Acta Universitatis Apulensis No. 63/2020
ISSN: 1582-5329 pp. 35-49
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.63.04

EXISTENCE AND ULAM STABILITY OF FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATIONS WITH TWO CAPUTO-HADAMARD DERIVATIVES

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Abstract. In this work, we study existence, uniqueness and Ulam stability type of solutions of fractional pantograph equations involving two Caputo-Hadamard fractional orders. The existence and uniqueness of solutions is established by contraction mapping principle, while the existence of solutions is derived by Leray-Schauder's alternative. We also present and discuss different types of Ulam-stability for our problem. Finally, we give some illustrative examples.

2010 *Mathematics Subject Classification*: 26A33, 34B15.

Keywords: Caputo-Hadamard derivative, fixed point, existence, pantograph equations, Ulam-Hyers stability.

1. Introduction

Pantograph equation have been the focus of many studies due to their application in various sciences, such as electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics. For further information and applications, see [3, 11, 12, 13, 19]. Recently, many studies on fractional pantograph differential equations involving different operators, such as Caputo operators [1], Riemann-Liouville operators [16], Hilfer operators [22], Hilfer-Hadamard operators [23], Caputo-Hdamard fractional derivative [9] have appeared during the past several years. Moreover, by applying different fixed point theorems, many mathematicians have obtained results of the existence and uniqueness of solutions for various classes of fractional pantograph differential equations, see [1, 2, 4, 6, 7]. Recently, Ulam's type stability problems have been taken up by several researchers and the study of this area has grown to be one of the most important subjects in the mathematical analysis area, for more details see [8, 10, 15, 18, 20] and references therein. Also the Ulam stability of pantograph differential equations with fractional derivative has been investigated by different authors, we refer the reader to the papers [9, 12, 21, 22, 23].

In the present work, we study existence, uniqueness and Ulam-stability of solutions for fractional pantograph differential equations involving two Caputo-Hadamard type derivatives:

$$\begin{cases} {}_{H}^{C}D^{\alpha}\left({}_{H}^{C}D^{\beta}+\gamma\right)u\left(t\right)=f\left(t,u\left(t\right),u\left(\lambda t\right)\right),t\in[1,T],\gamma\in\mathbb{R},0<\lambda<1, \\ u\left(1\right)=\theta, \quad u\left(T\right)=\vartheta, \quad \theta,\vartheta\in\mathbb{R}, \end{cases} \tag{1}$$

where $0<\alpha,\beta\leq 1, {}_{H}^{C}D^{\alpha}, {}_{H}^{C}D^{\beta}$ are Caputo-Hadamard fractional derivatives and $f:[1,T]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$, is given continuous function.

The paper is organized as follows. In Section 2, we recall some definitions and lemma which are used throughout the paper. In Section 3, we present our main results for existence and uniqueness of solutions for the proposed fractional problem (1). Section 4, we study the Ulam-stability type of solutions for the fractional problem (1). Some examples to illustrate our results are presented in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and lemmas which are used throughout this work.

Definition 1. *[17] The Hadamard fractional integral of order α for a continuous function $h:[a,+\infty)\rightarrow\mathbb{R}$ is defined as*

$$ {}_{H}I^{\alpha}f\left(t\right)=\tfrac{1}{\Gamma(\rho)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{\rho-1}\frac{f\left(s\right)}{s}ds,\rho>0, \tag{2}$$

where $\log\left(.\right)=\log_{e}\left(.\right)$*, provided that the integral exist.*

Definition 2. *[14] For at least $n-$times differentiable function $f:[a,\infty)\rightarrow\mathbb{R}$ the Caputo-Hadamard fractional derivative of order ρ is defined as*

$$ {}_{H}^{C}D^{\alpha}f(t)=\tfrac{1}{\Gamma(n-\rho)}\int_{a}^{t}\left(\log\frac{t}{s}\right)^{n-\rho-1}\delta^{n}\frac{f(s)}{s}ds, \tag{3}$$

where $n-1<\rho<n, n=[\rho]+1, \delta=t\frac{d}{dt}, [\rho]$ denotes the integer part of α and $\log\left(.\right)=\log_{e}\left(.\right)$.

Lemma 1. *[14] Let* $u\in C_{\delta}^{n}\left([a,b],\mathbb{R}\right).$ *Then*

$$
{}_{H}I^{\alpha}\left({}_{H}^{C}D^{\alpha}u\right)(t)=u(t)-\sum_{i=0}^{n-1}c_{i}(\log t)^{i},c_{i}\in\mathbb{R}, \tag{4}
$$

where $C_{\delta}^{n}\left([a,b],\mathbb{R}\right)=\left\{h:[a,b]\rightarrow\mathbb{R}:\delta^{n-1}h\in C\left([a,b],\mathbb{R}\right)\right\}.$

Lemma 2. *[5]. (Leray-Schauder alternative). Let* $S:F\rightarrow F$ *be a completely continuous operator (i.e.,a map that restricted to any bounded set in* S *is compact). Let*

$$
\Omega\left(S\right)=\left\{u\in F:u=\sigma S\left(u\right)\text{ for some }0<\sigma<1\right\}.
$$

Then either the set $\Omega\left(S\right)$ *is unbounded, or* S *has at least one fixed point.*

Now we prove the following auxiliary lemma.

Lemma 3. *Suppose that* $h\left(t\right)\in C\left(\left[1,T\right],\mathbb{R}\right)$ *and consider the fractional problem*

$$
{}_{H}^{C}D^{\beta}\left({}_{H}^{C}D^{\alpha}+\gamma\right)u\left(t\right)=h\left(t\right),t\in\left[1,T\right],0<\alpha,\beta\leq1, \tag{5}
$$

with the condition

$$
u\left(1\right)=\theta,\quad u\left(T\right)=\vartheta. \tag{6}
$$

Then, we have

$$
\begin{aligned}
u\left(t\right) &= \frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}h\left(s\right)\frac{ds}{s}-\frac{\gamma}{\Gamma\left(\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}u\left(s\right)ds \\
&\quad -\frac{(\log t)^{\beta}}{(\log T)^{\beta}}\left(\frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}h\left(s\right)\frac{ds}{s}\right. \\
&\quad \left.-\frac{\gamma}{\Gamma\left(\beta\right)}\int_{1}^{T}\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}u\left(s\right)ds-\vartheta+\theta\right)+\theta.
\end{aligned} \tag{7}
$$

Proof. Using Lemma 1, we can write

$$
\begin{aligned}
u\left(t\right) &= \frac{1}{\Gamma\left(\alpha+\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}h\left(s\right)\frac{ds}{s} \\
&\quad -\frac{\gamma}{\Gamma\left(\beta\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\beta-1}u\left(s\right)ds+\frac{c_{0}}{\Gamma\left(\beta+1\right)}\left(\log t\right)^{\beta}+c_{1}.
\end{aligned} \tag{8}
$$

where $c_{0},c_{1}\in\mathbb{R}$.

From (6), a simple calculation gives

$$\begin{aligned} c_1 &= \theta, \\ c_0 &= \frac{\Gamma(\beta+1)}{(\log T)^{\beta}} \left(\vartheta - \theta - \frac{1}{\Gamma(\alpha+\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha+\beta-1} h(s) \frac{ds}{s} \right. \\ & \left. + \frac{\gamma}{\Gamma(\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\beta-1} u(s)\, ds \right). \end{aligned}$$

Substituting the values of c_0 and c_1 in (8) yields the solution (7).This completes the proof.

3. Existence and uniqueness of solution

We denote by $X = C([1,T], \mathbb{R})$ the Banach space of all continuous functions from $[1,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [1,T]\}$.

In view of Lemma 3, we define an operator $P : X \to X$ as

$$\begin{aligned} Pu(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} f(s, u(s), u(\lambda s)) \frac{ds}{s} - \frac{\gamma}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} u(s)\, ds \\ & - \frac{(\log t)^{\beta}}{(\log T)^{\beta}} \left(\frac{1}{\Gamma(\alpha+\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha+\beta-1} f(s, u(s), u(\lambda s)) \frac{ds}{s} \right. \\ & \left. - \frac{\gamma}{\Gamma(\beta)} \int_1^T \left(\log \frac{T}{s} \right)^{\alpha+\beta-1} u(s)\, ds - \vartheta + \theta \right) + \theta. \end{aligned} \tag{9}$$

Our first main result is based on the Banach contraction principle.

Theorem 4. *Let $f : [1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H_1) : *There exists a constant $k > 0$ such that*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq k(|x_1 - y_1| + |x_2 - y_2|), t \in [1,T], x_i, y_i \in \mathbb{R}, i = 1,2.$$

If the inequality

$$\frac{4k}{\Gamma(\alpha+\beta+1)} (\log T)^{\alpha+\beta} < 1 - \frac{2|\gamma|}{\Gamma(\beta+1)} (\log T)^{\beta}, \tag{10}$$

is valid, then problem (1) has a unique solution on $[1,T]$.

Proof. Let us define $L=\sup_{t\in[1,T]}|f(t,0,0)|<\infty$. Setting

$$r\geq\frac{\frac{2L}{\Gamma(\alpha+\beta+)}(\log T)^{\alpha+\beta}+2|\theta|+|\vartheta|}{1-\left[\frac{4k}{\Gamma(\alpha+\beta+)}(\log T)^{\alpha+\beta}+\frac{2|\gamma|}{\Gamma(\beta+1)}(\log T)^{\beta}\right]}.$$

We show that $PB_r\subset B_r$, where $B_r=\{u\in X:\|u\|\leq r\}$. For $u\in B_r$, we find the following estimate based on the hypothesis (H_1) :

$$\begin{aligned}|f(t,u(t),u(\lambda t))| &\leq |f(t,u(t),u(\lambda t))-f(s,0,0)|+|f(s,0,0)|\\ &\leq 2k\|u\|+L\leq 2kr+L,\end{aligned}\tag{11}$$

Using (11), we get

$$\begin{aligned}\|u\| \leq\; & \frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|f(s,u(s),u(\lambda s))|\frac{ds}{s}+\frac{|\gamma|}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}|u(s)|\,ds\\ &+\frac{(\log t)^{\beta}}{(\log T)^{\beta}}\left(\frac{1}{\Gamma(\alpha+\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}|f(s,u(s),u(\lambda s))|\frac{ds}{s}\right.\\ &\left.+\frac{|\gamma|}{\Gamma(\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\beta-1}|u(s)|\,ds+|\vartheta|+|\theta|\right)+|\theta|\\ \leq\; & \frac{2(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+)}(2kr+L)+\frac{2|\gamma|(\log T)^{\beta}}{\Gamma(\beta+1)}r+2|\theta|+|\vartheta|.\end{aligned}$$

Thus

$$\begin{aligned}\|u\| \leq\; & \left(\frac{4k}{\Gamma(\alpha+\beta+)}(\log T)^{\alpha+\beta}+\frac{2|\gamma|}{\Gamma(\beta+1)}(\log T)^{\beta}\right)r\\ &+\frac{2L}{\Gamma(\alpha+\beta+)}(\log T)^{\alpha+\beta}+2|\theta|+|\vartheta|\leq r,\end{aligned}$$

which implies that $PB_r\subset B_r$. Now, for $u,v\in B_r$ and for any $t\in J$, we get

$$\begin{aligned}&|Pu(t)-Pv(t)|\\ \leq\; & \frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|f(s,u(s),u(\lambda s))-f(s,v(s),v(\lambda s))|\frac{ds}{s}\\ &+\frac{|\gamma|}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}|u(s)-v(s)|\,ds\\ &+\frac{(\log t)^{\beta}}{(\log T)^{\beta}}\left(\frac{1}{\Gamma(\alpha+\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}|f(s,u(s),u(\lambda s))-f(s,v(s),v(\lambda s))|\frac{ds}{s}\right.\\ &\left.+\frac{|\gamma|}{\Gamma(\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}|u(s)-v(s)|\,ds\right).\end{aligned}$$

By (H_1), we can write

$$\|P(u)-P(v)\| \leq \left(\frac{4k}{\Gamma(\alpha+\beta)}(\log T)^{\alpha+\beta}+\frac{2|\gamma|}{\Gamma(\beta+1)}(\log T)^{\beta}\right)\|u-v\|.$$

By (10), we see that P is a contractive operator. Consequently, by the Banach fixed point theorem, has a fixed point which is a solution of (1).

In the next result, we show the existence of solutions for the problem (1) by Lemma 2.

Theorem 5. *Let $f:[1,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that:*

(H_2) : *there exist real constants $\omega_i \geq 0$ $(i=1,2)$ and $\omega_0>0$ such that for all $x,y\in\mathbb{R}$, we have*

$$|f(t,x,y)| \leq \omega_0+\omega_1|x|+\omega_2|y|.$$

If

$$\frac{4(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega_1+\frac{2|\gamma|(\log T)^{\beta}}{\Gamma(\beta+1)}<1. \tag{12}$$

Then the problem (1) has at least one solution on J.

Proof. First, we show that the operator $P:X\to X$ is completely continuous. By continuity of the function f, it follows that the operator P is continuous.

Let $\Theta\subset X$ be bounded. Then we can find positive constant M such that

$$|f(t,u(t),u(\lambda t))| \leq M, \forall\ u\in\Theta.$$

Then for any $u\in\Theta$ and by (H_2), we have

$$\begin{aligned}\|Pu\| \leq\ & \frac{M}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}\frac{ds}{s}+\frac{\gamma r}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}ds\\ &+\frac{(\log t)^{\beta}}{(\log T)^{\beta}}\left(\frac{M}{\Gamma(\alpha+\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}\frac{ds}{s}\right.\\ &\left.+\frac{\gamma r}{\Gamma(\beta)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha+\beta-1}ds+|\vartheta|+|\theta|\right)+|\theta|.\end{aligned}$$

Hence, we obtain

$$\|Pu\| \leq \frac{2M(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2\gamma r(\log T)^{\beta}}{\Gamma(\beta+1)}+|\vartheta|+2|\theta|.$$

From the above inequality, it follows that the operator P is uniformly bounded.

Next, we show that P is equicontinuous. Let $t_1, t_2 \in [1, T]$ with $1 \leq t_1 < t_2 \leq T$. Then we have

$$
\begin{aligned}
& |Pu(t_2) - Pu(t_1)| \\
\leq\ & \frac{M}{\Gamma(\alpha+\beta)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\alpha+\beta-1} - \left(\log \frac{t_1}{s} \right)^{\alpha+\beta-1} \right] \frac{ds}{s} + \frac{M}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha+\beta-1} \\
& + \frac{\gamma r}{\Gamma(\beta)} \int_1^{t_1} \left[\left(\log \frac{t_1}{s} \right)^{\beta-1} - \left(\log \frac{t_2}{s} \right)^{\beta-1} \right] \frac{ds}{s} + \frac{\gamma r}{\Gamma(\beta)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\beta-1} \frac{ds}{s} \\
& + \frac{(\log t_1)^{\beta} - (\log t_2)^{\beta}}{(\log T)^{\beta}} \left(\frac{M (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\gamma r (\log T)^{\beta}}{\Gamma(\beta+1)} + |\vartheta| + |\theta| \right) \\
\leq\ & \frac{M}{\Gamma(\alpha+\beta+1)} \left[(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right] \\
& + \frac{\gamma r}{\Gamma(\beta+1)} \left[(\log t_1)^{\beta} - (\log t_2)^{\beta} + 2 \left(\log \frac{t_2}{t_1} \right)^{\beta} \right] \\
& + \frac{(\log t_1)^{\beta} - (\log t_2)^{\beta}}{(\log T)^{\beta}} \left(\frac{M (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\gamma r (\log T)^{\beta}}{\Gamma(\beta+1)} + |\vartheta| + |\theta| \right).
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of u. Therefore by the Arzelá-Ascoli theorem the operator P is completely continuous.

Finally, it will be verified that the set $\Omega = \{u \in X : u = \sigma P(u), 0 \leq \sigma \leq 1\}$ is bounded. Let $u \in \Omega$, then $u = \sigma P(u)$. For any $t \in [1, T]$, we have

$$
u(t) = \sigma Pu(t),
$$

then

$$
\begin{aligned}
& u(t) \\
\leq\ & \sigma \left(\frac{1}{\Gamma(\alpha+\beta)} \int_1^{t} \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} |f(s, u(s), u(\lambda s))| \frac{ds}{s} + \frac{|\gamma|}{\Gamma(\beta)} \int_1^{t} \left(\log \frac{t}{s} \right)^{\beta-1} |u(s)| \, ds \right. \\
& + \frac{(\log t)^{\beta}}{(\log T)^{\beta}} \left(\frac{1}{\Gamma(\alpha+\beta)} \int_1^{T} \left(\log \frac{T}{s} \right)^{\alpha+\beta-1} |f(s, u(s), u(\lambda s))| \frac{ds}{s} \right. \\
& \left. \left. + \frac{|\gamma|}{\Gamma(\beta)} \int_1^{T} \left(\log \frac{T}{s} \right)^{\alpha+\beta-1} u(s) \, ds + |\vartheta| + |\theta| \right) + |\theta| \right).
\end{aligned}
$$

Thanks to (H_2), we can write

$$u(t) \leq \frac{2(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}(\omega_0 + 2\omega_1 \|u\|) + \frac{2\gamma(\log T)^{\beta}}{\Gamma(\beta+1)}\|u\| + 2|\theta| + |\vartheta|.$$

Hence, we have

$$\|u\| \leq \left(\frac{4(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega_1 + \frac{2\gamma(\log T)^{\beta}}{\Gamma(\beta+1)}\right)\|u\| + \frac{2(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega_0 + 2|\theta| + |\vartheta|.$$

It follows that

$$\|u\| \leq \frac{\frac{2(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega_0 + 2|\theta| + |\vartheta|}{1 - \left(\frac{4(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\omega_1 + \frac{2\gamma(\log T)^{\beta}}{\Gamma(\beta+1)}\right)}.$$

This shows that the set Ω is bounded. Thus, by Lemma 2, the operator P has atleast one fixed point. Hence problem (1) has at least one solution on $[1, T]$.

4. Ulma-Stability

In this section, we define and study the Ulam–Hyers stability, the generalized Ulam–Hyers stability and Ulam-Hyers-Rassias stability for the problem (1).

Definition 3. *The fractional boundary value problem (1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in X$ of the inequality*

$$\left| {}_H^C D^{\alpha}\left({}_H^C D^{\beta} + \gamma\right) v(t) - f(t, v(t), v(\lambda t))\right| \leq \varepsilon,\ t \in [1, T], \tag{13}$$

there exists a solution $u \in X$ of fractional boundary value problem (1) with

$$|v(t) - u(t)| \leq c_f \varepsilon,\ t \in [1, T].$$

Definition 4. *The fractional boundary value problem (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+), \psi_f(0) = 0$, such that for each solution $v \in X$ of the inequality (13) there exists a solution $u \in X$ of the fractional boundary value problem (1) with*

$$|v(t) - u(t)| \leq \psi_f(\varepsilon),\ t \in [1, T].$$

Definition 5. *The fractional boundary value problem (1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in X$ if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in X$ of the inequality*

$$\left| {}_H^C D^{\alpha}\left({}_H^C D^{\beta} + \gamma\right) v(t) - f(t, v(t), v(\lambda t))\right| \leq \varepsilon\varphi(t),\ t \in [1, T], \tag{14}$$

there exists a solution $u \in X$ *of problem (1) with*

$$|v(t) - u(t)| \leq c_f \varepsilon \varphi(t), \ t \in [1, T].$$

Definition 6. *The fractional boundary value problem (1) is generalized Ulam-Hyers-Rassias stable with respect to* $\varphi \in X$ *if there exists a real number* $c_{f,\varphi} > 0$ *such that for each solution* $v \in X$ *of the inequality*

$$\left| {}_H^C D^{\alpha} \left({}_H^C D^{\beta} + \gamma \right) v(t) - f(t, v(t), v(\lambda t)) \right| \leq \varphi(t), \ t \in [1, T], \tag{15}$$

there exists a solution $u \in X$ *of problem (1) with*

$$|v(t) - u(t)| \leq c_{f,\varphi} \varphi(t), \ t \in [1, T].$$

Remark 1. *A function* $v \in X$ *is a solution of the inequality (13) if and only if there exists a function* $\psi : [1, T] \to \mathbb{R}$ *such that*

(1) : $|\psi(t)| \leq \varepsilon, t \in [1, T]$.

(2) : ${}_H^C D^{\alpha} \left({}_H^C D^{\beta} + \gamma \right) u(t) = f(t, u(t), u(\lambda t)) + \psi(t)$, $t \in [1, T], \gamma \in \mathbb{R}, 0 < \lambda < 1$.

Theorem 6. *Assume that* $f : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying* (H_1). *If*

$$\frac{2k}{\Gamma(\alpha + \beta + 1)} (\log T)^{\alpha+\beta} + \frac{|\gamma|}{\Gamma(\beta + 1)} (\log T)^{\beta} < 1, \tag{16}$$

then the fractional boundary value problem (1) is Ulam-Hyers stable and consequently, generalized Ulam-Hyers stable.

Proof. Let $v \in X$ be a solution of the inequality (13), i.e.

$$\left| {}_H^C D^{\alpha} \left({}_H^C D^{\beta} u(t) + \gamma \right) - f(t, u(t), u(\lambda t)) \right| \leq \varepsilon, t \in [1, T],$$

and let us denote by $u \in X$ the unique solution of the problem

$$\begin{cases} {}_H^C D^{\alpha} \left({}_H^C D^{\beta} u(t) + \gamma \right) = f(t, u(t), u(\lambda t)), \gamma \in \mathbb{R}, t \in J, 0 < \alpha, \beta < 1, 0 < \lambda < 1, \\ u(1) = v(1), u(T) = v(T), \end{cases}$$

By using Lemma 5, we have

$$u(t) = {}_H I^{\alpha+\beta} h_u(t) - \gamma {}_H I^{\beta} u(t) + \frac{c_0 (\log t)^{\beta}}{\Gamma(\beta + 1)} + c_1.$$

and by integration of the inequality (13), we obtain

$$\begin{aligned} \left| v(t) - {}_H I^{\alpha+\beta} h_v(t) - \gamma_H I^{\beta} v(t) + \frac{c_2 (\log t)^{\beta}}{\Gamma(\beta+1)} + c_3 \right| &\leq \frac{\varepsilon}{\Gamma(\alpha+\beta+1)} (\log t)^{\alpha+\beta} \\ &\leq \frac{\varepsilon}{\Gamma(\alpha+\beta+1)} (\log T)^{\alpha+\beta}. \end{aligned} \tag{17}$$

On the other hand, if $u(1) = v(1), u(T) = v(T)$, then $c_0 = c_2$ and $c_2 = c_3$

For any $t \in [1, T]$, we have

$$\begin{aligned} v(t) - u(t) &= v(t) - {}_H I^{\alpha+\beta} h_u(t) - \gamma_H I^{\beta} u(t) - \frac{c_2 (\log t)^{\beta}}{\Gamma(\beta+1)} - c_3 \\ &\quad + {}_H I^{\alpha+\beta} (h_v(t) - h_u(t)) - \gamma_H I^{\beta} (v(t) - u(t)), \end{aligned}$$

where,

$$h_u(t) = f(t, u(t), u(\lambda t)) \text{ and } h_v(t) = f(t, v(t), v(\lambda t)),$$

and

$${}_H I^{\alpha+\beta} (h_v(t) - h_u(t)) = {}_H I^{\alpha} [f(s, v(s), v(\lambda t)) - f(s, u(s), u(\lambda t))].$$

Using (H_1), we get

$$\begin{aligned} |v(t) - u(t)| &\leq \left| v(t) - {}_H I^{\alpha+\beta} h_u(t) - \gamma_H I^{\beta} u(t) - \frac{c_2 (\log t)^{\beta}}{\Gamma(\beta+1)} - c_3 \right| \\ &\quad + \frac{2k}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} \|v(s) - u(s)\| \frac{ds}{s} \\ &\quad + \frac{|\gamma|}{\Gamma(\beta)} \int_1^t (\log \frac{t}{s})^{\beta-1} \|v(s) - u(s)\| \frac{ds}{s}. \end{aligned} \tag{18}$$

By (17), we obtain

$$|v(t) - u(t)| \leq \frac{\varepsilon (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \left(\frac{2k (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma| (\log T)^{\beta}}{\Gamma(\beta+1)} \right) \|v(s) - u(s)\|,$$

which implies that

$$\|v(s) - u(s)\| \left(1 - \left[\frac{2k (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma| (\log T)^{\beta}}{\Gamma(\beta+1)} \right] \right) \leq \frac{\varepsilon (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

For each $t \in [1, e]$, we have

$$|v(t) - u(t)| \leq \frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)\left(1 - \left[\frac{2k(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|(\log T)^{\beta}}{\Gamma(\beta+1)}\right]\right)}\varepsilon = c_f \varepsilon.$$

Thus, the fractional boundary value problem (1) is Ulam-Hyers stable. By putting $\varphi(\varepsilon) = \gamma\varepsilon, \varphi(0) = 0$ yields that the fractional problem (1) generalized Ulam-Hyers stable.

Theorem 7. *Let $f : [1, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that (H_1) and (16) hold. In addition, the following hypothesis holds (H_3) : There exists an function $\varphi \in C([1, T], \mathbb{R}_+)$ and there exists $\eta_\varphi > 0$ such that for any $t \in [1, T]$*

$$\frac{1}{\Gamma(\alpha+\beta)} \int_1^t (\log \frac{t}{s})^{\alpha+\beta-1} \varphi(s) \frac{ds}{s} \leq \eta_\varphi \varphi(t). \tag{19}$$

Then the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

Proof. Let $v \in X$ be a solution of the inequality (15), i.e.

$$\left| {}_H^C D^\alpha \left({}_H^C D^\beta u(t) + \gamma \right) - f(t, u(t), u(\lambda t)) \right| \leq \varepsilon\varphi(t), \ t \in [1, T],$$

and let us denote by $u \in X$ the unique solution of the problem

$$\begin{cases} {}_H^C D^\alpha \left({}_H^C D^\beta u(t) + \gamma \right) = f(t, u(t), u(\lambda t)), \gamma \in \mathbb{R}, t \in J, 0 < \alpha, \beta < 1, 0 < \lambda < 1, \\ u(1) = v(1), \quad u(T) = v(T), \end{cases}$$

Thanks to Lemma 3, we obtain

$$u(t) = {}_H I^{\alpha+\beta} h_u(t) - \gamma {}_H I^\beta u(t) + \frac{c_0 (\log t)^\beta}{\Gamma(\beta+1)} + c_1,$$

and by integration of the inequality (15), we obtain

$$\begin{aligned} &\left| v(t) - {}_H I^{\alpha+\beta} h_v(t) - \gamma {}_H I^\beta v(t) + \frac{c_2 (\log t)^\beta}{\Gamma(\beta+1)} + c_3 \right| \qquad (20) \\ \leq\ & \frac{\varepsilon}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} \varphi(s) \frac{ds}{s} \leq \varepsilon \eta_\varphi \varphi(t). \end{aligned}$$

By (18) and (20), we have

$$\begin{aligned} |v(t)-u(t)| \leq\ & \varepsilon\eta_{\varphi}\varphi(t)+\frac{2k}{\Gamma(\alpha+\beta)}\int_1^t(\log\frac{t}{s})^{\alpha+\beta-1}|v(s)-u(s)|\frac{ds}{s} \\ & +\frac{|\gamma|}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}|v(s)-u(s)|\frac{ds}{s} \end{aligned}$$

Hence,

$$\|v(s)-u(s)\|\left(1-\frac{2\omega(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)\leq\frac{\varepsilon\eta_{\varphi}}{1-\left[\frac{2k(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\gamma|(\log T)^{\beta}}{\Gamma(\beta+1)}\right]}\varphi(t).$$

Then, for each $t\in[1,T]$

$$|v(t)-u(t)|\leq\frac{\varepsilon\eta_{\varphi}}{1-\left[\frac{2k(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\gamma|(\log T)^{\beta}}{\Gamma(\beta+1)}\right]}\varphi(t).$$

So, the fractional boundary value problem (1) is Ulam-Hyers-Rassias stable.

5. Examples

To illustrate our main results, we treat the following examples.

Example 1. *Consider the Caputo-Hadamard type fractional pantograph equation*

$$\begin{cases} {}^{C}_{H}D^{\frac{1}{2}}\left({}^{C}_{H}D^{\frac{1}{3}}+\frac{1}{15}\right)u(t)=\frac{2}{13e^{2t+3}}u(t)+\frac{2}{13e^{2t+3}}u\left(\frac{3}{4}t\right)+\frac{2}{9},\ t\in[1,e], \\ x(1)=\frac{1}{5},\quad x(e)=\sqrt{3}. \end{cases} \tag{21}$$

For this example, we have $\alpha=\frac{1}{2},\beta=\frac{1}{3},\gamma=\frac{1}{15},\lambda=\frac{3}{4}$ *and* $T=e$.

On the other hand,

$$f(t,u,v)=\frac{2}{13e^{2t+3}}u+\frac{2}{13e^{2t+3}}v+\frac{2}{9}.$$

For $t\in[1,e]$ *and* $(u_1,v_1),(u_2,v_2)\in\mathbb{R}^2$, *we have*

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq\frac{2}{13e^3}(|u_1-u_2|+|v_1-v_2|).$$

Hence the condition (H_1) *holds with* $k=\frac{2}{13e^3}$.

Thus conditions

$$\frac{4k}{\Gamma(\alpha+\beta+1)}(\log T)^{\alpha+\beta} \simeq 3.257\,1\times 10^{-2} < 1-\frac{2|\gamma|}{\Gamma(\beta+1)}(\log T)^{\beta} \simeq 0.850\,69,$$

and

$$\frac{2k}{\Gamma(\alpha+\beta+1)}(\log T)^{\alpha+\beta}+\frac{|\gamma|}{\Gamma(\beta+1)}(\log T)^{\beta} \simeq 9.094\,2\times 10^{-2} < 1,$$

are satisfied. It follows from Theorem 6, that the problem (21) has a unique solution on $[1,e]$ *and from Theorem 14, the fractional problem (21) is Ulam-Hyers stable.*

Example 2. *Let us consider the following fractional pantograph equation*

$$\begin{cases} {}_{H}^{C}D^{\frac{3}{4}}\left({}_{H}^{C}D^{\frac{2}{5}}+\frac{1}{10}\right)u(t)=\frac{1}{15}\sin(t)\,u(t)+\frac{1}{15}u\left(\frac{3}{4}t\right)+\frac{3}{7},\ t\in[1,e], \\ u(1)=2, \quad u(e)=3, \end{cases} \tag{22}$$

Consider fractional pantograph equation with $\alpha=\frac{3}{4}, \beta=\frac{2}{5}, \gamma=\frac{1}{10}, \lambda=\frac{3}{4}$ *and* $f(t,x,y)=\frac{1}{15}\sin(t)\,x+\frac{1}{15}y+\frac{3}{7}$.

For $(u_1,y_1),(u_2,v_2)\in\mathbb{R}^2$ *and* $t\in[1,e]$, *we have*

$$\begin{aligned}|f(t,u_1,v_1)-f(t,u_2,v_2)| &\le \frac{1}{15}|\sin(t)||u_1-u_2|+\frac{1}{15}|v_1-v_2| \\ &\le \frac{1}{15}(|u_1-u_2|+|v_1-v_2|).\end{aligned}$$

Hence hypothesis (H_1) *is satisfied with with* $k=\frac{1}{15}$.

We can show that

$$\frac{4k}{\Gamma(\alpha+\beta+1)}(\log T)^{\alpha+\beta}+\frac{2|\gamma|}{\Gamma(\beta+1)}(\log T)^{\beta} \simeq 0.473\,94 < 1.$$

Let $\varphi(t)=t^3$. *Then*

$$\frac{1}{\Gamma(\frac{3}{4}+\frac{2}{5}+1)}\int_1^t(\log\frac{t}{s})^{\frac{3}{4}+\frac{2}{5}-1}\varphi(s)\frac{ds}{s} \le \frac{6}{\Gamma\left(\frac{43}{20}\right)}t^3=\eta_\varphi\varphi(t).$$

Thus hypothesis (H_3) *is satisfied with* $\varphi(t)=t^3$ *and* $\eta_\varphi=\frac{6}{\Gamma(\frac{43}{20})}$. *It follows from Theorem 6 that the fractional problem (22) as a unique solution on* $[1,e]$, *and from Theorem 15 problem (22) is Ulam-Hyers-Rassias stable.*

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