Acta Universitatis Apulensis No. 62/2020
ISSN: 1582-5329 pp. 81-91
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.62.07

A SUBCLASS OF P-VALENTLY CLOSE-TO-CONVEX FUNCTIONS OF ORDER α

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Abstract. In the present paper we introduce and investigate an intresting subclass $\mathcal{K}_p^{(k)}(A, B, \alpha)$ analytic and p-valently close-to-convex functions in the open unit disk $\mathbb{U}$. For functions belonging to $\mathcal{K}_p^{(k)}(A, B, \alpha)$, we derive several properties including coefficient estimates, sufficient condition, distortion theorem. The connection with earliar works are also pointed out.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Analytic function, Close-to-Convex functions, Coefficient estimates, Sufficient condition, Distortion theorem.

1. Introduction and definitions

Let $\mathcal{A}_p$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N}), \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we write $\mathcal{A}_1 = \mathcal{A}$. For any two analytic functions f and g in $\mathbb{U}$, we say that f is subordinate to g in $\mathbb{U}$, written as $f \prec g$ if there exists a Schwarz function w such that $f(z) = g(w(z))$, for $z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then f is subordinate to g iff $f(0) = g(0)$ and $f(U) \subset g(U)$.

For $-1 \le B < A \le 1$, Janowski [6] introduced a class $P[A, B]$ consisting of analytic functions of the form

$$p_1(z) = 1 + b_1 z + b_2 z^2 + \cdots,$$

which satisfies

$$p_1(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in \mathbb{U}).$$

Further for $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, Aouf [1] studied the class $P[A, B, p, \alpha]$ consisting of analytic functions of the form

$$p(z) = p + c_1 z + c_2 z^2 + \cdots ,$$

satisfying

$$p(z) \prec \frac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz} \qquad (z \in \mathbb{U}).$$

A simple calculation shows that if $p \in P[A, B, p, \alpha]$ then there exist $p_1 \in P[A, B]$ such that

$$p(z) = (p - \alpha)p_1(z) + \alpha.$$

Let $\mathcal{S}_p(A, B, \alpha)$ be the class of the function of $f \in \mathcal{A}_p$ satisfying

$$\frac{zf'(z)}{f(z)} \in P[A, B, p, \alpha] \qquad (z \in \mathbb{U})$$

and $\mathcal{C}_p(A, B, \alpha)$ be the class of the function of $f \in \mathcal{A}_p$ for which

$$1 + \frac{zf''(z)}{f'(z)} \in P[A, B, p, \alpha] \qquad (z \in \mathbb{U}).$$

The classes $\mathcal{S}_p(A, B, \alpha)$ and $\mathcal{C}_p(A, B, \alpha)$ include several well known subclasses of p-valent starlike and convex functions as special cases. Specially for the class $\mathcal{S}_p(A, B, \alpha)$, we see that $\mathcal{S}_p(1, -1, \alpha) = \mathcal{S}_p^*(\alpha)$, $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$ and $\mathcal{S}_1(A, B, 0) = \mathcal{S}^*(A, B)$. Note that $\mathcal{S}_p^*(\alpha)$, the class of p-valent starlike function of order α, was studied by Goluzina [5], and $\mathcal{S}^*(A, B)$ was introduced by Janowaski [6]. Several properties including coefficient bounds for the class $\mathcal{S}_p(A, B, \alpha)$ are earlier studied and investigated by Aouf [1] and Sahoo and Sharma [10]. In a similar manner a class $\mathcal{K}_p(A, B, \alpha)$ for p-valent close-to-convex function can be defined as:

A function $f \in \mathcal{A}_p$ is said to belong to the class $\mathcal{K}_p(A, B, \alpha)$ if it satisfies the inclusion relation

$$\frac{zf'(z)}{g(z)} \in P[A, B, p, \alpha] \qquad (z \in \mathbb{U}), \tag{2}$$

where $g \in \mathcal{S}_p^*$.

Recently, Bulut [2] discussed a class $\mathcal{K}_s^{(k)}(\gamma, p)$ for analytic and p-valently close-to-convex functions. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_s^{(k)}(\gamma, p)$ if there exist a function $g \in S_p^*(\frac{(k-1)p}{k})$ ($k \in N$ is a fixed integer) such that

$$Re\Big(\frac{z^{(k-1)p+1}f'(z)}{g_k(z)}\Big) > \gamma \quad (z \in \mathbb{U};\ 0 \leq \gamma < p),$$

where g_k is defined by the equality

$$g_k(z) = \prod_{v=0}^{k-1} \varepsilon^{-vp} g(\varepsilon^v z); \quad \varepsilon = e^{\frac{2\pi\iota}{k}}. \tag{3}$$

Here assuming $g \in S_p^*(\frac{(k-1)p}{k})$ makes $\frac{g_k(z)}{z^{(k-1)p}}$ a p-valant starlike function which in turn implies the close-to-convexity of f. Recently several similar classes of $\mathcal{K}_s^{(k)}(\gamma, p)$ for analytic function have been defined and investigated, some of them we refer to [3, 7, 8, 12, 13, 14, 15, 16].

Motivated essentially by the works of Bulut [2] and Aouf [1], we introduce here a new class $\mathcal{K}_p^{(k)}(A, B, \alpha)$ for p-valently close-to-convex functions in the following manner:

Definition 1. *For $0 \le \alpha < 1$ and $-1 \le B < A \le 1$, a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{K}_p^{(k)}(A, B, \alpha)$, if there exist a function $g \in S_p^*(\frac{(k-1)p}{k})$ ($k \in N$ is a fixed integer) such that*

$$\frac{z^{(k-1)p+1} f'(z)}{g_k(z)} \in P[A, B, p, \alpha]$$

or equivalently

$$\frac{z^{(k-1)p+1} f'(z)}{g_k(z)} \prec \frac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz}, \tag{4}$$

where g_k is defined by the equality (3).

For k=1, the class $\mathcal{K}_p^{(k)}(A, B, \alpha)$ reduces to the class of functions $\mathcal{K}_p(A, B, \alpha)$. It may be pointed out here that, the class $\mathcal{K}_p^{(k)}(A, B, \alpha)$ generalizes several previously studied function classes. We deem it proper to demonstrate briefly the relevant connections with some of the well-known classes. Indeed, we have

(i) $\mathcal{K}_p^{(k)}(1, -1, \alpha) = \mathcal{K}_p^{(k)}(\alpha)$ (see [2])

(ii) $\mathcal{K}_1^{(2)}(1, -1, 0) = \mathcal{K}_s$ (see [4])

(iii) $\mathcal{K}_1^{(2)}(1, -1, \alpha) = \mathcal{K}_s(\alpha)$ (see [8])

(iv) $\mathcal{K}_1^{(k)}(1, -1, \alpha) = \mathcal{K}_s^{(k)}(\alpha)$ (see [11])

(v) $\mathcal{K}_1^{(k)}(\beta, -\beta\gamma, \alpha) = \mathcal{K}_s^{(k)}(\gamma, \alpha, \beta)$ (see [12]).

In the present investigation, we derive several properties including coefficient estimates, sufficient condition and distortion theorem for function belonging to the class $\mathcal{K}_p^{(k)}(A, B, \alpha)$.

2. Main results

In order to prove our main results for the function class $\mathcal{K}_p^{(k)}(A,B,\alpha)$, we need the following lemma:

Lemma 1. *[2] If*

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \in S_p^*\Big(\frac{(k-1)p}{k}\Big)$$

and

$$g_k(z) = \prod_{v=0}^{k-1} \varepsilon^{-vp} g(\varepsilon^v z); \quad \varepsilon = e^{\frac{2\pi\iota}{k}},$$

then

$$G(z) = \frac{g_k(z)}{z^{(k-1)p}} = z^p + \sum_{n=1}^{\infty} B_{p+n} z^{p+n} \in S_p^*. \tag{5}$$

Theorem 2. *Let $0 \le \alpha < 1$, $-1 \le B < A \le 1$, f given by (1) and $g \in S_p^*(\frac{(k-1)p}{k})$ are such that the condition (4) holds. Then for $n \ge 1$, we have*

$$\begin{aligned} &|(p+n)a_{p+n} - pB_{p+n}|^2 - (A-B)^2(p-\alpha)^2 \\ &\le \sum_{n=1}^{m-1} \Big\{(B^2-1)(p+n)^2|a_{p+n}|^2 - 2\big(B\{pB+(A-B)(p-\alpha)\}-p\big)(p+n)|a_{p+n}B_{p+n}| \\ &\qquad + [\big(pB+(A-B)(p-\alpha)\big)^2 - p^2]|B_{p+n}|^2\Big\}. \end{aligned} \tag{6}$$

Proof. Let $f \in \mathcal{K}_p^{(k)}(A,B,\alpha)$, then by definition of subordination, we have

$$\frac{zf'(z)}{G(z)} = \frac{p + [pB + (A-B)(p-\alpha)]w(z)}{1 + Bw(z)} \qquad (z \in \mathbb{U}),$$

where w is an analytic functions in $\mathbb{U}$ with $w(z) \le 1$ for $z \in \mathbb{U}$ and G is given by (5). From the above equality, we obtain

$$zf'(z) - pG(z) = \Big[\Big(pB + (A-B)(p-\alpha)\Big)G(z) - Bzf'(z)\Big]w(z)\,. \tag{7}$$

Now, we put $w(z) = \sum_{n=1}^{\infty} w_n z^n$, and substitute the series expansions (1) and (5), and cancel the factor z^p on both sides, we obtain

$$\sum_{n=1}^{\infty}[(p+n)a_{p+n} - pB_{p+n}]z^n$$

$$= \Big\{[pB+(A-B)(p-\alpha)]\big(z^p+\sum_{n=1}^{\infty}B_{p+n}z^{p+n}\big)-B\big(pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}\big)\Big\}\sum_{n=1}^{\infty}w_n z^n \ . \tag{8}$$

Equating the coefficient of z^m in (8), gives us
$(p+m)a_{p+m} - pB_{p+m}$

$$= (A-B)(p-\alpha)w_m + \big((pB+(A-B)(p-\alpha)\big)B_{p+1} - B(p+1)a_{p+1}\big)w_{m-1} + ...$$

$$+ \big((pB+(A-B)(p-\alpha)B_{p+m-1} - B(p+m-1)a_{p+m-1}\big)w_1$$

$$= (A-B)(p-\alpha)w_m + \sum_{n=1}^{m-1}\big((pB+(A-B)(p-\alpha)\big)B_{p+n} - B(p+n)a_{p+n}\big)w_{m-n}$$

which shows that $(p+m)a_{p+m} - pB_{p+m}$ on the right hand side of (8) depends only on $a_{p+1}, B_{p+1}, a_{p+2}, B_{p+2}, ..., a_{p+m-1}, B_{p+m-1}$, of left-hand side. Hence, for $n \geq 1$, we can write

$$\Big[(A-B)(p-\alpha) + \sum_{n=1}^{m-1}\big((pB+(A-B)(p-\alpha)\big)B_{p+n} - B(p+n)a_{p+n}\big)z^n\Big]\sum_{n=1}^{\infty}w_n z^n$$

$$= \sum_{n=1}^{m}\big((p+n)a_{p+n} - pB_{p+n}\big)z^n + \sum_{n=m+1}^{\infty}A_n z^n. \tag{9}$$

Using the fact that $|w(z)| < 1$ for all $z \in \mathbb{U}$ in (9), it reduce to inequality

$$\left|(A-B)(p-\alpha) + \sum_{n=1}^{m-1}\big((pB+(A-B)(p-\alpha)\big)B_{p+n} - B(p+n)a_{p+n}\big)z^n\right|$$

$$> \left|\sum_{n=1}^{m}\big((p+n)a_{p+n} - pB_{p+n}\big)z^n + \sum_{n=m+1}^{\infty}A_n z^n\right|.$$

On squaring the above inequality and integrating along $|z| = r < 1$, we obtain

$$\int_0^{2\pi}\left|(A-B)(p-\alpha) + \sum_{n=1}^{m-1}\big((pB+(A-B)(p-\alpha)\big)B_{p+n} - B(p+n)a_{p+n}\big)r^n e^{in\theta}\right|^2 d\theta$$

$$> \int_0^{2\pi}\left|\sum_{n=1}^{m}\big((p+n)a_{p+n} - pB_{p+n}\big)r^n e^{in\theta} + \sum_{n=m+1}^{\infty}A_n r^n e^{in\theta}\right|^2 d\theta \ .$$

Using the Paraseval's inequality, we get

$$|(A-B)(p-\alpha)|^2+\sum_{n=1}^{m-1}\left|\big(pB+(A-B)(p-\alpha)\big)B_{p+n}-B(p+n)a_{p+n}\right|^2 r^{2n}$$

$$>\sum_{n=1}^{m}\left|(p+n)a_{p+n}-pB_{p+n}\right|^2 r^{2n}+\sum_{n=m+1}^{\infty}|A_n|^2 r^{2n}.$$

Letting $r\to 1$ in this inequality, we reach to

$$\sum_{n=1}^{m}\left|(p+n)a_{p+n}-pB_{p+n}\right|^2\le|(A-B)(p-\alpha)|^2+\sum_{n=1}^{m-1}\left|\big(pB+(A-B)(p-\alpha)\big)B_{p+n}-B(p+n)a_{p+n}\right|^2.$$

Hence we deduce that

$$|(p+n)a_{p+n}-pB_{p+n}|^2-(A-B)^2(p-\alpha)^2$$

$$\le\sum_{n=1}^{m-1}\Big\{(B^2-1)(p+n)^2|a_{p+n}|^2-2\big(B\{pB+(A-B)(p-\alpha)\}-p\big)(p+n)|a_{p+n}B_{p+n}|$$

$$+\left[\big(pB+(A-B)(p-\alpha)\big)^2-p^2\right]|B_{p+n}|^2\Big\},$$

and thus we obtain the inequality (6). Which completes the proof of Theorem 2.

Theorem 3. *For $0\le\alpha<1$, $-1\le B<A\le 1$, f given by (1) and $g\in S_p^*(\frac{(k-1)p}{k})$ such that*

$$(1-B)\sum_{n=1}^{\infty}(p+n)|a_{p+n}|+[p+pB+(A-B)(p-\alpha)]\sum_{n=1}^{\infty}|B_{p+n}|<(A-B)(p-\alpha),\quad(10)$$

then $f\in\mathcal{K}_p^{(k)}(A,B,\alpha)$.

Proof. For f given by (1) and G defined by (5), we set

$$\Lambda=\left|zf^{'}(z)-pG(z)\right|-\left|[pB+(A-B)(p-\alpha)]G(z)-Bzf^{'}(z)\right|$$

$$=\Big|\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}-p\sum_{n=1}^{\infty}B_{p+n}z^{p+n}\Big|$$

$$-\Big\{\Big|-B\big(pz^p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n}\big)+[pB+(A-B)(p-\alpha)]\big(z^p+\sum_{n=1}^{\infty}B_{p+n}z^{p+n}\big)\Big|\Big\}$$

$$\Lambda \leq \sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n} + p\sum_{n=1}^{\infty}|B_{p+n}||z|^{p+n}$$
$$-\Big((A-B)(p-\alpha)|z|^p + B\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n} - [pB+(A-B)(p-\alpha)]\sum_{n=1}^{\infty}|B_{p+n}||z|^{p+n}\Big)$$
$$= -(A-B)(p-\alpha)|z|^p + (1-B)\sum_{n=1}^{\infty}(p+n)|a_{p+n}||z|^{p+n} + [p+pB+(A-B)(p-\alpha)]\sum_{n=1}^{\infty}|B_{p+n}||z|^{p+n}$$
$$= \Big(-(A-B)(p-\alpha) + (1-B)\sum_{n=1}^{\infty}(p+n)|a_{p+n}| + [p+pB+(A-B)(p-\alpha)]\sum_{n=1}^{\infty}|B_{p+n}|\Big)|z|^p.$$

From the inequality (10), we obtain that $\Lambda < 0$.
Thus we have

$$\Big|zf^{'}(z) - pG(z)\Big| < \Big|[pB + (A-B)(p-\alpha)]G(z) - Bzf^{'}(z)\Big|.$$

Hence $f \in \mathcal{K}_p^{(k)}(A, B, \alpha)$. This completes the proof of Theorem 3.

Theorem 4. *If $f \in \mathcal{K}_p^{(k)}(A, B, \alpha)$, then for $|z| = r$ $(0 \leq r < 1)$, we have*

$$(i)\ \frac{[p - (pB + (A-B)(p-\alpha))r]r^{p-1}}{(1-Br)(1+r)^{2p}} \leq |f^{'}(z)| \leq \frac{[p + (pB + (A-B)(p-\alpha))r]r^{p-1}}{(1+Br)(1-r)^{2p}} \quad (11)$$

$$(ii)\ \int_0^r \frac{[p - (pB + (A-B)(p-\alpha))\tau]\tau^{p-1}}{(1-B\tau)(1+\tau)^{2p}}d\tau \leq |f(z)| \leq \int_0^r \frac{[p + (pB + (A-B)(p-\alpha))\tau]\tau^{p-1}}{(1+B\tau)(1-\tau)^{2p}}d\tau. \quad (12)$$

Proof. If $f \in \mathcal{K}_p^{(k)}(A, B, \alpha)$, then there exist a function $g \in S_p^*(\frac{(k-1)p}{k})$ such that (4) holds.
(i) Since G given by (5) is p-valently starlike function. Hence from [1, Theorem 1], we have

$$\frac{r^p}{(1+r)^{2p}} \leq |G(z)| \leq \frac{r^p}{(1-r)^{2p}}, \qquad (|z| = r\ (0 \leq r < 1)). \quad (13)$$

Let us define Ψ by

$$\Psi(z) = \frac{zf^{'}(z)}{G(z)} \qquad (z \in \mathbb{U}),$$

then from (4), we have

$$\frac{[p - (pB + (A-B)(p-\alpha))r]}{(1-Br)} \leq |\Psi(z)| \leq \frac{[p + (pB + (A-B)(p-\alpha))r]}{(1+Br)} \qquad (z \in \mathbb{U}). \quad (14)$$

Thus from (13) and (14), we get the inequalities (11).
(ii) Let $z = re^{\iota\theta}$ $(0 < r < 1)$. If l denotes the closed line-segment in the complex $\zeta-$plane from $\zeta = 0$ and $\zeta = z$, i.e. $l = [0, re^{\iota\theta}]$, then we have

$$f(z) = \int_l f^{'}(\zeta)d\zeta = \int_0^r f^{'}(\tau e^{\iota\theta})e^{\iota\theta}d\tau.$$

Thus, by using the upper estimate in (11), we have

$$|f(z)| = \Big|\int_l f^{'}(\zeta)d\zeta\Big| \le \int_0^r |f^{'}(\tau e^{\iota\theta})|d\tau \le \int_0^r \frac{[p + (pB + (A-B)(p-\alpha))\tau]\tau^{p-1}}{(1+B\tau)(1-\tau)^{2p}}d\tau,$$

which yields the right hand of the inequality in (12).
In order to prove the lower bound in (12), let $z_0 \in \mathbb{U}$ with $|z_0| = r$ $(0 < r < 1)$, such that $|f(z_0)| = \min\{|f(z)| : |z| = r\}$.
It is sufficient to prove that the left-hand side inequality holds for this point z_0. Moreover, we have

$$|f(z)| \ge |f(z_0)| \qquad (|z| = r\ (0 \le r < 1)).$$

The image of the closed line-segment $l_0 = [0, f(z_0)]$ by f^{-1} is a piece of arc Γ included in the closed disc $\mathbb{U}_r = \{z : z \in \mathbb{C}\ \ and\ \ |z| \le r\ \ (0 \le r < 1)\}$, that is, $\Gamma = f^{-1}(l_0) \subset \mathbb{U}_r$. Hence, in accordance with (11),we obtain

$$|f(z_0)| = \int_{l_0} |dw| = \int_\Gamma |f^{'}(\zeta)||d\zeta| \ge \int_0^r \frac{[p - (pB + (A-B)(p-\alpha))\tau]\tau^{p-1}}{(1-B\tau)(1+\tau)^{2p}}d\tau.$$

This finishes the proof of the inequality (12).

Remark 1. *Note that the results obtained in above Theorems generalize several previously studied results, and we will show some of the interesting particular cases as follows:*
(i)For A = 1 and B = -1, Theorem 2, 3 and 4 give recent results by Bulut [2].
(ii)For p = 1, A = 1 and B = -1, Theorem 2, 3 and 4 provide results by seker [11].
(iii)For p = 1, A = β and B = $-\beta\gamma$, Theorem 2, 3 and 4 provide results by seker and Cho [12].

Let $f(z) = \Sigma_{n=1}^{\infty} a_n z^n$ and $g(z) = \Sigma_{n=1}^{\infty} b_n z^n$ be two analytic functions defined in $\mathbb{D}$. Then there Hadamard product (or convolution) is the function $(f * g)(z)$ defined by

$$(f * g)(z) = \Sigma_{n=1}^{\infty} a_n b_n z^n.$$

The classes of starlike and convex functions are closed under convolution with convex function. The following lemma is required for our next result.

Lemma 5. *[9] Let ψ and ϕ be convex in $\mathbb{U}$ and suppose $f \prec \psi$, then*

$$f * \phi = \psi * \phi.$$

Theorem 6. *If $f \in \mathcal{K}_p^{(k)}(A, B, \alpha)$, then there exists*

$$q(z) \prec \frac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz}$$

such that for all s and t with $|s| \leq 1$ and $|t| \leq 1$,

$$\frac{t^{p-1} f^{'}(sz) q(tz)}{s^{p-1} f^{'}(tz) q(sz)} \prec \left(\frac{1 - tz}{1 - sz}\right)^{2p}. \tag{15}$$

Proof. Let $f \in \mathcal{K}_p^{(k)}(A, B, \alpha)$, then there exist $g \in S_p^*(\frac{(k-1)p}{k})$. Suppose

$$q(z) = \frac{zf^{'}(z)}{G(z)}\ , \tag{16}$$

where G given by (5). Then by (3), we have

$$q(z) \prec \frac{p + [pB + (A - B)(p - \alpha)]z}{1 + Bz}.$$

Logarithmic derivative of (16), implies

$$\frac{zf^{''}(z)}{f^{'}(z)} - \frac{zq^{'}(z)}{q(z)} + 1 - p = \frac{zG^{'}(z)}{G(z)} - p\ . \tag{17}$$

Since $G \in \mathcal{S}_p^*$, so that

$$\frac{1}{p} \frac{zG^{'}(z)}{G(z)} \prec \frac{1 + z}{1 - z}. \tag{18}$$

From (17) and (18), we have

$$\frac{zf^{''}(z)}{f^{'}(z)} - \frac{zq^{'}(z)}{q(z)} + 1 - p \prec \frac{2pz}{1 - z}\ .$$

For s and t such that $|s| \leq 1$ and $|t| \leq 1$, the function

$$h(z) = \int_0^z \frac{s}{1 - su} - \frac{t}{1 - tu} du$$

is convex in $\mathbb{U}$. Applying Lemma 5, we have

$$\left(\frac{zf^{''}(z)}{f^{'}(z)} - \frac{zq^{'}(z)}{q(z)} + 1 - p\right) * h(z) \prec \frac{2pz}{1 - z} * h(z).$$

Given any function k analytic in $\mathbb{U}$, with $k(0) = 0$, we have

$$(k * h)(z) = \int_{tz}^{sz} k(u)\frac{du}{u} \qquad (z \in \mathbb{U}),$$

which implies that

$$log\Big[\frac{(sz)^{1-p}f'(sz)q(tz)}{(tz)^{1-p}f'(tz)q(sz)}\Big] \prec \; log\Big[\frac{1-tz}{1-sz}\Big]^{2p},$$

which is equivalent to (15). This completes the proof of Theorem 6.

Conflicts of Interest

The author declare that there is no conflict of interest regarding the publication of this paper.

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