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CONFORMAL η-RICCI SOLITON ON A TYPE OF $(\mathrm{LCS})_N$ MANIFOLD

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Abstract. An n-dimensional Lorentzian concircular structural manifold (in short $(\mathrm{LCS})_n$ manifold) has enormous applications in Mathematical Physics as it has Lorentzian metric g as well as a contact form η. In this note we have established some results regarding conformal η-Ricci soliton and conformal Ricci soliton on $(\mathrm{LCS})_n$ manifold satisfying some curvature conditions like ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-Quasi-conformally semi-symmetric and obtained the nature of the soliton as well as the nature of the structural vector field ξ.

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1. Introduction

Richard S. Hamilton introduced the concept of Ricci flow (for details see [17]) which was named after great Italian mathematician Gregorio Ricci-Curbastro. If we take a smooth closed (compact without boundary) Riemannian manifold M equipped with a smooth Riemannian metric g then the Ricci flow is defined by the geometric evolution equation,

$$\frac{\partial g(t)}{\partial t} = -2S(g(t)) \tag{1}$$

where S is the Ricci curvature tensor of the manifold and $g(t)$ is a one-parameter family of metrices on M.

A Riemannian manifold (M, g) is called a Ricci soliton if there exists a vector field V and a constant λ such that the following equation holds,

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0 \tag{2}$$

where $\mathcal{L}_V$ denotes Lie derivative along the direction V and λ is a non-zero constant. The vector field V is called potential vector field and λ is called soliton constant. Ricci soliton which is a natural extension of Einstein manifold is a self-similar solution of Ricci flow. The potential vector field V and soliton constant λ play vital roles while determining the nature of the soliton. A soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Now if V is Killing then the Ricci soliton reduces to Einstein manifold. Compact Ricci solitons are the fixed points of the Ricci flow (1.1) projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

In 2005, A. E. Fischer [2] has introduced conformal Ricci flow which is a variation of the classical Ricci flow equation (1.1) that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by,

$$\begin{aligned} \frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) &= -pg \\ r(g) &= -1 \end{aligned} \tag{3}$$

where $r(g)$ is the scalar curvature of the manifold, p is scalar non-dynamical field and n is the dimension of the manifold. Corresponding to the aforementioned conformal Ricci flow equation N. Basu and A. Bhattacharyya [15] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation given by,

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{4}$$

In 2009, J. T. Cho and M. Kimura [11] introduced the concept of η-Ricci soliton which is another generalization of classical Ricci soliton and is given by,

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{5}$$

where μ is a real constant, η is a 1-form defined as $\eta(X) = g(X, V)$ for any $X \in \chi(M)$. Clearly it can be noted that if $\mu = 0$ then the η-Ricci soliton (g, V, λ, μ) reduces to Ricci soliton.

Recently Md. D. Siddiqi [14] established the notion of conformal η-Ricci soliton which generalizes both conformal Ricci soliton and η-Ricci soliton. The equation for conformal η-Ricci soliton is given by,

$$\mathcal{L}_X g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0. \tag{6}$$

If we take $\mu = 0$ in (1.6) then it reduces to conformal Ricci soliton (1.4).

Ricci solitons have been studied in many contexts: on Kähler manifolds[16], on contact and Lorentzian manifolds [5],[6], on K-contact manifolds [18] etc. by many authors. Nagaraja and Premalatha [10] studied the nature of Ricci soliton on Kenmotsu manifold; Călin and Crasmareanu [7] on f-Kenmotsu manifold; He and Zhu [8] on Sasakian manifold; Ingalahalli and Bagewadi [9] on α-Sasakian manifold; Y. Wang [24] on 3-dimensional cosymplectic manifold and S. Pahan and A. Bhattacharyya on 3-dimensional trans-Sasakian manifold [21].In 2016, T. Dutta, N. Basu and A. Bhattacharyya studied conformal Ricci soliton on 3-dimensional trans-Sasakian manifold[23].

S. R. Ashoka, C. S. Bagewadi and G. Ingalahalli [22] gave some insight on Ricci soliton in $(\mathrm{LCS})_n$ manifold. Many authors have developed several results on many context of $(\mathrm{LCS})_n$ manifolds like: Yadav, Chaubey, Suthar[20]; Hui and Chakraborty [19]; Baishya [12]; Blaga [3] etc. on η-Ricci Soliton. Chaubey and Siddiqi have studied almost conformal η-Ricci solitons in 3-dimensional $(\mathrm{LCS})_3$ manifolds.

Motivated from above mentioned well praised works we have studied behaviour of conformal η-Ricci soliton on n-dimensional Lorentzian concircular structure manifold (briefly $(\mathrm{LCS})_n$ manifold) satisfying certain curvature properties such as ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-Quasi-conformally semi-symmetric which are represented by,

$$R(\xi, X).H = 0, \qquad R(\xi, X).C = 0, \qquad R(\xi, X).\tilde{C} = 0$$

respectively. In the later section we have revisited some definitions and important properties of $(\mathrm{LCS})_n$ manifold and there after the main results of this paper have been described.

2. Some preliminaries on $(\mathrm{LCS})_n$ manifold

The notion of Lorentzian concircular structure manifold, briefly $(\mathrm{LCS})_n$ manifold is first introduced in 2003 by Shaikh (for details see [1]). An n-dimensional smooth connected paracontact Hausdorff manifold is called a *Lorentzian manifold* if it admits a Lorentzian metric. Lorentzian metric is named after great Dutch Physicist Hendrik Lorentz. A *Lorentzian metric tensor* g is a smooth symmetric tensor field of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non degenerate inner product of signature $(-,+,......,+)$, where T_pM is the tangent space of M at p and $\mathbb{R}$ is the real number space. A non-zero tangent vector $v \in T_pM$ is said to be *timelike*, *non-spacelike*, *null* or *spacelike* if it satisfies $g_p(v, v) < 0$, ≤ 0, $= 0$ or > 0 respectively.

In a Lorentzian manifold (M, g) a vector field ρ is defined by $g(X, \rho) = \eta(X)$, is said to be *concircular vector field* if,

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \omega(X)\eta(Y)\} \tag{7}$$

is satisfied where α is a non-zero scalar field, ω is a closed 1-form and ∇ is the covariant derivative operator w.r.t. Lorentzian metric g.

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called *charecteristic vector field* or the *generator* of the manifold, then we have,

$$g(\xi,\xi) = -1. \tag{8}$$

Since ξ is a concircular vector field there must exists a non-zero 1-form η, such that,

$$g(X,\xi) = \eta(X) \tag{9}$$

$$(\nabla_X \eta)(Y) = \alpha g(X,Y) + \eta(X)\eta(Y) \tag{10}$$

hold for arbitrary vector fields $X, Y \in \chi(M)$ and α is a non-zero scalar field which satisfies,

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{11}$$

ρ being certain scalar function which is given by $\rho = -(\xi\alpha)$. If we define $\phi X = \frac{1}{\alpha}\nabla_X \xi$, then from (10) we can deduce that,

$$\phi X = X + \eta(X)\xi. \tag{12}$$

Clearly ϕ is a symmetric (1,1) tensor which is called *structure tensor* of the manifold. Thus the n-dimensional Lorentzian manifold M together with the unit timelike concircular vector field ξ, 1-form η and (1,1) tensor ϕ is said to be Lorentzian concircular structure (briefly $(\mathrm{LCS})_n$) manifold. If we take $\alpha = 1$, then the manifold reduces to LP-Sasakian manifold of Matsumoto [13].

A $(\mathrm{LCS})_n$ manifold satisfies the following properties,

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \eta\circ\phi = 0, \quad \phi\xi = 0 \tag{13}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{14}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \tag{15}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y) \tag{16}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)(g(X,Y)\xi - \eta(Y)X) \tag{17}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha(g(X,Y) + \eta(X)\eta(Y)) \tag{18}$$

where R is the Riemannian curvature tensor. Furthermore if (g, V, λ, p, μ) is a conformal η-Ricci soliton then we can deduce the following,

$$S(X,Y) = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)g(X,Y) - (\mu + \alpha)\eta(X)\eta(Y) \tag{19}$$

$$QX = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)X - (\mu + \alpha)\eta(X)\xi \tag{20}$$

$$r = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)n + (\mu + \alpha) \tag{21}$$

where S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold. We now want to recall some useful definitions [4],

Definition 1. *A vector field ξ is called $torseforming$ if it satisfies*

$$\nabla_X \xi = fX + \gamma(X)\xi \tag{22}$$

for a smooth function $f \in C^\infty(M)$, 1-form γ and for all vector field X on M. A torse forming vector field is called recurrent if $f = 0$.

3. Main results

Theorem 1. *A conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-conharmonically semi-symmetric curvature property, satisfies the following properties,*

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

b) ξ is a geodesic vector field,

c) $\nabla_\xi S = 0$ and $\nabla_\xi Q = 0$.

Proof. The conharmonic curvature tensor H is defined by,

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{23}$$

Now taking inner product w.r.t. ξ and using (15), (19) and (20) we have,

$$\eta(H(X,Y)Z) = (\alpha^2 - \rho - \frac{p}{(n-2)} - \frac{2}{n(n-2)} + \frac{2\lambda}{(n-2)} + \frac{\alpha}{(n-2)} - \frac{\mu}{(n-2)}) (g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{24}$$

Here we have considered ξ-conharmonically semi-symmetric curvature property, i.e., $R(\xi, X).H = 0$, which yields,

$$R(\xi,X)H(Y,Z)W - H(R(\xi,X)Y,Z)W - H(Y,R(\xi,X)Z)W - H(Y,Z)R(\xi,X)W = 0. \tag{25}$$

Applying (17) in the above equation we have,

$$g(X,H(Y,Z)W)\xi - \eta(H(Y,Z)W)X - g(X,Y)H(\xi,Z)W + \eta(Y)H(X,Z)W - g(X,Z)H(Y,\xi)W + \eta(Z)H(Y,X)W - g(X,W)H(Y,Z)\xi + \eta(W)H(Y,Z)W = 0. \tag{26}$$

By taking inner product of the previous equation with ξ we get,

$$\begin{aligned} &g(X,H(Y,Z)W) + \eta(H(Y,Z)W)\eta(X) + g(X,Y)\eta(H(\xi,Z)W) - \\ &\eta(Y)\eta(H(X,Z)W) + g(X,Z)\eta(H(Y,\xi)W) - \eta(Z)\eta(H(Y,X)W) \\ &+g(X,W)\eta(H(Y,Z)\xi) - \eta(W)\eta(H(Y,Z)W) = 0. \end{aligned} \tag{27}$$

After using (24) the equation reduces to,

$$\begin{aligned} &g(X,H(Y,Z)W) + (\alpha^2 - \rho - \frac{p}{(n-2)} - \frac{2}{n(n-2)} + \frac{2\lambda}{(n-2)} + \frac{\alpha}{(n-2)} \\ &-\frac{\mu}{(n-2)})(g(X,Z)g(Y,W) - g(Z,W)g(X,Y)) = 0. \end{aligned} \tag{28}$$

Let us consider the set $\{e_i\}_{i=1}^n$ as a basis of the manifold. Then replacing $X = Y = e_i$ in the above equation yields,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}. \tag{29}$$

Hence (a) is proved.

Now considering $X = \xi$ we can rewrite (6) as,

$$g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi) + 2S(Y,Z) + [2\lambda - (p + \frac{2}{n})]g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0 \tag{30}$$

for all $Y, Z \in \chi(M)$. Simplifying using (19), the above equation reduces to,

$$g(\nabla_Y\xi, Z) + g(Y, \nabla_Z\xi) - 2\alpha[g(Y,Z) + \eta(Y)\eta(Z)] = 0. \tag{31}$$

Considering $Z = \xi$ in the above equation, we get,

$$g(\nabla_\xi\xi, Y) = 0. \tag{32}$$

Since the aforementioned relation holds for any $Y \in \chi(M)$, so $\nabla_\xi\xi = 0$. This concludes that ξ is a geodesic vector field. Thus (b) is proved.

Taking covariant derivative of (19) and (20) we can find the general expressions of ∇S and ∇Q as,

$$(\nabla_X S)(Y,Z) = -(\mu+\alpha)[g(Y,\nabla_X\xi)\eta(Z) + g(Z,\nabla_X\xi)\eta(Y)] \tag{33}$$

$$(\nabla_X Q)Y = -(\mu+\alpha)[g(Y,\nabla_X\xi)\xi + \eta(Y)\nabla_X\xi] \tag{34}$$

for any $Y, Z \in \chi(M)$. Letting $X = \xi$ in (33) and (34) we get, $\nabla_\xi S = 0$ and $\nabla_\xi Q = 0$. It completes our results.

Theorem 2. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-conharmonically semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. Let ξ be a torse forming vector field. Then we have from (22) that $\nabla_X\xi = fX + \gamma(X)\xi$, for a smooth function $f \in C^\infty(M)$, 1-form γ and for all vector field X on M. Taking inner product w.r.t. ξ it yields,

$$g(\nabla_X\xi, \xi) = f\eta(X) - \gamma(X).$$

Hence we get $f\eta = \gamma$. After applying this result, (22) becomes,

$$\nabla_X\xi = f[X + \eta(X)\xi]. \tag{35}$$

Applying (35) in (31) we get,

$$2(f - \alpha)[g(Y, Z) - \eta(Y)\eta(Z)] = 0,$$

for all vector fields Y and Z and hence we get $f = \alpha$. Thus (35) reduces to,

$$\nabla_X\xi = \alpha[X + \eta(X)\xi] = \alpha\phi^2(X), \tag{36}$$

i.e., $\nabla_X\xi$ is collinear to $\phi^2(X)$ for all X. Hence we get $d\eta = 0$, which means that η is colsed.

Now let us consider ξ to be recurrent vector field. So, $f = \alpha = 0$. Thus (35) yields that ξ is a concurrent vector field i.e., $\nabla_X\xi = 0$ for all vector field X on M. Also we have,

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 0 \tag{37}$$

for all X and Y on M. Thus we can conclude ξ is Killing vector field.

Remark 1. *We know conformal η-Ricci soliton reduces to conformal Ricci soliton if we consider μ to be zero in (6). Accordingly the results of theorem 1 change while the results of theorem 2 remain the same for conformal Ricci soliton. We can state the modified results of theorem 1 as:*

A conformal Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-conharmonically semi-symmetric curvature property, satisfies the following properties,

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

b) ξ is a geodesic vector field,

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Theorem 3. *A conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-concircularly semi-symmetric curvature property satisfies the following properties,*

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. The concircular curvature tensor C is defined by,

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{38}$$

Now taking inner product w.r.t. ξ and using (15) we have,

$$\eta(C(X,Y)Z) = (\alpha^2 - \rho - \frac{r}{n(n-1)})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{39}$$

Here we have considered ξ-concircularly semi-symmetric curvature property i.e., $R(\xi, X).C = 0$, which yields,

$$R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W - C(Y,R(\xi,X)Z)W - C(Y,Z)R(\xi,X)W = 0. \tag{40}$$

Applying (17) in the above equation we have,

$$\begin{aligned} &g(X,C(Y,Z)W)\xi - \eta(C(Y,Z)W)X - g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W - \\ &g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W - g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)W = 0. \end{aligned} \tag{41}$$

By taking inner product in the previous equation with ξ we get,

$$\begin{aligned} &g(X,C(Y,Z)W) + \eta(C(Y,Z)W)\eta(X) + g(X,Y)\eta(C(\xi,Z)W) - \\ &\eta(Y)\eta(C(X,Z)W) + g(X,Z)\eta(C(Y,\xi)W) - \eta(Z)\eta(C(Y,X)W) \\ &+g(X,W)\eta(C(Y,Z)\xi) - \eta(W)\eta(C(Y,Z)W) = 0. \end{aligned} \tag{42}$$

After using (33) the equation reduces to,

$$g(X,C(Y,Z)W) + (\alpha^2 - \rho - \frac{r}{n(n-1)})(g(X,Z)g(Y,W) - g(Z,W)g(X,Y)) = 0. \tag{43}$$

Then replacing $X = Y = e_i$ in the above equation, where the set $\{e_i\}_{i=1}^n$ is a basis of the manifold, yields,

$$[\frac{p}{2} + \frac{1}{n} - \lambda - \alpha - (n-1)(\alpha^2 - \rho)]g(Z,W) - (\mu + \alpha)\eta(Z)\eta(W) = 0. \tag{44}$$

Since this holds for arbitrary $Z, W \in \chi(M)$, setting $Z = W = \xi$ we have,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}. \tag{45}$$

This proves (a) and the expression is identical with (29). Other two outcomes (b) and (c) are immediate consequences and can be proved similarly like theorem 1.

Theorem 4. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-concircularly semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. Since the results of theorem 3 for concircular curvature tensor are same as of theorem 1 for conharmonic curvature tensor, the proof of this theorem is identical with the proof of theorem 2.

Remark 2. *We know conformal η-Ricci soliton is a mere generalisation conformal Ricci soliton. If we let μ to be zero in (6) then it reduces to conformal Ricci soliton. The results of theorem 3 change while the results of theorem 4 remain the same for conformal Ricci soliton. We can modify the results of theorem 3 as:*

A conformal Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-concircularly semi-symmetric curvature property, satisfies the following properties,

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Theorem 5. *A conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-Quasi-conformally semi-symmetric curvature property, admits the following properties,*

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. The Quasi-conformal curvature tensor $\tilde{C}$ is defined by,

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}(\frac{a}{(n-1)} + 2b)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{46}$$

where a and b are non-zero constants. Now taking inner product w.r.t. ξ and using (15) we have,

$$\begin{aligned} \eta(\tilde{C}(X,Y)Z) &= [a(\alpha^2 - \rho) + b(p + \frac{2}{n} - 2\lambda - \alpha + \mu) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \\ &\quad (g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \end{aligned} \tag{47}$$

Here we have considered ξ-Quasi-conformally semi-symmetric curvature property i.e., $R(\xi, X).\tilde{C} = 0$, which yields,

$$R(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(R(\xi, X)Y, Z)W - \tilde{C}(Y, R(\xi, X)Z)W - \tilde{C}(Y, Z)R(\xi, X)W = 0. \tag{48}$$

Applying (17) in the above equation we have,

$$\begin{aligned} &g(X, \tilde{C}(Y, Z)W)\xi - \eta(\tilde{C}(Y, Z)W)X - g(X, Y)\tilde{C}(\xi, Z)W + \eta(Y)\tilde{C}(X, Z)W - \\ &g(X, Z)\tilde{C}(Y, \xi)W + \eta(Z)\tilde{C}(Y, X)W - g(X, W)\tilde{C}(Y, Z)\xi + \eta(W)\tilde{C}(Y, Z)W = 0. \end{aligned} \tag{49}$$

By taking inner product in the previous equation w.r.t. ξ we get,

$$\begin{aligned} &g(X, \tilde{C}(Y, Z)W) + \eta(\tilde{C}(Y, Z)W)\eta(X) + g(X, Y)\eta(\tilde{C}(\xi, Z)W) - \\ &\eta(Y)\eta(\tilde{C}(X, Z)W) + g(X, Z)\eta(\tilde{C}(Y, \xi)W) - \eta(Z)\eta(\tilde{C}(Y, X)W) \\ &+g(X, W)\eta(\tilde{C}(Y, Z)\xi) - \eta(W)\eta(\tilde{C}(Y, Z)W) = 0. \end{aligned} \tag{50}$$

After using (43) the above equation reduces to,

$$\begin{aligned} &g(X, \tilde{C}(Y, Z)W) + [a(\alpha^2 - \rho) + b(p + \frac{2}{n} - 2\lambda - \alpha + \mu) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \\ &(g(X, Z)g(Y, W) - g(Z, W)g(X, Y)) = 0. \end{aligned} \tag{51}$$

Then replacing $X = Y = e_i$ in the above equation, where the set $\{e_i\}_{i=1}^n$ is a basis of the manifold, yields,

$$\begin{aligned} &[a + (n-2)b]S(Z, W) + [br - \frac{r}{n}(a + 2(n-1)b)]g(Z, W) + [a(\alpha^2 - \rho) + \\ &b(p + \frac{2}{n} - 2\lambda - \alpha + \mu) - \frac{r}{n}(\frac{a}{n-1} + 2b)]g(Z, W) = 0. \end{aligned} \tag{52}$$

Since this holds for arbitrary $Z, W \in \chi(M)$, setting $Z = W = \xi$ we have,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}. \tag{53}$$

Hence (a) is proved.

(b) and (c) can be proved in similar manner like the proof of theorem 1.

Theorem 6. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-Quasi-conformally semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. The proof can be done in similar fashion like theorem 2.

Remark 3. *To get conformal Ricci soliton from conformal η-Ricci soliton we assume $\mu = 0$ in (6). Consequently the results of theorem 5 change while the results of theorem 6 remain unchanged for conformal Ricci soliton. We can revise the results of theorem 5 as:*

A conformal Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-Quasi-conformally semi-symmetric curvature property satisfies the following properties,

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

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