamma_i = 0$, we get the same result proven in [9].*

Putting $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 9. *Let* $\gamma, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\gamma > 0$ *and* $h_i, k_i \in \mathcal{S}$, $h_i(z) = z + c_{2i}z^2 + ...$, $k_i(z) = z + d_{2i}z^2 + ...$, $i = \overline{1,n}$. *If*

$$\left|\frac{zh_i'(z) - h_i(z)}{zh_i(z)}\right| \le 1, \quad \left|\frac{zk_i'(z) - k_i(z)}{zh_i(z)}\right| \le 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le 1, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le 1, \tag{33}$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\frac{\sum_{i=1}^{n}(|\beta_i| + |\gamma_i|)}{\prod_{i=1}^{n}(|\beta_i|\,|\gamma_i|)} < 1, \tag{34}$$

$$\prod_{i=1}^{n}(|\beta_i|\,|\gamma_i|) \le \frac{1}{\underset{|z|\le 1}{max}\left[2\left(1-|z|^2\right)|z|\,\frac{|z|+|k|}{1+|k||z|}\right]}, \tag{35}$$

where

$$|k| = \frac{\left|\sum_{i=1}^{n}\left[\beta_i\left(c_{2i}+d_{2i}\right) + 2\gamma_i\left(c_{2i}+d_{2i}\right)\right]\right|}{2\prod_{i=1}^{n}(|\beta_i|\,|\gamma_i|)}, \tag{36}$$

then the integral operator $\mathcal{I}_n$, *defined by*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i-1}^{n}\left[\left(\frac{h_i(t)}{k_i(t)}\right)^{\beta_i}\left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\gamma_i}\right]dt\ , \tag{37}$$

is in the class $\mathcal{S}$.

Remark 5. *The integral operator given by (37) is another known result proven in [15].*

The next corollary is a consequence of Theorem 2.2:

Corollary 10. *Let* $\delta, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $c = Re\delta > 0$ *and* $f_i, g_i, h_i, k_i \in \mathcal{S}$, $f_i(z) = z + a_{2i}z^2 + ...$, $g_i(z) = z + b_{2i}z^2 + ...$, $h_i(z) = z + c_{2i}z^2 + ...$, $k_i(z) = z + d_{2i}z^2 + ...$, $i = \overline{1,n}$ *and* M_i, N_i, P_i *positive real numbers. If*

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \le 1, \quad \left|\frac{zg_i'(z)}{g_i(z)}\right| \le M_i, \quad |g_i(z)| \le 1,$$

$$\left|\frac{zh_i'(z)}{h_i(z)}\right| \le N_i, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \le P_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \le 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \le 1, \tag{38}$$

and

$$\left|\sum_{i=1}^{n}\left[(\alpha_i - 1)(2a_{2i} + b_{2i}) + \beta_i(c_{2i} + d_{2i} + 2) + 2\gamma_i(c_{2i} + d_{2i})\right]\right| \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \tag{39}$$

$$\left|\sum_{i=1}^{n}\left[(\alpha_i-1)\left(2a_{2i}+b_{2i}\right)+\beta_i\left(c_{2i}+d_{2i}+2\right)+2\gamma_i\left(c_{2i}+d_{2i}\right)\right]\right|=$$

$$=\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right], \tag{40}$$

then the function $\mathcal{G}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. If we consider (14) and (40) we have $|k|=1$.

Using the inequality (13) we obtain

$$\sum_{i=1}^{n}\left[\left|\alpha_i-1\right|\left(M_i+1\right)+\left|\beta_i\right|\left(N_i+P_i+2\right)+2\left|\gamma_i\right|\right]\le\frac{1}{\max\limits_{|z|\le1}\left[\frac{1-|z|^{2c}}{c}\right]}. \tag{41}$$

We have

$$\max_{|z|\le1}\left[\frac{1-|z|^{2c}}{c}\right]=\frac{2}{(2c+1)^{\frac{2c+1}{2c}}} \tag{42}$$

and from (13), (14) and (42) we obtain (39), the conditions of Theorem 2.2 are satisfied.

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Camelia Bărbatu
Faculty of Mathematics and Computer Science,
"Babeş-Bolyai" University,
Cluj Napoca, Romania
email: camipode@yahoo.com

Daniel Breaz
Department of Mathematics, Faculty of Science,
"1 Decembrie 1918" University,
Alba Iulia, Romania
email: dbreaz@uab.ro