Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 63/2020
pp. 65-80
doi: 10.17114/j.aua.2020.63.06

ON THE φ-ORDER AND THE φ-TYPE OF ANALYTIC AND MEROMORPHIC FUNCTIONS IN THE UNIT DISC

M. A. Kara and B. Belaïdi

Abstract. The paper is devoted to deriving some estimates of sum, product and derivative of analytic or meromorphic functions in the unit disc involving the concepts of the φ-order and the φ-type. Our results generalise and improve many previous results that have used the usual order [7, 16, 17, 21, 23] and the iterated p-order [4, 13, 14].

2010 *Mathematics Subject Classification*: 34M10, 30D35.

Keywords: entire function, meromorphic function, φ-order, φ-type.

1. Introduction

In order to study the growth of solutions of the following higher order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1}$$

where $k \geq 2$, $A_0 \not\equiv 0$ and the coefficients $A_j (j = 0, \ldots, k-1)$ are analytic functions (respectively, meromorphic functions) in unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ or in the whole complex plane $\mathbb{C}$, it is very important to determine the order and the type of the coefficients $A_j (j = 0, \ldots, k-1)$. Through this paper, the reader is assumed to be familiar with the standard notations and the fundamental results of Nevanlinna value distribution theory of meromorphic functions in the complex plane and in the unit disc (see [7, 9, 17, 21, 22]) and can refer to ([3, 4, 8, 10, 11, 15]) for more details about the growth of solutions of equation (1).

For all $r \in [0, 1)$, we define $\exp_1 r = \exp r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r)$, $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Inductively, for all r in $(0, 1)$, $\log^+ r = \max\{0, \log r\}$, $\log_1^+ r = \log^+ r$ and $\log_{p+1}^+ r = \log^+(\log_p^+ r)$, $p \in \mathbb{N} \cup \{0\}$. We also make the conventions $\exp_0 r = 1 = \log_0^+ r$, $\exp_{-1} r = \log_1^+ r$ and $\log_{-1}^+ r = \exp_1 r$.

Definition 1. *[4] The iterated p-order of an analytic function f in Δ is defined by*

$$\tilde{\rho}_p(f) := \limsup_{r \longrightarrow 1^-} \frac{\log_{p+1}^+ M(r,f)}{-\log(1-r)}, \quad p \in \mathbb{N},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus of f. For a meromorphic function f in Δ, the iterated p-order is defined by

$$\rho_p(f) := \limsup_{r \longrightarrow 1^-} \frac{\log_p^+ T(r,f)}{-\log(1-r)}, \quad p \in \mathbb{N},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

Definition 2. *[10] The iterated p-type of an analytic function f in Δ with $0 < \tilde{\rho}_p(f) < +\infty$ is defined by*

$$\tilde{\tau}_p(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\tilde{\rho}_p(f)} \log_p^+ M(r,f).$$

For a meromorphic function f in Δ with $(0 < \rho_p(f) < +\infty)$, the iterated p-type is defined by

$$\tau_p(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\rho_p(f)} \log_{p-1}^+ T(r,f).$$

It is clear that $\rho_1(f)$ and $\tau_1(f)$ coincide with the usual order $\rho(f)$ and the usual type $\tau(f)$ respectively. In [14], Lahiri proved that the iterated p-order of the derivative of an entire function or a meromorphic function in the complex plane is the same as that of the function, i.e., $\rho_p(f') = \rho_p(f), p \geq 1$. Later, Cao-Yi [4] gave the version of this result in the unit disc and obtained other properties that are well-known in the complex plane [7, 17, 23].

Theorem A. *[4] Let f_1 and f_2 be two meromorphic functions in Δ. Then, for $p \in \mathbb{N}$ we have*

(i) $\rho_p(f_1) = \rho_p(\frac{1}{f_1})$, *where* $f_1 \not\equiv 0$, $\rho_p(af_1) = \rho_p(f_1)$, *where* $a \in \mathbb{C}^*$.

(ii) $\rho_p(f_1') = \rho_p(f_1)$.

(iii) $\max\{\rho_p(f_1 + f_2), \rho_p(f_1 f_2)\} \leq \max\{\rho_p(f_1), \rho_p(f_2)\}$.

(iv) *If* $\rho_p(f_1) < \rho_p(f_2)$, *then* $\rho_p(f_1 + f_2) = \rho_p(f_1 f_2) = \rho_p(f_2)$.

Theorem B. *[4] Let f_1 and f_2 be two analytic functions in Δ. Then, for $p \in \mathbb{N}$ we have*

(i) $\tilde{\rho}_p(af_1) = \tilde{\rho}_p(f_1)(a \in \mathbb{C}^*)$.

(ii) $\tilde{\rho}_p(f_1') = \tilde{\rho}_p(f_1)$.

(iii) $\max\{\tilde{\rho}_p(f_1+f_2), \tilde{\rho}_p(f_1\,f_2)\} \leq \max\{\tilde{\rho}_p(f_1), \tilde{\rho}_p(f_2)\}$.

(iv) *If* $\tilde{\rho}_p(f_1) < \tilde{\rho}_p(f_2)$, *then* $\tilde{\rho}_p(f_1+f_2) = \tilde{\rho}_p(f_2)$.

Latreuch-Belaïdi [16] obtained under some conditions similar results for the usual type ($p = 1$).

Theorem C. *[16] If* f_1 *and* f_2 *are two analytic functions in* Δ *satisfying* $0 < \tilde{\rho}(f_1) = \tilde{\rho}(f_2) = \rho < +\infty$ *and* $\tilde{\tau}(f_1) \neq \tilde{\tau}(f_2)$, *then* $\tilde{\rho}(f_1+f_2) = \rho$ *and* $\tilde{\tau}(f_1+f_2) = \max\{\tilde{\tau}(f_1), \tilde{\tau}(f_2)\}$.

As it is shown by Chyzhykov-Semochko in [6, Example 1.4], the iterated p-order does not cover an arbitrary growth of entire solutions of equation (1) and the concept of the φ-order (cf. [20]) was investigated in the same paper. This concept is quickly adopted by Semochko [18] and by Belaïdi [1, 2], where the φ-type's concept is given.

Definition 3. *[18] Let* φ *be an increasing unbounded function in the unit disc* Δ. *The* φ*-orders of an analytic function* f *in* Δ *are defined by*

$$\tilde{\rho}_\varphi^0(f) = \limsup_{r\longrightarrow 1^-} \frac{\varphi(M(r,f))}{-\log(1-r)}, \quad \tilde{\rho}_\varphi^1(f) = \limsup_{r\longrightarrow 1^-} \frac{\varphi(\log M(r,f))}{-\log(1-r)}.$$

If f *is meromorphic in* Δ, *then the* φ*-orders are defined by*

$$\rho_\varphi^0(f) = \limsup_{r\longrightarrow 1^-} \frac{\varphi(e^{T(r,f)})}{-\log(1-r)}, \quad \rho_\varphi^1(f) = \limsup_{r\longrightarrow 1^-} \frac{\varphi(T(r,f))}{-\log(1-r)}.$$

We can see that if $\varphi(r) = \log_p^+ r$, $p \in \mathbb{N}$ and f is an analytic function in Δ, then $\tilde{\rho}_\varphi^1(f) = \tilde{\rho}_p(f)$. By Φ we define the class of positive unbounded increasing functions in the unit disc Δ such that $\varphi(e^t)$ is slowly growing, i.e., $\forall c > 0 : \lim\limits_{t\to+\infty} \dfrac{\varphi(e^{ct})}{\varphi(e^t)} = 1$. As examples, $\varphi(r) = \log_p^+ r$, ($p \geq 2$) belongs to the class Φ and $\varphi(r) = \log^+ r \notin \Phi$.

Now, we introduce by analogous manner the definitions of the φ-types in the unit disc Δ related to the φ-order.

Definition 4. *Let* φ *be an increasing unbounded function in the unit disc* Δ. *We define the* φ*-types of an analytic function* f *in* Δ *with* $0 < \tilde{\rho}_\varphi^i(f) < +\infty$ $(i = 0, 1)$ *by*

$$\tilde{\tau}_\varphi^0(f) = \limsup_{r\longrightarrow 1^-} (1-r)^{\tilde{\rho}_\varphi^0(f)} \exp\{\varphi(M(r,f))\},$$

$$\tilde{\tau}_\varphi^1(f) = \limsup_{r\longrightarrow 1^-} (1-r)^{\tilde{\rho}_\varphi^1(f)} \exp\{\varphi(\log M(r,f))\}.$$

If f is meromorphic in Δ, then we define the φ-types with $0 < \rho_\varphi^i(f) < +\infty$ $(i = 0, 1)$ by

$$\tau_\varphi^0(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\rho_\varphi^0(f)} \exp\{\varphi(e^{T(r,f)})\},$$
$$\tau_\varphi^1(f) = \limsup_{r \longrightarrow 1^-} (1-r)^{\rho_\varphi^1(f)} \exp\{\varphi(T(r,f))\}.$$

2. Main results

The main purpose of this paper is to investigate under suitable conditions the φ-order and the φ-type of $f_1 + f_2$, $f_1 f_2$ and f_1', where f_1, f_2 are analytic or meromorphic functions in Δ.

Theorem 1. *Let $\varphi \in \Phi$ and f_1, f_2 be two meromorphic functions in Δ. Then*

(i) $\rho_\varphi^j(f_1 + f_2) \leq \max\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\}$, $(j = 0, 1)$.

(ii) $\rho_\varphi^j(f_1 f_2) \leq \max\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\}$, $(j = 0, 1)$.

(iii) $\rho_\varphi^j(af_1) = \rho_\varphi^j(f_1)$ *and* $\tau_\varphi^j(af_1) = \tau_\varphi^j(f_1)$, $(a \in \mathbb{C}^*; j = 0, 1)$.

(iv) $\rho_\varphi^j\left(\frac{1}{f_1}\right) = \rho_\varphi^j(f_1)$ *and* $\tau_\varphi^j\left(\frac{1}{f_1}\right) = \tau_\varphi^j(f_1)$, $(f_1 \not\equiv 0; j = 0, 1)$.

Theorem 2. *Let $\varphi \in \Phi$ and f_1, f_2 be two meromorphic functions in Δ. If $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$, $(j = 0, 1)$, then $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$, $(j = 0, 1)$.*

Theorem 3. *Let $\varphi \in \Phi$ and f_1, f_2 be two analytic functions in Δ. Then*

(i) $\tilde{\rho}_\varphi^j(af_1) = \tilde{\rho}_\varphi^j(f_1)$ *and* $\tilde{\tau}_\varphi^j(af_1) = \tilde{\tau}_\varphi^j(f_1)$, $(a \in \mathbb{C}^*; j = 0, 1)$.

(ii) $\max\{\tilde{\rho}_\varphi^j(f_1 + f_2), \tilde{\rho}_\varphi^j(f_1 f_2)\} \leq \max\{\tilde{\rho}_\varphi^j(f_1), \tilde{\rho}_\varphi^j(f_2)\}$, $(j = 0, 1)$.

(iii) If $\tilde{\rho}_\varphi^j(f_1) < \tilde{\rho}_\varphi^j(f_2)$, *then* $\tilde{\rho}_\varphi^j(f_1 + f_2) = \tilde{\rho}_\varphi^j(f_2)$, $(j = 0, 1)$.

Theorem 4. *Let $\varphi \in \Phi$ and f_1, f_2 be two meromorphic functions in Δ. Then*

(i) If $0 < \rho_\varphi^j(f_1) < \rho_\varphi^j(f_2) < +\infty$ *and* $\tau_\varphi^j(f_1) < \tau_\varphi^j(f_2), (j = 0, 1)$, *then*

$$\tau_\varphi^j(f_1 + f_2) = \tau_\varphi^j(f_1 f_2) = \tau_\varphi^j(f_2), j = 0, 1. \tag{2}$$

(ii) If $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 + f_2) < +\infty, (j = 0, 1)$, *then*

$$\tau_\varphi^j(f_1 + f_2) \leq \max\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\}.$$

Moreover, if $\tau_\varphi^j(f_1) \neq \tau_\varphi^j(f_2)$, $(j = 0, 1)$, *then*

$$\tau_\varphi^j(f_1 + f_2) = \max\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\}. \tag{3}$$

(iii) *If* $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho_\varphi^j(f_1 f_2) < +\infty, (j = 0, 1),$ *then*

$$\tau_\varphi^j(f_1 f_2) \leq \max\{\tau_\varphi^j(f), \tau_\varphi^j(f_2)\}.$$

Moreover, if $\tau_\varphi^j(f_1) \neq \tau_\varphi^j(f_2),$ $(j = 0, 1),$ *then*

$$\tau_\varphi^j(f_1 f_2) = \max\{\tau_\varphi^j(f_1), \tau_\varphi^j(f_2)\}. \tag{4}$$

Theorem 5. *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two meromorphic functions in* Δ*. If* $0 < \rho_\varphi^j(f_1) = \rho_\varphi^j(f_2) = \rho < +\infty$ *and* $\tau_\varphi^j(f_1) < \tau_\varphi^j(f_2),$ $(j = 0, 1),$ *then*

$$\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho, \tag{5}$$

$$\tau_\varphi^j(f_1 + f_2) = \tau_\varphi^j(f_1 f_2) = \tau_\varphi^j(f_2). \tag{6}$$

Theorem 6. *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two analytic functions in* Δ*. Then*

(i) *If* $0 < \tilde{\rho}_\varphi^j(f_1) < \tilde{\rho}_\varphi^j(f_2) < +\infty$ *and* $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2), (j = 0, 1),$ *then*

$$\tilde{\tau}_\varphi^j(f_1 + f_2) = \tilde{\tau}_\varphi^j(f_2). \tag{7}$$

(ii) *If* $0 < \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2) = \tilde{\rho}_\varphi^j(f_1 + f_2) < +\infty, (j = 0, 1),$ *then*

$$\tilde{\tau}_\varphi^j(f_1 + f_2) \leq \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$$

Moreover, if $\tilde{\tau}_\varphi^j(f_1) \neq \tilde{\tau}_\varphi^j(f_2),$ *then* $\tilde{\tau}_\varphi^j(f_1 + f_2) = \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$

(iii) *If* $0 < \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2) = \tilde{\rho}_\varphi^j(f_1 f_2) < +\infty, (j = 0, 1),$ *then*

$$\tilde{\tau}_\varphi^j(f_1 f_2) \leq \max\{\tilde{\tau}_\varphi^j(f_1), \tilde{\tau}_\varphi^j(f_2)\}.$$

Theorem 7. *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two analytic functions in* Δ*. If* $0 < \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2) < +\infty$ *and* $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2),$ $(j = 0, 1),$ *then*

$$\tilde{\rho}_\varphi^j(f_1 + f_2) = \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2), \tag{8}$$

$$\tilde{\tau}_\varphi^j(f_1 + f_2) = \tilde{\tau}_\varphi^j(f_2). \tag{9}$$

Theorem 8. *If* f *is a meromorphic function in* Δ *and* $\varphi \in \Phi$*, then* $\rho_\varphi^j(f') = \rho_\varphi^j(f),$ $(j = 0, 1).$

Theorem 9. *If* f *is an analytic function in* Δ *and* $\varphi \in \Phi$*, then* $\tilde{\rho}_\varphi^j(f') = \tilde{\rho}_\varphi^j(f),$ $(j = 0, 1).$

Remark 1. *For some related results in the complex plane, see [12].*

3. Basic properties and lemmas

In this section, we recall some basic properties of functions from the class Φ and give only lemmas that we need to prove our results.

Proposition 1. *[6] If $\varphi \in \Phi$, then*

$$\forall m > 0, \forall k \geq 0: \quad \frac{\varphi^{-1}(\log x^m)}{x^k} \longrightarrow +\infty, \quad x \longrightarrow +\infty, \tag{10}$$

$$\forall \delta > 0: \quad \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \longrightarrow +\infty, \quad x \longrightarrow +\infty. \tag{11}$$

Remark 2. *[6] One can show that* (11) *implies that*

$$\forall c > 0, \varphi(ct) \leq \varphi(t^c) \leq (1+o(1))\varphi(t), \quad t \longrightarrow +\infty. \tag{12}$$

Proposition 2. *[18] Let $\varphi \in \Phi$ and f be an analytic function in the unit disc Δ. Then $\rho_\varphi^1(f) = \tilde{\rho}_\varphi^1(f)$.*

Remark 3. *For an analytic function f in the unit disc Δ and $\varphi \in \Phi$, the equality $\rho_\varphi^0(f) = \tilde{\rho}_\varphi^0(f)$ is not always verified. As a counter-example, we consider the function $\varphi(.) = \log_2(.) = \log\log(.)$ which belongs to the class Φ and satisfies $\rho_{\log_2}^0(f) = \rho_1(f)$ and $\tilde{\rho}_{\log_2}^0(f) = \tilde{\rho}_1(f)$. Since $\rho_1(f) \leq \tilde{\rho}_1(f) \leq \rho_1(f) + 1$ (see [21] or [15, Proposition 2.2.2]), then $\rho_{\log_2}^0(f) \leq \tilde{\rho}_{\log_2}^0(f) \leq \rho_{\log_2}^0(f) + 1$.*

Lemma 10. *[11] Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $F \subset [0,1)$ with $\int_F \frac{dr}{1-r} < +\infty$. Then, there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

Lemma 11. *[11, 21] Let f be a meromorphic function in the unit disc Δ and $k \in \mathbb{N}$. Then we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log^+ T(r,f) + \log\frac{1}{1-r}\right)$$

possibly outside of an exceptional set $F \subset [0,1)$ with $\int_F \frac{dr}{1-r} < +\infty$. If $\rho_1(f) < +\infty$, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\frac{1}{1-r}\right).$$

4. Proofs of the main results

Proof of Theorem 1. (i) The definition of ρ^1_φ-order implies that for any given $\varepsilon > 0$ and for r sufficiently large in $[0,1)$, we have

$$T(r, f_i) \leq \varphi^{-1}\left((\rho^1_\varphi(f_i) + \varepsilon)\log\frac{1}{1-r}\right), (i = 1, 2). \tag{13}$$

Hence

$$T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + O(1)$$
$$= O\left(\varphi^{-1}\left((\max\{\rho^1_\varphi(f_1), \rho^1_\varphi(f_2)\} + \varepsilon)\log\frac{1}{1-r}\right)\right).$$

By the monotonicity of φ and (12), we obtain

$$\frac{\varphi(T(r, f_1 + f_2))}{\log\frac{1}{1-r}} \leq \max\{\rho^1_\varphi(f_1), \rho^1_\varphi(f_2)\} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then we get $\rho^1_\varphi(f_1 + f_2) \leq \max\{\rho^1_\varphi(f_1), \rho^1_\varphi(f_2)\}$.
$(ii) - (iv)$ follow immediately from the properties

$$T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2),$$

$$T(r, af_1) = T(r, f_1) + O(1), (a \in \mathbb{C}^*),$$

$$T\left(r, \frac{1}{f_1}\right) = T(r, f_1) + O(1)$$

and the definitions of φ-order and φ-type. Similar proofs for $j = 0$.

Proof of Theorem 2. By Theorem 1, we have $\rho^j_\varphi(f_1 + f_2) \leq \rho^j_\varphi(f_2)$ and then

$$\rho^j_\varphi(f_2) = \rho^j_\varphi(f_1 + f_2 - f_1) \leq \max\{\rho^j_\varphi(f_1 + f_2), \rho^j_\varphi(f_1)\}.$$

Suppose that $\rho^j_\varphi(f_1) > \rho^j_\varphi(f_1 + f_2)$, then

$$\rho^j_\varphi(f_2) = \rho^j_\varphi(f_1 + f_2 - f_1) \leq \max\{\rho^j_\varphi(f_1 + f_2), \rho^j_\varphi(f_1)\} = \rho^j_\varphi(f_1)$$

this contradicts the hypothesis $\rho^j_\varphi(f_1) < \rho^j_\varphi(f_2)$. Then $\rho^j_\varphi(f_2) \leq \rho^j_\varphi(f_1 + f_2)$ and therefore $\rho^j_\varphi(f_1 + f_2) = \rho^j_\varphi(f_2), (j = 0, 1)$. Similarly, again by Theorem 1 and the fact that

$$\rho^j_\varphi(f_2) = \rho^j_\varphi\left(f_1\, f_2\, \frac{1}{f_1}\right) \leq \max\{\rho^j_\varphi(f_1\, f_2), \rho^j_\varphi(f_1)\} = \rho^j_\varphi(f_1),$$

we obtain $\rho_{\varphi}^{j}(f_1\, f_2) = \rho_{\varphi}^{j}(f_2), (j = 0, 1)$.

Proof of Theorem 3. (i) It is obvious from the definitions of $\tilde{\rho}_{\varphi}^{j}(f_1)$ and $\tilde{\tau}_{\varphi}^{j}(f_1)$, $(j = 0, 1)$. (ii) holds by using the known inequalities $|f_1 + f_2| \leq 2\max\{|f_1|, |f_2|\}$ and $|f_1\, f_2| \leq [\max\{|f_1|, |f_2|\}]^2$ and applying (12). We can obtain (iii) by analogous discussion as in the proof of the first part of Theorem 2.

Proof of Theorem 4. We only give the proofs for $j = 1$, the proofs for $j = 0$ are analogous. (i) The definition of τ_{φ}^{1}-type implies that there exists a sequence $\{r_n, n \geq 1\}$ tending to 1^- satisfying $1 - (1 - \frac{1}{n})(1 - r_n) < r_{n+1}$ such that for any given $\varepsilon > 0$ we have

$$T(r_n, f_2) \geq \varphi^{-1}\left(\log\left(\frac{\tau_{\varphi}^{1}(f_2) - \varepsilon}{(1 - r_n)^{\rho_{\varphi}^{1}(f_2)}}\right)\right) \tag{14}$$

and for $r \longrightarrow 1^-$ there holds

$$T(r, f_i) \leq \varphi^{-1}\left(\log\left(\frac{\tau_{\varphi}^{1}(f_i) + \varepsilon}{(1 - r)^{\rho_{\varphi}^{1}(f_i)}}\right)\right), \; (i = 1, 2). \tag{15}$$

By Proposition 1 and the fact that

$$T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - \log 2, \tag{16}$$

we obtain

$$\begin{aligned} T(r_n, f_1 + f_2) &\geq \varphi^{-1}\left(\log\left(\frac{\tau_{\varphi}^{1}(f_2) - \varepsilon}{(1 - r_n)^{\rho_{\varphi}^{1}(f_2)}}\right)\right) - \varphi^{-1}\left(\log\left(\frac{\tau_{\varphi}^{1}(f_1) + \varepsilon}{(1 - r_n)^{\rho_{\varphi}^{1}(f_1)}}\right)\right) - \log 2 \\ &\geq \varphi^{-1}\left(\log\left(\frac{\tau_{\varphi}^{1}(f_2) - 2\varepsilon}{(1 - r_n)^{\rho_{\varphi}^{1}(f_2)}}\right)\right) \end{aligned} \tag{17}$$

provided ε such that $0 < 2\varepsilon < \tau_{\varphi}^{1}(f_2) - \tau_{\varphi}^{1}(f_1)$. It follows from Theorem 2 that $\rho_{\varphi}^{1}(f_1 + f_2) = \rho_{\varphi}^{1}(f_2)$. Then, by (17) and the monotonicity of φ, we have

$$(1 - r_n)^{\rho_{\varphi}^{1}(f_1 + f_2)} \exp\{\varphi(T(r_n, f_1 + f_2))\} \geq \tau_{\varphi}^{1}(f_2) - 2\varepsilon.$$

By arbitrariness of $\varepsilon\, (0 < 2\varepsilon < \tau_{\varphi}^{1}(f_2) - \tau_{\varphi}^{1}(f_1))$, we obtain

$$\tau_{\varphi}^{1}(f_1 + f_2) \geq \tau_{\varphi}^{1}(f_2). \tag{18}$$

On the other hand, since $\rho_\varphi^1(f_1+f_2)=\rho_\varphi^1(f_2)>\rho_\varphi^1(f_1)=\rho_\varphi^1(-f_1)$, then it follows from (18) that $\tau_\varphi^1(f_2)=\tau_\varphi^1(f_1+f_2-f_1)\geq\tau_\varphi^1(f_1+f_2)$ and therefore $\tau_\varphi^1(f_1+f_2)=\tau_\varphi^1(f_2)$.
By a similar discussion as in the above proof such that

$$T(r,f_1\,f_2)\geq T(r,f_1)-T(r,f_2)+O(1), \tag{19}$$

we obtain

$$\tau_\varphi^1(f_1\,f_2)\geq\tau_\varphi^1(f_2). \tag{20}$$

Since $\rho_\varphi^1(f_1\,f_2)=\rho_\varphi^1(f_2)>\rho_\varphi^1(f_1)=\rho_\varphi^1\left(\frac{1}{f_1}\right)$, then by (20) we have $\tau_\varphi^1(f_2)=\tau_\varphi^1\left(f_1\,f_2\,\frac{1}{f_1}\right)\geq\tau_\varphi^1(f_1\,f_2)$ and therefore $\tau_\varphi^1(f_1\,f_2)=\tau_\varphi^1(f_2)$.
(ii) It follows from the assumption $0<\rho_\varphi^1(f_1)=\rho_\varphi^1(f_2)=\rho_\varphi^1(f_1+f_2)<+\infty$, (15) and Proposition 1 that

$$\begin{aligned}
T(r,f_1+f_2)&\leq T(r,f_1)+T(r,f_2)+O(1)\\
\leq\varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_1)+\varepsilon}{(1-r)^{\rho_\varphi^1(f_1)}}\right)\right)&+\varphi^{-1}\left(\log\left(\frac{\tau_\varphi^1(f_2)+\varepsilon}{(1-r)^{\rho_\varphi^1(f_2)}}\right)\right)+O(1)\\
&\leq\varphi^{-1}\left(\log\left(\frac{\max\{\tau_\varphi^1(f_1),\tau_\varphi^1(f_2)\}+3\varepsilon}{(1-r)^{\rho_\varphi^1(f_1+f_2)}}\right)\right).
\end{aligned}$$

By the monotonicity of φ and arbitrariness of $\varepsilon>0$ we obtain

$$\tau_\varphi^1(f_1+f_2)\leq\max\{\tau_\varphi^1(f_1),\tau_\varphi^1(f_2)\}. \tag{21}$$

Moreover, we may suppose without loss of generality that $\tau_\varphi^1(f_1)<\tau_\varphi^1(f_2)$. Then, by (21) and since $\rho_\varphi^1(f_1+f_2)=\rho_\varphi^1(f_1)=\rho_\varphi^1(-f_1)$, we have

$$\tau_\varphi^1(f_2)=\tau_\varphi^1(f_1+f_2-f_1)\leq\max\{\tau_\varphi^1(f_1+f_2),\tau_\varphi^1(f_1)\}=\tau_\varphi^1(f_1+f_2)$$

and therefore $\tau_\varphi^1(f_1+f_2)=\max\{\tau_\varphi^1(f_1),\tau_\varphi^1(f_2)\}$.
(iii) Since $T(r,f_1\,f_2)\leq T(r,f_1)+T(r,f_2)$, we obtain by a similar discussion as in (21) that

$$\tau_\varphi^1(f_1\,f_2)\leq\max\{\tau_\varphi^1(f_1),\tau_\varphi^1(f_2)\}. \tag{22}$$

Moreover, if we suppose that $\tau_\varphi^1(f_1)<\tau_\varphi^1(f_2)$, then by (22) and since $\rho_\varphi^1(f_1\,f_2)=\rho_\varphi^1(f_1)=\rho_\varphi^1\left(\frac{1}{f_1}\right)$, we obtain

$$\tau_\varphi^1(f_2)=\tau_\varphi^1\left(f_1\,f_2\,\frac{1}{f_1}\right)\leq\max\{\tau_\varphi^1(f_1\,f_2),\tau_\varphi^1(f_1)\}=\tau_\varphi^1(f_1\,f_2).$$

Hence, $\tau^1_\varphi(f_1\, f_2) = \max\{\tau^1_\varphi(f_1), \tau^1_\varphi(f_2)\}$.

Proof of Theorem 5. We have from Theorem 1 that

$$\rho^j_\varphi(f_1 + f_2) \le \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2) \quad \text{and} \quad \rho^j_\varphi(f_1\, f_2) \le \rho^j_\varphi(f_1) = \rho^j_\varphi(f_2).$$

Analogous reasoning as in the proof of Theorem 4 especially (16) and (19) leads to

$$\rho^j_\varphi(f_1 + f_2) \ge \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1) \quad \text{and} \quad \rho^j_\varphi(f_1\, f_2) \ge \rho^j_\varphi(f_2) = \rho^j_\varphi(f_1).$$

Hence, (5) holds. On the other hand, (6) is obvious from (5) and Theorem 4.

Proof of Theorem 6. We will prove the theorem for $j = 0$, the proofs for $j = 1$ are analogous and follow also from Proposition 2 and Theorem 4. (i) The definition of the $\tilde{\tau}^0_\varphi$-type implies that for any $\varepsilon > 0$ there exists a sequence $\{r_n : n \ge 1\}$ tending to 1^- satisfying $1 - (1 - \frac{1}{n})(1 - r_n) < r_{n+1}$ such that for any given $\varepsilon > 0$ we have

$$M(r_n, f_2) \ge \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}^0_\varphi(f_2) - \varepsilon}{(1 - r_n)^{\tilde{\rho}^0_\varphi(f_2)}}\right)\right) \tag{23}$$

and for all $r \longrightarrow 1^-$

$$M(r, f_i) \le \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}^0_\varphi(f_i) + \varepsilon}{(1 - r)^{\tilde{\rho}^0_\varphi(f_i)}}\right)\right),\ (i = 1, 2). \tag{24}$$

We consider a sequence $\{z_n : n \ge 1\}$ satisfying $|f_2(z_n)| = M(r_n, f_2)$ in each circle $|z| = r_n$. Then, by (23), (24) and Proposition 1 we obtain

$$\begin{aligned} M(r_n, f_1 + f_2) &\ge |f_1(z_n) + f_2(z_n)| \ge |f_2(z_n)| - |f_1(z_n)| \\ &\ge M(r_n, f_2) - M(r_n, f_1) \\ &\ge \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}^0_\varphi(f_2) - \varepsilon}{(1 - r_n)^{\tilde{\rho}^0_\varphi(f_2)}}\right)\right) - \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}^0_\varphi(f_1) + \varepsilon}{(1 - r_n)^{\tilde{\rho}^0_\varphi(f_1)}}\right)\right) \\ &\ge \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}^0_\varphi(f_2) - 2\varepsilon}{(1 - r_n)^{\tilde{\rho}^0_\varphi(f_2)}}\right)\right) \end{aligned} \tag{25}$$

provided ε such that $0 < 2\varepsilon < \tilde{\tau}^0_\varphi(f_2) - \tilde{\tau}^0_\varphi(f_1)$ and $r_n \longrightarrow 1^-$. We have from Theorem 3 that $\tilde{\rho}^0_\varphi(f_1 + f_2) = \tilde{\rho}^0_\varphi(f_2)$. Thus, by the monotonicity of φ, (25) and (12), we obtain

$$(1 - r_n)^{\tilde{\rho}^0_\varphi(f_1 + f_2)} \exp\{\varphi(M(r_n, f_1 + f_2))\} \ge \tilde{\tau}^0_\varphi(f_2) - 2\varepsilon.$$

By arbitrariness of ε $(0 < 2\varepsilon < \tilde{\tau}_{\varphi}^{0}(f_2) - \tilde{\tau}_{\varphi}^{0}(f))$, we get

$$\tilde{\tau}_{\varphi}^{0}(f_1 + f_2) \geq \tilde{\tau}_{\varphi}^{0}(f_2). \tag{26}$$

On the other hand, by applying (26) and since $\tilde{\rho}_{\varphi}^{0}(f_1 + f_2) = \tilde{\rho}_{\varphi}^{0}(f_2) > \tilde{\rho}_{\varphi}^{0}(f_1) = \tilde{\rho}_{\varphi}^{0}(-f_1)$, then we obtain

$$\tilde{\tau}_{\varphi}^{0}(f_2) = \tilde{\tau}_{\varphi}^{0}(f_1 + f_2 - f_1) \leq \tilde{\tau}_{\varphi}^{0}(f_1 + f_2)$$

and therefore $\tilde{\tau}_{\varphi}^{0}(f_1 + f_2) = \tilde{\tau}_{\varphi}^{0}(f_2)$.
(ii) It follows from the assumption $0 < \tilde{\rho}_{\varphi}^{0}(f_1) = \tilde{\rho}_{\varphi}^{0}(f_2) = \tilde{\rho}_{\varphi}^{0}(f_1 + f_2) < +\infty$, (24) and Proposition 1 that

$$\begin{aligned} M(r, f_1 + f_2) &\leq M(r, f_1) + M(r, f_2) \\ &\leq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_{\varphi}^{0}(f_1) + \varepsilon}{(1-r)^{\tilde{\rho}_{\varphi}^{0}(f_1+f_2)}}\right)\right) + \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_{\varphi}^{0}(f_2) + \varepsilon}{(1-r)^{\tilde{\rho}_{\varphi}^{0}(f_1+f_2)}}\right)\right) \\ &\leq \varphi^{-1}\left(\log\left(\frac{\max\{\tilde{\tau}_{\varphi}^{0}(f_1), \tilde{\tau}_{\varphi}^{0}(f_2)\} + 2\varepsilon}{(1-r)^{\tilde{\rho}_{\varphi}^{0}(f_1+f_2)}}\right)\right). \end{aligned}$$

The monotonicity of φ and arbitrariness of $\varepsilon > 0$ yield

$$\tilde{\tau}_{\varphi}^{0}(f_1 + f_2) \leq \max\{\tilde{\tau}_{\varphi}^{0}(f_1), \tilde{\tau}_{\varphi}^{0}(f_2)\}. \tag{27}$$

Moreover, we may suppose without loss of generality that $\tilde{\tau}_{\varphi}^{0}(f_1) < \tilde{\tau}_{\varphi}^{0}(f_2)$. Since $\tilde{\rho}_{\varphi}^{0}(f_1 + f_2) = \tilde{\rho}_{\varphi}^{0}(f_1) = \tilde{\rho}_{\varphi}^{0}(-f_1)$, then by applying (27) we get

$$\tilde{\tau}_{\varphi}^{0}(f_2) = \tilde{\tau}_{\varphi}^{0}(f_1 + f_2 - f_1) \leq \max\{\tilde{\tau}_{\varphi}^{0}(f_1 + f_2), \tilde{\tau}_{\varphi}^{0}(f_1)\} = \tilde{\tau}_{\varphi}^{0}(f_1 + f_2). \tag{28}$$

We deduce from (27) and (28) that $\tilde{\tau}_{\varphi}^{0}(f_1 + f_2) = \max\{\tilde{\tau}_{\varphi}^{0}(f_1), \tilde{\tau}_{\varphi}^{0}(f_2)\}$.
(iii) By the assumption $0 < \tilde{\rho}_{\varphi}^{0}(f_1) = \tilde{\rho}_{\varphi}^{0}(f_2) = \tilde{\rho}_{\varphi}^{0}(f_1\, f_2) < +\infty$ and (24) it follows that

$$\begin{aligned} M(r, f_1\, f_2) &\leq M(r, f_1)\, M(r, f_2) \\ &\leq \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_{\varphi}^{0}(f_1) + \varepsilon}{(1-r)^{\tilde{\rho}_{\varphi}^{0}(f_1\, f_2)}}\right)\right) \varphi^{-1}\left(\log\left(\frac{\tilde{\tau}_{\varphi}^{0}(f_2) + \varepsilon}{(1-r)^{\tilde{\rho}_{\varphi}^{0}(f_1\, f_2)}}\right)\right) \\ &\leq \left[\varphi^{-1}\left(\log\left(\frac{\max\{\tilde{\tau}_{\varphi}^{0}(f_1), \tilde{\tau}_{\varphi}^{0}(f_2)\} + \varepsilon}{(1-r)^{\tilde{\rho}_{\varphi}^{0}(f_1\, f_2)}}\right)\right)\right]^2. \end{aligned}$$

By the monotonicity of φ and (12), we obtain

$$\varphi(M(r,f_1\,f_2)) \leq (1+o(1))\log\left(\frac{\max\{\tilde{\tau}_\varphi^0(f_1),\tilde{\tau}_\varphi^0(f_2)\}+\varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1\,f_2)}}\right)$$

$$\leq \log\left(\frac{\max\{\tilde{\tau}_\varphi^0(f_1),\tilde{\tau}_\varphi^0(f_2)\}+2\varepsilon}{(1-r)^{\tilde{\rho}_\varphi^0(f_1\,f_2)}}\right).$$

Since $\varepsilon > 0$ is arbitrary, we deduce $\tilde{\tau}_\varphi^0(f_1\,f_2) \leq \max\{\tilde{\tau}_\varphi^0(f_1),\tilde{\tau}_\varphi^0(f_2)\}$.

Proof of Theorem 7. We have from Theorem 3 that $\tilde{\rho}_\varphi^j(f_1+f_2) \leq \tilde{\rho}_\varphi^j(f_1) = \tilde{\rho}_\varphi^j(f_2)$. To prove the converse inequality, we assume the contrary $\tilde{\rho}_\varphi^j(f_1+f_2) < \tilde{\rho}_\varphi^j(f_1)$. Then, by (7) we have $\tilde{\tau}_\varphi^j(f_1) = \tilde{\tau}_\varphi^j(f_1+f_2-f_2) = \tilde{\tau}_\varphi^j(f_2)$ which contradicts our hypothesis $\tilde{\tau}_\varphi^j(f_1) < \tilde{\tau}_\varphi^j(f_2)$. Hence, $\tilde{\rho}_\varphi^j(f_1+f_2) \geq \tilde{\rho}_\varphi^j(f_1)$ and therefore (8) holds. It is clear that (9) follows immediately from our assertion (8) and the second part of Theorem 6.

Proof of Theorem 8. Denote $\rho_\varphi^1(f) = \rho$ and $\rho_\varphi^1(f') = \rho'$. By (13), Lemma 11 and (10), for all $r \longrightarrow 1^-$, $r \notin F$, where F is a set satisfying $\int_F \frac{dr}{1-r} < +\infty$, we have

$$T(r,f') = m(r,f') + N(r,f') \leq m(r,\frac{f'}{f}) + m(r,f) + 2N(r,f)$$

$$\leq m(r,\frac{f'}{f}) + 2T(r,f) = O\left(\log\varphi^{-1}\left((\rho+\varepsilon)\log\frac{1}{1-r}\right) + \log\frac{1}{1-r}\right)$$

$$+O\left(\varphi^{-1}\left((\rho+\varepsilon)\log\frac{1}{1-r}\right)\right) = O\left(\varphi^{-1}\left((\rho+\varepsilon)\log\frac{1}{1-r}\right)\right).$$

By the monotonicity of φ and (12), we obtain

$$\varphi(T(r,f')) \leq (1+o(1))\left((\rho+\varepsilon)\log\frac{1}{1-r}\right) \leq (\rho+2\varepsilon)\log\frac{1}{1-r},\ r \notin F.$$

By Lemma 10, since $\varepsilon > 0$ is arbitrary, there holds $\rho_\varphi^1(f') \leq \rho_\varphi^1(f) = \rho$.

Now, we prove $\rho_\varphi^1(f) \leq \rho_\varphi^1(f')$. Indeed, by following analogous demonstration techniques as in [22, Theorem 4.1] or in [5], we can obtain the following estimate

$$T(r,f) < O\left(T(\frac{1+r}{2},f') + \log\frac{1}{1-r}\right),\ r \longrightarrow 1^-.$$

By (13) and (10) we have that $T(r,f') \leq \varphi^{-1}\left((\rho'+\varepsilon)\log\frac{1}{1-r}\right)$ and then

$$T(r,f) \leq O\left(\varphi^{-1}\left((\rho'+\varepsilon)\log\frac{2}{1-r}\right) + \log\frac{1}{1-r}\right)$$
$$= O\left(\varphi^{-1}\left((\rho'+2\varepsilon)\log\frac{1}{1-r}\right)\right).$$

By the monotonicity of φ and (12), we obtain

$$\varphi(T(r,f)) \leq (1+o(1))\left((\rho'+2\varepsilon)\log\frac{1}{1-r}\right) \leq (\rho'+3\varepsilon)\log\frac{1}{1-r}.$$

By arbitrariness of $\varepsilon > 0$ we deduce that $\rho^1_\varphi(f) \leq \rho^1_\varphi(f') = \rho'$ and therefore Theorem 8 is proved.

Proof of Theorem 9. It is well-known that for an analytic function f we have

$$f(z) = f(0) + \int_0^z f'(\zeta)d\zeta.$$

Then, for $|z| = r < 1$ we obtain

$$M(r,f) \leq |f(0)| + rM(r,f') \leq |f(0)| + M(r,f').$$

Clearly, it follows from the monotonicity of φ that $\tilde{\rho}^0_\varphi(f) \leq \tilde{\rho}^0_\varphi(f')$. Now, we prove the reverse inequality. Taking $z_0 = re^{i\theta}$ for any $|z| = r \in [0,1)$ such that $M(r,f') = |f'(z_0)|$ and $R = \frac{1+r}{2}$. By Cauchy's integral formula, we have

$$f'(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta,$$

where $\Gamma = \{\zeta : |\zeta - z_0| = R - r\}$. Set $\zeta - z_0 = (R-r)\,e^{i\theta}\ (0 \leq \theta \leq 2\pi),\ d\zeta = (R-r)\,ie^{i\theta}d\theta$. Since $\max\{|f(\zeta)| : \zeta \in \Gamma\} \leq M(R,f)$, then we obtain

$$M(r,f') = |f'(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(\zeta)|}{|\zeta - z_0|^2}(R-r)d\theta$$
$$\leq \frac{M(R,f)}{R-r} = \frac{2}{1-r}M(\frac{1+r}{2},f). \tag{29}$$

We consider now a set $E \subset [0,1)$ such that

$$E = \left\{r \in [0,1) : \log\frac{2}{1-r} < \log M(\frac{1+r}{2},f)\right\}.$$

Then, by the monotonicity of φ, (29) and (12) we obtain

$$\begin{aligned}\frac{\varphi(M(r,f'))}{\log\frac{1}{1-r}} &\leq \frac{\varphi\left(\exp\left\{\log\frac{2}{1-r}+\log M(\frac{1+r}{2},f)\right\}\right)}{\log\frac{1}{1-r}}\\ &\leq \frac{\varphi\left(\exp\left\{2\log M(\frac{1+r}{2},f)\right\}\right)}{\log\frac{1}{1-r}} = \frac{\varphi\left(\exp\left\{2\log M(R,f)\right\}\right)}{\log\frac{2}{1-R}}\\ &\leq \frac{(1+o(1))\varphi(M(R,f))}{\log\frac{1}{1-R}}.\end{aligned}$$

Thus,

$$\tilde{\rho}^0_\varphi(f') = \limsup_{r\longrightarrow 1^-}\frac{\varphi(M(r,f'))}{-\log(1-r)} \leq \limsup_{R\longrightarrow 1^-}\frac{(1+o(1))\varphi(M(R,f))}{\log\frac{1}{1-R}} = \tilde{\rho}^0_\varphi(f)$$

holds on E. It remains to estimate $\dfrac{\varphi(M(r,f'))}{\log\frac{1}{1-r}}$ on E^c such that

$$E^c = \left\{r\in[0;1) : \log\frac{2}{1-r} \geq \log M(\frac{1+r}{2},f)\right\}.$$

In fact, by Karamata's theorem (see [19]) we have $\varphi(e^t) = t^{o(1)}$ as $t \longrightarrow +\infty$. Then, for $r \longrightarrow 1^-$ there holds

$$\frac{\varphi(M(r,f'))}{\log\frac{1}{1-r}} \leq \frac{\varphi\left(\exp\left\{\log\frac{2}{1-r}+\log\frac{2}{1-r}\right\}\right)}{\log\frac{1}{1-r}} = \frac{\left(2\log\frac{2}{1-r}\right)^{o(1)}}{\log\frac{1}{1-r}}.$$

Hence,

$$\tilde{\rho}^0_\varphi(f') := \limsup_{r\longrightarrow 1^-}\frac{\varphi(M(r,f'))}{-\log(1-r)} = 0.$$

We deduce that $\tilde{\rho}^0_\varphi(f') \leq \tilde{\rho}^0_\varphi(f)$ and therefore $\tilde{\rho}^0_\varphi(f') = \tilde{\rho}^0_\varphi(f)$.
The case for $j=1$ follows immediately from Proposition 2 and Theorem 8.

5. Conclusion

In our present paper, we have investigated the concepts of the φ-order and the φ-type and discussed their nice properties. We consider our results very helpful and will certainly contribute in future possible studies of the growth of solutions of equation (1). Moreover, the general concept of (α,β)-order introduced by Sheremeta [20] shows that there are still many things can be done in this subject.

Acknowledgements. This paper is supported by University of Mostaganem (UMAB) (PRFU Project Code C00L03UN270120180005).

References

[1] B. Belaïdi, *Growth of ρ_φ-order solutions of linear differential equations with entire coefficients*, PanAmer. Math. J., 27, 4 (2017), 26–42.

[2] B. Belaïdi, *Fast growing solutions to linear differential equations with entire coefficients having the same ρ_φ-order*, J. Math. and Applications, 42 (2019), 63–77.

[3] T. B. Cao, J. F. Xu, Z. X. Chen, *On the meromorphic solutions of linear differential equations on the complex plane*, J. Math. Anal. Appl., 364, 1 (2010), 130–142.

[4] T. B. Cao, H. X. Yi, *The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc*, J. Math. Anal. Appl., 319, 1 (2006), 278–294.

[5] C. T. Chuang, *Sur la comparison de la croissance d'une fonction méromorphe et de celle de sa dérivée*, Bull. Sci. Math., 75, 2 (1951), 171–190.

[6] I. Chyzhykov, N. Semochko, *Fast growing entire solutions of linear differential equations*, Math. Bull. Shevchenko Sci. Soc., 13 (2016), 68–83.

[7] A. A. Goldberg, I. V. Ostrovskii, *Value distribution of meromorphic functions*, Translated by M. Ostrovskii with an appendix by A. Eremenko and J. Langley, Transl. Math. Monog., 236, Amer. Math. Soc., Providence RI, 2008.

[8] S. Hamouda, *Iterated order of solutions of linear differential equations in the unit disc*, Comput. Methods Funct. Theory, 13, 4 (2013), 545–555.

[9] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[10] J. Heittokangas, R. Korhonen, J. Rättyä, *Fast growing solutions of linear differential equations in the unit disc*, Results Math., 49 (2006), 265–278.

[11] J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss., 122 (2000), 1–54.

[12] M. A. Kara, B. Belaïdi, *Some estimates of the φ-order and the φ-type of entire and meromorphic functions*, Int. J. Open Prob. Complex Anal., 10, 3 (2019), 42–58.

[13] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math., 22, 4 (1998), 385–405.

[14] I. Lahiri, *Generalised order of the derivative of a meromorphic function*, Soochow J. Math., 14, 1 (1988), 85–92.

[15] I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics 15, Walter de Gruyter & Co., Berlin, 1993.

[16] Z. Latreuch, B. Belaïdi, *Further estimations on the order of growth and the type of meromorphic functions in the unit disc*, J. Interdis. Math., 16, 6 (2013), 373–383.

[17] B. Ya Levin, *Lectures on entire functions*, In collaboration with Yu. Lyubarskii, M. Sodin and V. Tkachenko, Transl. Math. Monog., 150, Amer. Math. Soc., Providence, RI, 1996.

[18] N. Semochko, *On the solution of linear differential equations of arbitrary fast growth in the unit disc*, Mat. Stud., 45, 1 (2016), 3–11.

[19] E. Seneta, *Regularly varying functions*, Lecture Notes in Mathematics, 508, Springer-Verlag, Berlin-New York, 1976.

[20] M. N. Sheremeta, *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*, Izv. Vyssh. Uchebn. Zaved. Mat., 2 (1967), 100–108. (in Russian).

[21] M. Tsuji, *Potential theory in modern function theory*, Chelsea Publishing Co., New York, 1975.

[22] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.

[23] C. C. Yang, H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Beijing, 2003.

Mohamed Abdelhak Kara
Department of Mathematics, Laboratory of Pure and Applied Mathematics,
University of Mostaganem (UMAB), B. P. 227 Mostaganem, Algeria
email: *kara.abdelhak@gmail.com*

Benharrat Belaïdi
Department of Mathematics, Laboratory of Pure and Applied Mathematics,
University of Mostaganem (UMAB), B. P. 227 Mostaganem, Algeria
email: *benharrat.belaidi@univ-mosta.dz*