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A NOTE ON HYPERELASTIC CURVES

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Abstract. We study a constrained variational problem whose solutions are called hyperelastic curves. We derive a differential equation for critical points of the hyperelastic curvature energy action in $2-$dimensional null cone and we completely solve this equation. Finally, we construct some coordinate systems and express hyperelastic curves in the null cone.

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1. Introduction

Elastic curve or elastica determined by Euler in 1744 is extremal of the bending energy functional $\int \left(\kappa^2+\lambda\right) ds$, where κ and λ are respectively the curvature of a curve and the Lagrange multiplier, acting on suitable space of curves [6, 10]. Over the last two decades, this curve model has been studied and developed by many authors. In particular, the method presented by Langer and Singer [6] is a very useful method for studying and classifying elastic curves. Using their approach, elastic curve has been generalized in various ways. One of these generalizations is hyperelastic curve (or known $r-$elastic curve). The hyperelastic curve is defined as a critical point of the functional $\int (\kappa^r+\lambda)\, ds$ for any natural number $r \geq 2$, if $\lambda = 0$, then the critical point is called free hyperelastic curve [1, 2]. The (free) hyperelastic curve has been employed to furnish reduction methods in constructing Chen-Willmore submanifolds (see for detail information [1, 2, 3, 4].

Elastic curves and some generalization of elastic curves have been considered in non-Euclidean spaces. For example, the authors in [11] studied elastic curves in a $2-$dimensional null cone. However, there are no articles in the literature about the generalization of elastic curves in the null cone. Therefore, the main purpose of this paper is to examine hyperelastic curves as a generalization of elastic curves in the $2-$dimensional null cone. In accordance with this purpose, we present the variational problem associated with the cone curvature energy functional in the null cone. Then, we derive the Euler-Lagrange equation for hyperelastic curve in the null cone and completely solve the differential equation by quadratures. Therefore, we find a Killing vector field along the critical curve. Finally, using the constructed Killing vector field, we seek solutions of the derivative equations of asymptotic orthonormal frame field according to choosing different coordinates in the null cone.

2. Geometrical Set up

In this section we give some explanatory materials including short information about frame fields of curves in a $2-$dimensional null cone $\mathbb{Q}^2$ in Minkowski $3-$space $\mathbb{R}^3_1$.

Minkowski $3-$space $\mathbb{R}^3_1$ is the metric space equipped with Minkowski metric given by $\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3$ for vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in Euclidean $3-$space $\mathbb{R}^3$. Because of the structure of Minkowski metric, a vector u in $\mathbb{R}^3_1$ is said a spacelike vector if $\langle u, u\rangle > 0$ (or $u = 0$), a timelike vector if $\langle u, u\rangle < 0$ and a null (or lightlike) vector if $\langle u, u\rangle = 0$ and $u \neq 0$. In relation to this case, a smooth curve in $\mathbb{R}^3_1$ is a spacelike (resp., timelike and lightlike), if its velocity vector is a spacelike (resp., timelike and lightlike). On the other hand, a surface in $\mathbb{R}^3_1$ is non-degenerate (or degenerate) if induced metric on its tangent plane is non-degenerate (or degenerate). We will deal with a $2-$dimensional null cone, which is a degenerate surface in Minkowski $3-$space $\mathbb{R}^3_1$. The $2-$dimensional null cone of $\mathbb{R}^3_1$ is the set of all null vectors of $\mathbb{R}^3_1$ is given by

$$\mathbb{Q}^2 = \mathbb{Q}^2_1(0) = \{u \in \mathbb{R}^3_1 : \langle u, u\rangle = 0\} - \{(0, 0, 0)\}.$$

As is well known, all curves in a $2-$dimensional null cone $\mathbb{Q}^2$ are spacelike [7, 8, 9].

Let γ be a spacelike curve in a $2-$dimensional null cone $\mathbb{Q}^2$. If we take the arc length parameter s of the curve γ as the parameter, then we have the spacelike unit tangent vector field $T(s) = \gamma'(s) = \frac{d\gamma(s)}{ds}$. Now we can choose the normal vector field $N(s)$ satisfying the following conditions

$$\langle \gamma(s), N(s)\rangle = 1,\ \langle N(s), N(s)\rangle = \langle T(s), N(s)\rangle = 0. \tag{2.1}$$

Then, we can find $N(s) = -\gamma''(s) - \frac{1}{2} < \gamma''(s), \gamma''(s) > \gamma(s)$. The frame field $\{\gamma, T, N\}$ is called an asymptotic orthonormal frame field along the curve γ in $\mathbb{Q}^2$. Derivative equations of the asymptotic orthonormal frame field are given by

$$\begin{gathered} \gamma'(s) = T(s), \\ T'(s) = \kappa(s)\gamma(s) - N(s), \\ N'(s) = -\kappa(s)T(s) \end{gathered} \tag{2.2}$$

where $\kappa(s) = -\frac{1}{2}\langle \gamma''(s), \gamma''(s)\rangle$ is the cone curvature function [7].

Let $\gamma(t) : I \to \mathbb{Q}^2$ be a spacelike curve in the null cone $\mathbb{Q}^2$ parametrized by an arbitrary parameter t. $V = V(t)$ will denote the tangent vector field to γ, T the unit tangent vector field and v the speed $v(t) = \|V(t)\| =< V(t), V(t) >^{\frac{1}{2}}$. Now, we will also denote by γ a variation

$$\begin{array}{rcl} \gamma : (-\varepsilon, \varepsilon) \times I & \to & \mathbb{Q}^2 \\ (w,t) & \longrightarrow & \gamma(w,t) = \gamma_w(t) \end{array} \tag{2.3}$$

with $\gamma(0,t) = \gamma(t)$. Associated with such a variation is the variation vector field $W = W(t) = (\frac{\partial\gamma}{\partial w})(0,t)$ along the curve γ. We can write $V = V(w,t) = (\partial\gamma/\partial t)(w,t)$, $W = W(w,t) = (\partial\gamma/\partial w)(w,t)$, $T = T(w,t)$, $v = v(w,t)$, etc. with the obvious meaning. Then we can write $\gamma(s)$, $\kappa(w,s)$, $V(s)$ etc. for the arc length parameter s, where $s \in [0,\ell]$ and ℓ is the arc length of γ.

By a direct computation, we have the following lemma (similar to that of [6]).

Lemma 1. *Let γ be a reparametrized curve with arc length s, $0 \le s \le \ell$, in a $2-$dimensional null cone $\mathbb{Q}^2$ and γ_w be a variation with a variation vector field W. Then the following formulas are satisfied:*

i) $< W, \gamma >= 0$,
ii) $[V, W] = 0$,
iii) $W(v) =< W_s, T > v$,
iv) $[W, T] = - < W_s, T > T$,
iv) $W(\kappa) = -4 < W_s, T > \kappa - < W_{ss}, \kappa\gamma - N >$.

3. The Euler-Lagrange Equation Characterizing Hyperelastic Curve

We will now give that how to find hyperelastic curve between spacelike curves in a $2-$dimensional null cone $\mathbb{Q}^2$. For this purpose, we first consider the family of C^∞ spacelike curves as follows

$$\begin{array}{rl} \mathcal{L}_{v_0,v_1} = & \{\gamma | \gamma : [0,\ell] \to \mathbb{Q}^2 \\ & \gamma(i\ell) = p_i,\ p_i \in \mathbb{Q}^2,\ \gamma'(i\ell) = v_i, \\ & v_i \in T_{p_i}\mathbb{Q}^2,\ \ \|\gamma'\| = 1,\ i = 0,1\}. \end{array}$$

Therefore a hyperelastic curve is a critical point of the following functional

$$\begin{array}{rcl} \mathcal{F}^\lambda : \ \mathcal{L}_{v_0,v_1} & \to & [0,\infty) \\ \gamma & \to & \mathcal{F}^\lambda(\gamma) = \int\limits_0^\ell (\kappa^r + \lambda)\, ds, \end{array} \tag{3.1}$$

where λ is the Lagrange multiplier.

We suppose that γ is an extremal of the functional $\mathcal{F}^{\lambda}$. Since W is a variation vector field along γ, we have

$$\partial\mathcal{F}^{\lambda}(W) = \frac{d}{dw}\mathcal{F}^{\lambda}(\gamma_w)|_{w=0} = 0,$$

that is

$$\begin{aligned} 0 &= \int_0^{\ell} \left(r\kappa^{r-1}\left(-4 < W_s, T > \kappa - < W_{ss}, \kappa\gamma - N >\right)\right. \\ &\quad \left. + \left(\kappa^r + \lambda\right) < W_s, T >\right) ds. \end{aligned}$$

By the integrating by parts, we get

$$0 = \int_0^{\ell} < E, W > ds + S(\gamma, W)_0^{\ell}, \tag{3.2}$$

where

$$S(\gamma, W) = -[< W, ((2r-1)\,\kappa^r - \lambda)T + r\,(r-1)\,\kappa^{r-2}\kappa_s N > + < W_s, r\kappa^{r-1}(\kappa\gamma - N) >]$$

and

$$E(\gamma) = \left(r\,(r-1)\,\kappa^{r-3}\left((r-2)\,(\kappa_s)^2 + \kappa\kappa_{ss}\right) - (2r-1)\,\kappa^r + \lambda\right) N.$$

Since γ is an extremal of the functional $\mathcal{F}^{\lambda}$, $E(\gamma)$ vanishes identically [5]. Then we obtain the Euler-Lagrange equation

$$r\,(r-1)\,\kappa^{r-3}\left((r-2)\,(\kappa_s)^2 + \kappa\kappa_{ss}\right) - (2r-1)\,\kappa^r + \lambda = 0. \tag{3.3}$$

Therefore, we can give the characterization of a unit speed hyperelastic curve in $\mathbb{Q}^2$ as follows.

Theorem 2. *A unit speed hyperelastic curve in a $2-$dimensional null cone $\mathbb{Q}^2$ is characterized by the Euler-Lagrange equation* (3.3)*.*

4. Integration of the Hyperelastic Curve

In this section we seek the solutions of the Euler$-$Lagrange equation (3.3). If the cone curvature κ is constant, then the constant value found as follows

$$\kappa^r = \frac{\lambda}{(2r-1)}$$

and so, Eq. (2.2) is a system of linear ordinary differential equations with constant coefficients. Therefore, we can directly solve it.

Suppose that the cone curvature κ is not constant. If we multiply $\left(r\left(r-1\right)\kappa^{r-2}\kappa_s\right)$ both sides of Eq. (3.3) and then the the first integral, we are found

$$\left(r\left(r-1\right)\kappa^{r-2}\kappa_s\right)^2+2r\kappa^{r-1}\left(\kappa^r+\lambda\right)-2r^2\kappa^{2r-1}=C, \tag{4.1}$$

where C is a integral constant. Thus the cone curvature κ can be expressed by quadratures

$$\pm\int\frac{r\left(r-1\right)\kappa^{r-2}d\kappa}{\sqrt{C-2r\kappa^{r-1}\left(\kappa^r+\lambda\right)+2r^2\kappa^{2r-1}}}=\int ds.$$

Then, we can give the following theorem.

Theorem 3. *The Euler-Lagrange equation* (3.3) *can be completely solved by quadratures.*

Now, we deal with solve derivative equations of the asymptotic orthonormal frame field (2.2) for a hyperelastic curve in $\mathbb{Q}^2$, and thus we define the Killing vector field along γ in $\mathbb{Q}^2$.

Definition 4. *If a vector field W along a regular curve γ in $\mathbb{Q}^2$ satisfies the conditions $W(v)=W(\kappa)=0$, then W is called a Killing vector field along γ.*

By using Eq. (3.3) and Lemma 1, we can see that the vector field $J_\gamma = -r\left(r-1\right)\kappa^{r-2}\kappa_s\gamma+r\kappa^{r-1}T$ is a Killing vector field along the hyperelastic curve γ. The solution space of a linear system constituted by the equations $W(v)=W(\kappa)=0$ is a 3−dimension. This dimension agrees with the dimension of the isometry group of a 2−dimensional null cone $\mathbb{Q}^2$. Therefore a Killing vector field along a hyperelastic curve γ can extend to a Killing vector field in $\mathbb{Q}^2$ at least locally.

Since the hyperelastic curve is invariant along any Killing vector field W, $S\left(\gamma, W\right)$ is constant along γ. Then, we have

$$S(\gamma,J_\gamma)=-\left(r\left(r-1\right)\kappa^{r-2}\kappa_s\right)^2-2r\kappa^{r-1}\left(\kappa^r+\lambda\right)+2\kappa<J_\gamma,J_\gamma>$$

for Killing vector field J_γ, where

$$<J_\gamma,J_\gamma>=r^2\kappa^{2r-2}.$$

We can give the following theorem.

Theorem 5. If γ *is a hyperelastic curve in a* 2−*dimensional null cone* $\mathbb{Q}^2$, *then it satisfies the Euler-Lagrange equation*

$$-\left(r\left(r-1\right)\kappa^{r-2}\kappa_s\right)^2-2r\kappa^{r-1}\left(\kappa^r+\lambda\right)+2\kappa<J_\gamma,J_\gamma>=-C,$$

for a constant C.

By using the Killing vector field J_γ, we can solve Eq. (2.2) for a hyperelastic curve. We can classify solutions according to the sign of the constant C. Firstly, we suppose that $C > 0$. Now we consider the system of coordinate in $\mathbb{Q}^2$ given by

$$\psi(\rho,\theta) = (\rho, \rho\sinh\theta, \rho\cosh\theta),$$

where $\rho \in (0,+\infty)$, $\theta \in (-\infty,+\infty)$, with the metric $ds^2 = \rho^2 d\theta^2$. Therefore the vector field $\psi_\theta(\rho,\theta) = (0, \rho\cosh\theta, \rho\sinh\theta)$ is a Killing vector field. By using a proper Lorentzian transformation in $\mathbb{Q}^2$, we can suppose

$$J_\gamma = -r\,(r-1)\,\kappa^{r-2}\kappa_s\gamma + r\kappa^{r-1}T = a\psi_\theta,$$

for a pending constant a. The inner product J_γ is found as $< J_\gamma, J_\gamma >= r^2\kappa^{2r-2} = a^2\rho^2$. If γ write as $\gamma(s) = \psi(\rho(s),\theta(s))$, we have

$$T(s) = \gamma'(s) = \rho'(s)\psi_\rho + \theta'(s)\psi_\theta.$$

From the inner product of J_γ and T vector fields, we get $r\kappa^{r-1}\,(\kappa) = a\rho^2\theta'(s)$. On the other hand, we have

$$1 =< \gamma'\,(s)\,, \gamma'\,(s) >= \rho^2\theta'\,(s)^2\,.$$

Computing the cone curvature of $\gamma(s)$ gives

$$\left(r\,(r-1)\,\kappa^{r-2}\kappa_s\right)^2 + 2r\kappa^{r-1}\,(\kappa^r + \lambda) - 2r^2\kappa^{2r-1} = a^2.$$

We can see from Eq. (4.1), $a^2 = C$. Then, we obtain

$$\theta'(s) = \frac{\sqrt{C}}{r\kappa^{r-1}}, \qquad \rho = \frac{r\kappa^{r-1}}{\sqrt{C}}.$$

Secondly, we assume that $C < 0$. Then, for $\theta \in (0, 2\pi)$, $\rho \in (0,+\infty)$, the system in the following

$$\psi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho)$$

gives a coordinate system in a $2-$dimensional null cone $\mathbb{Q}^2$ with the metric $ds^2 = \rho^2 d\theta^2$. So, $\psi_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$ denotes a Killing vector field. By using a proper Lorentzian transformation in $\mathbb{Q}^2$, we can suppose

$$J_\gamma = -\left(r\,(r-1)\,\kappa^{r-2}\kappa_s\right)\gamma + r\kappa^{r-1}T = a\psi_\theta,$$

for a pending constant a. The inner product J_γ is found as $< J_\gamma, J_\gamma >= r^2\kappa^{2r-2} = a^2\rho^2$, where $\rho = \frac{r\kappa^{r-1}}{a}$. We write the curve with regard to the local coordinates as $\gamma(s) = \psi(\rho(s),\theta(s))$, then

$$T = \gamma'(s) = \rho'(s)\psi_\rho + \theta'(s)\psi_\theta.$$

Calculating the inner product of J_γ and T vector fields give $r\kappa^{r-1} = \theta'(s)a\rho^2$and $r\kappa^{r-1}\theta'(s) = a$. Then we find

$$\theta'(s)^2\rho^2 = 1$$

since T is a unit spacelike vector field. Computing the cone curvature of $\gamma(s)$ gives

$$\left(r\left(r-1\right)\kappa^{r-2}\kappa_s\right)^2 + 2r\kappa^{r-1}\left(\kappa+\lambda\right) - 2r^2\kappa^{2r-1} = -a^2.$$

Comparing this formula with the equation (4.1) implies $a^2 = -C$. Taking into consideration ψ_θ and J_γ, we obtain

$$\theta'(s) = \frac{\sqrt{-C}}{r\kappa^{r-1}}, \qquad \rho = \frac{r\kappa^{r-1}}{\sqrt{-C}}.$$

Finally, we suppose that $C = 0$ and so we select the coordinate system in $\mathbb{Q}^2$ given by

$$\psi(\rho,\theta) = (\rho\theta, \frac{1}{2}\rho(1-\theta^2), \frac{1}{2}\rho(1+\theta^2)),$$

where, $\rho \in (0, +\infty)$, $\theta \in (-\infty, +\infty)$ with the metric $ds^2 = \rho^2 d\theta^2$. In this case, the vector fields $\psi_\theta = (\rho, -\rho\theta, \rho\theta)$ is a Killing vector field in $\mathbb{Q}^2$ and we can suppose

$$J_\gamma = -\left(r\left(r-1\right)\kappa^{r-2}\kappa_s\right)\gamma + r\kappa^{r-1}T = \psi_\theta.$$

With the methodology above, we obtain

$$\rho = r\kappa^{r-1}, \qquad \theta'(s) = \frac{1}{r\kappa^{r-1}}.$$

Theorem 6. *The hyperelastic curve in a* $2-$*dimensional null cone* $\mathbb{Q}^2$ *can be expressed by integral explicitly.*

5. Summary and Conclusion

We examined the variational problem of the hyperelastic curve known as a generalization of the elastic curve model in Minkowski $3-$space. There exist different ways to deal with the variational problem for the bending energy functional. Although the study is a generalization of the reference [11], we develop this theory in the $2-$dimensional null cone following the approach using [6]. Killing vector fields on the null cone are also noteworthy in this paper since there is very limited information about null geometry. The solutions of the derived differential equation are solved by using the constructed Killing fields. The results can be summarized as follows:

Corollary 7. *Let γ be a hyperelastic curve with the cone curvature κ in a $2-$dimensional null cone $\mathbb{Q}^2$. Then γ can be parameterized as the following:*

i) If the constant $C > 0$, then

$$\gamma(s) = \frac{r}{\sqrt{C}}(\kappa^{r-1}(s), \kappa^{r-1}(s)\sinh(\frac{\sqrt{C}}{r}\int\frac{ds}{\kappa^{r-1}(s)}), \kappa^{r-1}(s)\cosh(\frac{\sqrt{C}}{r}\int\frac{ds}{\kappa^{r-1}(s)})),$$

ii) If the constant $C < 0$, then

$$\gamma(s) = \frac{r}{\sqrt{-C}}(\kappa^{r-1}(s)\cos(\frac{\sqrt{-C}}{r}\int\frac{ds}{\kappa^{r-1}(s)}), \kappa^{r-1}(s)\sin(\frac{\sqrt{-C}}{r}\int\frac{ds}{\kappa^{r-1}(s)}), \kappa^{r-1}(s)),$$

iii) If the constant $C = 0$, then

$$\gamma(s) = r(\kappa^{r-1}(s)(\frac{1}{r}\int\frac{ds}{\kappa^{r-1}(s)}), \frac{1}{2}\kappa^{r-1}(1-(\frac{1}{r}\int\frac{ds}{\kappa^{r-1}(s)})^2), \frac{1}{2}\kappa^{r-1}(1+(\frac{1}{r}\int\frac{ds}{\kappa^{r-1}(s)})^2)).$$

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