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COEFFICIENT INEQUALITIES FOR CLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH (P,Q)-DERIVATIVE OF SALAGEAN OPERATOR

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Abstract. In this paper, the authors introduce the newly constructed subclass of bi-univalent functions defined by the Jackson (p,q)-derivative operator of Salagèan type associated with Chebyshev polynomial. The initial coefficient bounds and Fekete-Szegö inequalities for the function class are obtained. Moreover, certain special cases are also pointed out. The results present in this paper generalizes and improves the results due to Altinkaya and Yalcin.

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1. Introduction and Motivation

Let $\mathcal{A}$ represent the class of functions analytic in $\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ satisfying normalized condition $f(0) = f'(0) - 1 = 0$. Then each $f \in \mathcal{A}$ has the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions of the form (1) which are univalent in Δ.

If f and g are analytic functions in Δ, then f is said to be subordinate to g, written as $f(z) \prec g(z) \quad (z \in \Delta)$, if there exists a Schwarz function $\phi(z)$, analytic in Δ with $\phi(0) = 0$ and $|\phi(z)| < 1 \quad (z \in \Delta)$ such that $f(z) = g(\phi(z)) \ \ (z \in \Delta)$. If g is univalent in Δ then (see [6, 19])

$$f(z) \prec g(z) \ \ (z \in \Delta) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The well-known Koebe One-Quarter-Theorem (see [12]) ensures that the image of Δ under every function f in the normalized univalent function class $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every such univalent function has an inverse f^{-1} satisfies the condition:

$$f^{-1}(f(z)) = z \quad (z \in \Delta),$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let σ denote the class of such functions. For a brief history of function in the class, see [4, 5, 16, 21]. Recently, many researchers (see [1, 17, 22, 24, 25]), mention a few have put their efforts to introduce various subclasses of bi-univalent function and obtained initial non-sharp coefficient bounds. Determining general coefficient estimates $|a_n|$ $(n \in \mathbb{N})$ for analytic and bi-univalent problem is still an open problem.

The Chebyshev polynomials named after Pafnuty Chebyshev [10] play an important role in geometric function theory. These polynomials are a sequence of orthogonal polynomials that are related to De Movire formula. Their need and importance in the area of numerical analysis and approximation theory have increased day by day both theoretical as well as practical view point. There are four types of Chebyshev polynomials. However, majority of research papers deal with orthogonal polynomials of Chebyshev family, which contains many results of Chebyshev polynomials of first kind $T_n(t)$ and second kind $U_n(t)$ (see, for details[11, 18]).
In case of a real variable t on $(-1, 1)$, the Chebyshev polynomials of first and second kinds are defined by

$$T_n(t) = cos n\theta, \qquad U_n(t) = \frac{sin(n+1)\theta}{sin\theta},$$

where the subscript n denotes the polynomials degree and $t = cos\theta$.
Now, we consider the function which is the generating function of a Chebyshev polynomial of second type

$$\psi(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in \Delta).$$

If we take $t = cos\alpha, \quad \alpha \in (-\frac{\pi}{3}, \frac{\pi}{3})$, then

$$\psi(z,t) = \frac{1}{1 - 2cos\alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n$$
$$= 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \cdots \quad (z \in \Delta).$$

Therefore, we can write

$$\psi(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in \Delta, \quad t \in (-1,1)), \tag{3}$$

where $U_{n-1} = \frac{sin(ncos^{-1}t)}{\sqrt{1-t^2}} \quad (n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind. It is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \quad U_4(t) = 16t^4 - 12t^2 + 1, \cdots. \tag{4}$$

In geometric function theory, different researchers all over the globe have constructed various subclasses of analytic and bi-univalent functions from different view points. In the recent years, applications of quantum calculus (q-calculus) in the fields of ordinary fractional calculus, optimal control, q-difference equation, q-integral equation, q-transform analysis, approximation theory and number theory are an active area of research. q-calculus is a generalization of many fields, such as hypergeometric series, complex analysis, and particle physics. Jackson [14, 15] was the first to develop q integral and q-derivative in a systematic way and later geometrical interpretation of the q-analysis has been recognized through studies of quantum groups. It has attracted the attention of various researchers. Researchers have applied it to construct and investigated several classes of analytic and bi-univalent functions and their interesting results are too numerous to discuss. The extension of the q-calculus to post-quantum calculus denoted by the (p,q)-calculus is possible. Such an extension of quantum calculus cannot be obtained directly by substitution of q by q/p in q-calculus. When the case $p = 1$ in (p,q)-calculus, the q-calculus may be obtained (see [13, 20]).

We provide some basic concept of q-calculus. We assume throughout our discussion that $0 < q < p \leq 1$. We recall the definitions of fractional (p,q)-calculus operator of complex-valued function $f(z)$ as follows:

Definition 1. *(see [9]) The (p,q)-derivative operator of a function f is defined by*

$$(D_{p,q}f)(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \neq 0 \tag{5}$$

and $(D_{p,q}f)(0) = f'(0)$ *provided that the function* f *is differentiable at 0.*

The so-called (p,q)-bracket or twin-basic number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \quad (p \neq q). \tag{6}$$

It is clear that $D_{p,q}z^n = [n]_{p,q}z^{n-1}$. For $p = 1$, the Jackson (p,q)-derivative reduces to the Hahn derivative given by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0). \tag{7}$$

The twin-basic number $[n]_{p,q}$ is a natural generalization of the q-number i.e.

$$\lim_{p \longrightarrow 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q} \quad (q \neq 1). \tag{8}$$

Thus, for a function f of the form (1), it follows from Definition 1 that

$$(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1} \quad (z \in \Delta), \tag{9}$$

where $[n]_{p,q}$ is defined as (6). Note that q-number is the extension of ordinary derivative i.e.

$$\lim_{q \longrightarrow 1^-} (D_q f)(z) = f'(z). \tag{10}$$

The (p,q) analogue of Salagèan differential operator $R^k_{p,q} : \mathcal{A} \longrightarrow \mathcal{A} \quad (k \in \mathbb{N})$ is defined as follows:

$$\begin{aligned} R^0_{p,q}f(z) &= f(z) \\ R^1_{p,q}f(z) &= z(D_{p,q}f)(z)) \\ &\cdots \\ &\cdots \\ &\cdots \\ R^k_{p,q}f(z) &= R^1_{p,q}(R^{k-1}_{p,q}f(z)). \end{aligned} \tag{11}$$

Therefore, for a function $f(z)$ of the form (1), we have

$$R^k_{p,q}f(z) = z + \sum_{n=2}^{\infty} [n]^k_{p,q} a_n z^n \quad (z \in \Delta). \tag{12}$$

From (12), we observe that

$$\begin{aligned} lim_{p\longrightarrow 1,q\longrightarrow 1^-} R^k_{p,q} f(z) &= z + lim_{p\longrightarrow 1,q\longrightarrow 1^-} \sum_{n=2}^{\infty} [n]^k_{p,q} a_n z^n \\ &= z + \sum_{n=2}^{\infty} n^k a_n z^n = S^k f(z), \end{aligned} \tag{13}$$

where S^k is the Salagèan differential operator which was defined by Salagèan (see [23]) and later studied by various researchers in this direction.
Very recently, several authors (see[3, 7, 8]) have introduced and investigated various subclasses of bi-univalent functions and obtained the initial coefficient bounds and Fekete-Szegö inequality by making use of Chebyshev polynomials instead of class of Caratheodory functions. Motivated by aforementioned works and making use of (p,q)-derivative operator of Salagèan type, we introduce a new subclass of bi-univalent function associated with the Chebyshev polynomials as follows:

Definition 2. *A function $f \in \sigma$ given by (1) is said to be in the class $M^k_\sigma(p, q, \lambda, t)$ $(0 < q < p \leq 1;\quad k \in \mathbb{N};\quad 0 < \lambda \leq 1;\quad t \in (\frac{1}{2}, 1])$ if the following subordination conditions are satisfied.*

$$\frac{1}{2}\left[\frac{R^k_{p,q} f(z)}{f(z)} + \left(\frac{R^k_{p,q} f(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] \prec L(z,t) = \frac{1}{1-2tz+z^2} \quad (z \in \Delta), \tag{14}$$

and

$$\frac{1}{2}\left[\frac{R^k_{p,q} g(w)}{g(w)} + \left(\frac{R^k_{p,q} g(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right] \prec L(w,t) = \frac{1}{1-2tw+w^2} \quad (w \in \Delta), \tag{15}$$

where $g = f^{-1}$.

By specializing the parameters p, q, λ and k in Definition 2, we obtain the following subclasses of σ mention below.

Remark 1. *For $p \longrightarrow 1$ and $k = 1$, we get the class $M^1_\sigma(1, q, \lambda, t) = R_\sigma(q, \lambda, t)$ consist of functions $f \in \sigma$ satisfying the condition*

$$\frac{1}{2}\left[\frac{z(D_q f)(z)}{f(z)} + \left(\frac{z(D_q f)(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] \prec L(z,t) = \frac{1}{1-2tz+z^2},$$

and

$$\frac{1}{2}\left[\frac{w(D_q g)(w)}{g(w)}+\left(\frac{w(D_q g)(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right] \prec L(w,t)=\frac{1}{1-2tw+w^2},$$

where the function $g=f^{-1}$ is defined by (2).

Remark 2. *For $p \longrightarrow 1$ and $q \longrightarrow 1^-$, we get the class $M_\sigma^k(1,1^-,\lambda,t)=V_\sigma^k(\lambda,t)$ consist of functions $f\in\sigma$ satisfying the condition*

$$\frac{1}{2}\left[\frac{S^k f(z)}{f(z)}+\left(\frac{S^k f(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] \prec L(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\frac{1}{2}\left[\frac{S^k g(w)}{g(w)}+\left(\frac{S^k g(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right] \prec L(w,t)=\frac{1}{1-2tw+w^2},$$

where the function $g=f^{-1}$ is defined by (2).

Remark 3. *For $p \longrightarrow 1$, $q \longrightarrow 1^-$ and $k=1$ we get the class $M_\sigma^1(1,1^-,\lambda,t)=\mathcal{S}_\sigma(\lambda,t)$ consist of functions $f\in\sigma$ satisfying the condition*

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] \prec L(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right] \prec L(w,t)=\frac{1}{1-2tw+w^2},$$

where the function $g=f^{-1}$ is defined by (2).

Remark 4. *For $p \longrightarrow 1$, $q \longrightarrow 1^-$ and $k=\lambda=1$ we get the class $M_\sigma^1(1,1^-,1,t)=\mathcal{S}_\sigma(t)$ consist of functions $f\in\sigma$ satisfying the condition*

$$\left[\frac{zf'(z)}{f(z)}\right] \prec L(z,t)=\frac{1}{1-2tz+z^2},$$

and

$$\left[\frac{wg'(w)}{g(w)}\right] \prec L(w,t)=\frac{1}{1-2tw+w^2},$$

where the function $g=f^{-1}$ is defined by (2).

Remark 5. *The classes in Remarks 3 and 4 mentioned above improve the classes discussed in Definition 1.1 and 1.3 studied by Altinkaya and Yalcin (see [2]) respectively.*

The object of the present investigation is to obtain initial coefficient bounds and Fekete-Szegö inequalities for the newly constructed subclass of bi-univalent functions $M_\sigma^k(p,q,\lambda,t)$ associated with Chebyshev polynomials.

2. Coefficient bounds for the class $M_\sigma^k(p,q,\lambda,t)$

Theorem 1. *Let the function f given by (1) be in the class $M_\sigma^k(p,q,\lambda,t)$. Then*

$$|a_2| \le \frac{4\lambda t\sqrt{2t}}{\sqrt{|[2\lambda(1+\lambda)([3]_{p,q}^k - [2]_{p,q}^k) - (\lambda^2+3\lambda)([2]_{p,q}^k - 1)^2]4t^2 + (1+\lambda)^2([2]_{p,q}^k - 1)^2|}} \tag{16}$$

and

$$|a_3| \le \frac{16\lambda^2 t^2}{(1+\lambda)^2([2]_{p,q}^k - 1)^2} + \frac{4\lambda t}{(1+\lambda)([3]_{p,q}^k - 1)}. \tag{17}$$

Proof. Let $f \in M_\sigma^k(p,q,\lambda,t)$ and g be the analytic extension of f^{-1} in Δ. Then by Definition 2, we have

$$\frac{1}{2}\left[\frac{R_{p,q}^k f(z)}{f(z)} + \left(\frac{R_{p,q}^k f(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] = L(u(z),t) \tag{18}$$

and

$$\frac{1}{2}\left[\frac{R_{p,q}^k g(w)}{g(w)} + \left(\frac{R_{p,q}^k g(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right] = L(v(w),t). \tag{19}$$

Define the functions $u(z)$ and $v(w)$ by

$$u(z) = c_1 z + c_2 z^2 + \ldots, \tag{20}$$

$$v(w) = d_1 w + d_2 w^2 + \cdots, \tag{21}$$

that are analytic in Δ with $u(0) = v(0) = 0$ and $|u(z)| < 1$ and $|v(w)| < 1$ for all $z,\ \ w \in \Delta$. Making use of (20) and (21) in (18) and (19) respectively, we obtain

$$\frac{1}{2}\left[\frac{R_{p,q}^k f(z)}{f(z)} + \left(\frac{R_{p,q}^k f(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right] = 1 + U_1(t)u(z) + U_2(t)u^2(z) + \cdots, \tag{22}$$

and

$$\frac{1}{2}\left[\frac{R^k_{p,q}g(w)}{g(w)}+\left(\frac{R^k_{p,q}g(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right]=1+U_1(t)v(w)+U_2(t)v^2(w)+\cdots. \tag{23}$$

Now, for a function $f(z)$ of the form (1), we have

$$\frac{R^k_{p,q}f(z)}{f(z)}=1+([2]^k_{p,q}-1)a_2z+[([3]^k_{p,q}-1)a_3-([2]^k_{p,q}-1)a_2^2]z^2+\cdots, \tag{24}$$

and

$$\left[\frac{R^k_{p,q}f(z)}{f(z)}\right]^{\frac{1}{\lambda}}=1+\frac{([2]^k_{p,q}-1)}{\lambda}a_2z+\left[\left\{\frac{1-\lambda}{2\lambda^2}([2]^k_{p,q}-1)^2-\frac{([2]^k_{p,q}-1)}{\lambda}\right\}a_2^2+\frac{([3]^k_{p,q}-1)}{\lambda}a_3\right]z^2+\cdots. \tag{25}$$

From (24) and (25), it follows that

$$\frac{1}{2}\left[\frac{R^k_{p,q}f(z)}{f(z)}+\left(\frac{R^k_{p,q}f(z)}{f(z)}\right)^{\frac{1}{\lambda}}\right]=1+\frac{1+\lambda}{2\lambda}([2]^k_{p,q}-1)a_2z+\left[\left(\frac{(1-\lambda)([2]^k_{p,q}-1)^2}{4\lambda^2}-\frac{(1+\lambda)([2]^k_{p,q}-1)}{2\lambda}\right)\right.$$
$$\left.a_2^2+\frac{1+\lambda}{2\lambda}([3]^k_{p,q}-1)a_3\right]z^2+\cdots. \tag{26}$$

Similarly, for a function $g(w)$ of the form (2), we have

$$R^k_{p,q}g(w)=w-[2]^k_{p,q}a_2w^2+(2a_2^2-a_3)[3]^k_{p,q}w^3+\cdots.$$

Proceeding similar manner as above, we obtain

$$\frac{1}{2}\left[\frac{R^k_{p,q}g(w)}{g(w)}+\left(\frac{R^k_{p,q}g(w)}{g(w)}\right)^{\frac{1}{\lambda}}\right]=1-\frac{1+\lambda}{2\lambda}([2]^k_{p,q}-1)a_2w$$
$$+\left[\left\{\frac{1+\lambda}{2\lambda}(2[3]^k_{p,q}-[2]^k_{p,q}-1)+\frac{1-\lambda}{4\lambda^2}([2]^k_{p,q}-1)^2\right\}a_2^2-\frac{1+\lambda}{2\lambda}([3]^k_{p,q}-1)a_3\right]w^2+\cdots. \tag{27}$$

Using (20),(26) in (22) and (21), (27) in (23) we get

$$1+\frac{1+\lambda}{2\lambda}([2]^k_{p,q}-1)a_2z+\left[\left\{\frac{(1-\lambda)([2]^k_{p,q}-1)^2}{4\lambda^2}-\frac{(1+\lambda)}{2\lambda}([2]^k_{p,q}-1)\right\}a_2^2+\frac{1+\lambda}{2\lambda}([3]^k_{p,q}-1)a_3\right]z^2+\cdots.$$
$$=1+U_1(t)c_1z+[U_1(t)c_2+U_2(t)c_1^2]z^2+\cdots. \tag{28}$$

and

$$1-\frac{1+\lambda}{2\lambda}([2]^k_{p,q}-1)a_2w+\left[\left\{\frac{1+\lambda}{2\lambda}(2[3]^k_{p,q}-[2]^k_{p,q}-1)+\frac{1-\lambda}{4\lambda^2}([2]^k_{p,q}-1)^2\right\}a_2^2\right.$$
$$\left.-\frac{1+\lambda}{2\lambda}([3]^k_{p,q}-1)a_3\right]w^2+\cdots.=1+U_1(t)d_1w+[U_1(t)d_2+U_1(t)d_1^2]w^2+\cdots. \tag{29}$$

From (28) and (29), we have

$$\frac{1+\lambda}{2\lambda}([2]_{p,q}^k-1)a_2=U_1(t)c_1, \tag{30}$$

$$\left\{\frac{1-\lambda}{4\lambda^2}([2]_{p,q}^k-1)^2-\frac{1+\lambda}{2\lambda}([2]_{p,q}^k-1)\right\}a_2^2+\frac{1+\lambda}{2\lambda}([3]_{p,q}^k-1)a_3=U_1(t)c_2+U_2(t)c_1^2, \tag{31}$$

and

$$-\frac{1+\lambda}{2\lambda}([2]_{p,q}^k-1)a_2=U_1(t)d_1, \tag{32}$$

$$\left\{\frac{1+\lambda}{2\lambda}(2[3]_{p,q}^k-[2]_{p,q}^k-1)+\frac{1-\lambda}{4\lambda^2}([2]_{p,q}^k-1)^2\right\}a_2^2-\frac{1+\lambda}{2\lambda}([3]_{p,q}^k-1)a_3=U_1(t)d_2+U_2(t)d_1^2. \tag{33}$$

Equations (30) and (32) give

$$c_1=-d_1, \tag{34}$$

and

$$\frac{(1+\lambda)^2}{2\lambda^2}([2]_{p,q}^k-1)^2a_2^2=U_1^2(t)(c_1^2+d_1^2). \tag{35}$$

If we add (31) and (33), we get

$$\left[\frac{(1-\lambda)([2]_{p,q}^k-1)^2}{2\lambda^2}-\frac{1+\lambda}{2\lambda}([2]_{p,q}^k-1)+\frac{1+\lambda}{2\lambda}(2[3]_{p,q}^k-[2]_{p,q}^k-1)\right]a_2^2=U_1(t)(c_2+d_2)+U_2(t)(c_1^2+d_1^2). \tag{36}$$

Using (35) in the equation (36) gives

$$a_2^2=\frac{2\lambda^2U_1^3(t)(c_2+d_2)}{\left[(1-\lambda)([2]_{p.q}^k-1)^2+2\lambda(1+\lambda)([3]_{p,q}^k-[2]_{p,q}^k)\right]U_1^2(t)-(1+\lambda)^2([2]_{p,q}^k-1)^2U_2(t)}. \tag{37}$$

It is well-known that if $|u(z)|<1$ and $|v(w)|<1$, then

$$|c_j|\leq 1 \quad\text{and}\quad |d_j|\leq 1 \quad \text{for all} \quad j\in\mathbb{N}. \tag{38}$$

Making use of (4) and applying (38) to the coefficients c_2 and d_2 in (37), we obtain the desire estimate for $|a_2|$ as stated in (16). Further, for finding bounds for a_3 we proceed as follows.
Subtracting (33) from (31), we get

$$\frac{1+\lambda}{\lambda}([3]_{p,q}^k-1)a_3-\frac{1+\lambda}{\lambda}([3]_{p,q}^k-1)a_2^2=U_1(t)(c_2-d_2)+U_2(t)(c_1^2-d_1^2). \tag{39}$$

Using (34) and (35) in (39) give

$$a_3=\frac{2\lambda^2(c_1^2+d_1^2)U_1^2(t)}{(1+\lambda)^2([2]_{p,q}^k-1)^2}+\frac{\lambda(c_2-d_2)U_1(t)}{(1+\lambda)([3]_{p,q}^k-1)}. \tag{40}$$

Putting the value of $U_1(t)$ from (4) and using the coefficient inequalities (38) in (40), we obtain

$$|a_3| \leq \frac{16\lambda^2 t^2}{(1+\lambda)^2([2]_{p,q}^k - 1)^2} + \frac{4\lambda t}{(1+\lambda)([3]_{p,q}^k - 1)}.$$

This completes the proof of Theorem 1.

Taking $p \longrightarrow 1$ and $k = 1$ in the above theorem, we get the following result:

Corollary 2. *Let the function f given by (1) be in the class $R_\sigma(q, \lambda, t)$. Then we have*

$$|a_2| \leq \frac{4\lambda t\sqrt{2t}}{\sqrt{|[2\lambda(1+\lambda)([3]_q - [2]_q) - (\lambda^2 + 3\lambda)([2]_q - 1)^2]4t^2 + (1+\lambda)^2([2]_q - 1)^2|}},$$

and

$$|a_3| \leq \frac{16\lambda^2 t^2}{(1+\lambda)^2([2]_q - 1)^2} + \frac{4\lambda t}{(1+\lambda)([3]_q - 1)}.$$

Putting $p \longrightarrow 1$ and $q \longrightarrow 1^-$ in Theorem 1, we obtain the following result in form of corollary:

Corollary 3. *Let the function f given by (1) be in the class $V_\sigma^k(\lambda, t)$. Then*

$$|a_2| \leq \frac{4\lambda t\sqrt{2t}}{\sqrt{|[2\lambda(1+\lambda)(3^k - 2^k) - (\lambda^2 + 3\lambda)(2^k - 1)^2]4t^2 + (1+\lambda)^2(2^k - 1)^2|}},$$

and

$$|a_3| \leq \frac{16\lambda^2 t^2}{(1+\lambda)^2(2^k - 1)^2} + \frac{4\lambda t}{(1+\lambda)(3^k - 1)}.$$

Letting $k = 1$ in Corollary 3 we obtain the result for the class $S_\sigma(\lambda, t)$.

Corollary 4. *(see [[2], Theorem 2.1) Let f given by (1) be in the class $S_\sigma(\lambda, t)$. Then*

$$|a_2| \leq \frac{4\lambda t\sqrt{2t}}{\sqrt{|(\lambda^2 - \lambda)4t^2 + (1+\lambda)^2|}},$$

and

$$|a_3| \leq \frac{16\lambda^2 t^2}{(1+\lambda)^2} + \frac{2\lambda t}{(1+\lambda)}.$$

Taking $\lambda = 1$ in the Corollary 4 we reach at the following conclusion.

Corollary 5. *(see [2], Corollary 2.2) Let the function f given by (1) be in the class $S_\sigma(t)$. Then*

$$|a_2| \leq 2t\sqrt{2t},$$

and

$$|a_3| \leq 4t^2 + t.$$

3. Fekete-Szegö Inequalities for the function class $M_\sigma^k(p,q,\lambda,t)$

Theorem 6. *Let the function f given by (1) be in the class $M_\sigma^k(p,q,\lambda,t)$. Then for any real number α, we have*

$$|a_3-\alpha a_2^2| \leq \begin{cases} \frac{4\lambda t}{(1+\lambda)([3]_{p,q}^k-1)} \quad |\alpha-1| \leq \frac{\left|\frac{(1+\lambda)^2([2]_{p,q}^k-1)^2}{4t^2}+2\lambda(1+\lambda)([3]_{p,q}^k-[2]_{p,q}^k)-(\lambda^2+3\lambda)([2]_{p,q}^k-1)^2\right|}{2\lambda(1+\lambda)([3]_{p,q}^k-1)}, \\ \frac{32\lambda^2t^3|1-\alpha|}{|[2\lambda(1+\lambda)([3]_{p,q}^k-[2]_{p,q}^k)-(\lambda^2+3\lambda)([2]_{p,q}^k-1)^2]4t^2+(1+\lambda)^2([2]_{p,q}^k-1)^2|} \\ \quad |\alpha-1| \geq \frac{\left|\frac{(1+\lambda)^2([2]_{p,q}^k-1)^2}{4t^2}+2\lambda(1+\lambda)([3]_{p,q}^k-[2]_{p,q}^k)-(\lambda^2+3\lambda)([2]_{p,q}^k-1)^2\right|}{2\lambda(1+\lambda)([3]_{p,q}^k-1)}. \end{cases} \tag{41}$$

Proof. Taking the values of a_2 and a_3 from (37) and (39) and after simplification, we get

$$a_3-\alpha a_2^2 = (1-\alpha)\frac{2\lambda^2U_1^3(t)(c_2+d_2)}{[(1-\lambda)([2]_{p,q}^k-1)^2+2\lambda(1+\lambda)([3]_{p,q}^k-[2]_{p,q}^k)]U_1^2(t)-(1+\lambda)^2([2]_{p,q}^k-1)^2U_2(t)}$$

$$+\frac{\lambda(c_2-d_2)U_1(t)}{(1+\lambda)([3]_{p,q}^k-1)} = U_1(t)\left[\left(s(\alpha)+\frac{\lambda}{(1+\lambda)([3]_{p,q}^k-1)}\right)c_2+\left(s(\alpha)-\frac{\lambda}{(1+\lambda)([3]_{p,q}^k-1)}\right)d_2\right], \tag{42}$$

where

$$s(\alpha) = \frac{2\lambda^2(1-\alpha)U_1^2(t)}{[(1-\lambda)([2]_{p,q}^k-1)^2+2\lambda(1+\lambda)([3]_{p,q}^k-[2]_{p,q}^k)]U_1^2(t)-(1+\lambda)^2([2]_{p,q}^k-1)^2U_2(t)}. \tag{43}$$

Thus, in view of (4), we conclude that

$$|a_3-\alpha a_2^2| \leq \begin{cases} \frac{4\lambda t}{(1+\lambda)([3]_{p,q}^k-1)} & 0 \leq |s(\alpha)| \leq \frac{\lambda}{(1+\lambda)([3]_{p,q}^k-1)} \\ 4t|s(\alpha)| & |s(\alpha)| \geq \frac{\lambda}{(1+\lambda)([3]_{p,q}^k-1)}. \end{cases} \tag{44}$$

The assertion (41) now follows from (44). This complete the proof of Theorem 6.

Taking $p \longrightarrow 1$ and $k=1$ in Theorem 6 we obtain the following result.

Corollary 7. *Let the function f given by (1) be in the class $R_\sigma(q,\lambda,t)$. For any $\alpha \in \mathbb{R}$,*

$$|a_3-\alpha a_2^2| \leq \begin{cases} \frac{4\lambda t}{(1+\lambda)([3]_q-1)} \quad |\alpha-1| \leq \frac{|\frac{(1+\lambda)^2([2]_q-1)^2}{4t^2}+2\lambda(1+\lambda)([3]_q-[2]_q)-(\lambda^2+3\lambda)([2]_q-1)^2|}{2\lambda(1+\lambda)([3]_q-1)}, \\ \frac{32\lambda^2t^3(1-\alpha)}{|[2\lambda(1+\lambda)([3]_q-[2]_q)-(\lambda^2+3\lambda)([2]_q-1)^2]4t^2+(1+\lambda)^2([2]_q-1)^2|}, \\ \quad |\alpha-1| \geq \frac{|\frac{(1+\lambda)^2([2]_q-1)^2}{4t^2}+2\lambda(1+\lambda)([3]_q-[2]_q)-(\lambda^2+3\lambda)([2]_q-1)^2|}{2\lambda(1+\lambda)([3]_q-1)}. \end{cases}$$

Putting $p \longrightarrow 1$ and $q \longrightarrow 1^-$ in Theorem 6 we obtain Fekete-Szegö inequality for the function class $V_\sigma^k(\lambda,t)$ as follows:

Corollary 8. *Let the function f given by (1) be in the function class $V_\sigma^k(\lambda,t)$. Then for any $\alpha \in \mathbb{R}$, we have*

$$|a_3-\alpha a_2^2| \leq \begin{cases} \frac{4\lambda t}{(1+\lambda)(3^k-1)} \\ \qquad |\alpha-1| \leq \frac{\left|\frac{(1+\lambda)^2(2^k-1)^2}{4t^2}+2\lambda(1+\lambda)(3^k-2^k)-(\lambda^2+3\lambda)(2^k-1)^2\right|}{2\lambda(1+\lambda)(3^k-1)} \\ \frac{32\lambda^2t^3|1-\alpha|}{|[2\lambda(1+\lambda)(3^k-2^k)-(\lambda^2+3\lambda)(2^k-1)^2]4t^2+(1+\lambda)^2(2^k-1)^2|} \\ \qquad |\alpha-1| \geq \frac{\left|\frac{(1+\lambda)^2(2^k-1)^2}{4t^2}+2\lambda(1+\lambda)(3^k-2^k)-(\lambda^2+3\lambda)(2^k-1)^2\right|}{2\lambda(1+\lambda)(3^k-1)} \end{cases}$$

Letting $k=1$, $p \longrightarrow 1$ and $q \longrightarrow 1^-$ in the Theorem 6, we get the following result due to Altinkaya and Yalcin.

Corollary 9. *([2], Theorem 6) Let the function f given by (1) be in the class $S_\sigma(\lambda,t)$. Then*

$$|a_3-\alpha a_2^2| \leq \begin{cases} \frac{2\lambda t}{1+\lambda} & |\alpha-1| \leq \frac{\left|\frac{(1+\lambda)^2}{4t^2}+\lambda^2-\lambda\right|}{4\lambda(1+\lambda)} \\ \frac{32\lambda^2t^3|1-\alpha|}{(\lambda^2-\lambda)4t^2+(1+\lambda)^2} & |\alpha-1| \geq \frac{\left|\frac{(1+\lambda)^2}{4t^2}+\lambda^2-\lambda\right|}{4\lambda(1+\lambda)}. \end{cases}$$

Taking $\alpha=1$ in Corollary 9, we get the following result due to Altinkaya and Yalcin.

Corollary 10. *([2], Corollary 7) Let f given by (1) be in the class $S_\sigma(\lambda,t)$. Then*

$$|a_3-a_2^2| \leq \frac{2\lambda t}{1+\lambda}.$$

Putting $\lambda=1$ in Corollary 10 we obtain the result for class $S_\sigma(t)$.

Corollary 11. *([2], Corollary 8) Let the function f given by (1) be in the class $S_\sigma(t)$. Then*

$$|a_3-a_2^2| \leq t.$$

Concluding Remarks: Various authors have considered the q-derivative operator to defined bi-univalent function classes. The results present in this paper has a new generalization for q-derivative operator of Salagèan type. Many corollaries have been generated by varying some parameters involved in the above classes of functions defined. Researchers can make use of Faber polynomial and Sigmoid functions instead of class of Chebyshev polynomials for finding bounds for the above mentioned classes.

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