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ON TWO MULTIPLICATIVE FUNCTIONS DEFINED BY THE NUMBER OF SOLUTIONS TO $X_1X_2\cdots X_K = N$

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Abstract. In this short note we present two arithmetic functions related to the number of solutions of a certain Diophantine equation and we show that these satisfy some interesting properties. In particular, we show that the two functions are multiplicative and that they are related to other well-known arithmetic functions. Upper bounds and formulas involving binomial coefficients for these functions are also provided. In the last section, we give a link between the Dirichlet series of the two functions and the well-known Riemann zeta function.

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1. Introduction

The following interesting multiplicative function $S : \mathbb{N}^* \to \mathbb{N}$ caught our eye in Problem O124 proposed in Mathematical Reflections **3**(2009). For a positive integer n, the quantity $S(n)$ is defined as the number of pairs consisting of positive integers (x, y) such that $xy = n$ and $gcd(x, y) = 1$. The problem asks to show the relation

$$\sum_{d|n} S(d) = \tau(n^2) \qquad (1)$$

holds, where $\tau(s)$ is the number of divisors of the positive integer s. A simple proof of the equality (1) relies on the observation that the function S is multiplicative, that is for any relatively prime integers m and n we have $S(mn) = S(m)S(n)$. Using this, it is sufficient to note that for any prime p and any positive integer α, $S(p^\alpha) = 2$, hence we get $S(n) = 2^s$, where $n = p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ is the prime factorization of n.

Another problem published in the same journal, namely J108 in Mathematical Reflections **1**(2009), asks to show that the number of ordered pairs (a, b) of relatively prime positive divisors of n is equal to $\tau(n^2)$, the number of divisors of n^2. The function which counts these pairs is obviously related to S.

After this introductory section, the paper starts with the study some of the properties of arithmetic multiplicative functions S_k, $k \geq 1$, a family of functions which naturally extend the multiplicative function S above. The functions appear sporadically through the literature, as if some authors rediscovered them at different times. The curious reader can consult the articles [3], [7] or the books [5] and [6], where these functions appear under different notations. Some properties of the function S_k are also presented in [2]. There is an immense literature on the theory of multiplicative functions. For some details, one could consult [1] and [6].

In section 3, we fix the positive integer k and present some upper bounds for $S_k(n)$ which are related to upper bounds on the function counting the prime divisors of n. Section 4 is dedicated to the exploration of the properties of another arithmetic family of functions M_k, closely related to S_k, where $k \in \mathbb{N} \setminus \{0\}$. Finally, in the last section we prove identities relating the Dirichlet series of S_k and M_k to the well-known Riemann zeta function.

2. The multiplicative functions S_k

Denote by $S_k(n)$ the number of representations of the positive integer n as a product of k positive integers, that is the number of solutions in positive integers of the equation

$$x_1x_2\cdots x_k = n \tag{2}$$

In this way, for a fixed positive integer k, we define the arithmetic function

$$n \mapsto S_k(n).$$

It is clear that $S_1 = \mathbf{1}$, the constant function 1. A first result concerning the function S_k is the following.

Theorem 1. *The function S_k is multiplicative.*

Proof. Let m and n be two relatively prime integers. Consider $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ solutions in positive integers of the corresponding equations to m and n, that is we have the relations $x_1x_2\cdots x_k = m$ and $y_1y_2\cdots y_k = n$. Then by multiplication we get $(x_1y_1)(x_2y_2)\cdots(x_ky_k) = mn$, that is the product of two solutions (component by component) gives a solution to the corresponding equation to mn. Conversely, let $(z_1, \cdots, z_k)$ be any solution to the equation $z_1z_2\cdots z_k = mn$. Define $x_i = gcd(z_i, m)$ and $y_i = gcd(z_i, n)$, $i = 1, \cdots, k$. It is clear that $x_1x_2\cdots x_k = m$, $y_1y_2\cdots y_k = n$ and $(x_1y_1)(x_2y_2)\cdots(x_ky_k) = mn$, hence $S_k(mn) = S_k(m)S_k(n)$. □

Theorem 2. *S_k is the summation function of S_{k-1}, that is for any positive integer n the following relation holds:*

$$S_k(n) = \sum_{d|n} S_{k-1}(d) \tag{3}$$

Proof. For a fixed divisor d of n consider all solutions $(x_1, \cdots, x_k)$ to equation (2) such that $x_1 = d$. The number of such solutions is $S_{k-1}(\frac{n}{d})$. It follows that

$$S_k(n) = \sum_{d|n} S_{k-1}(\frac{n}{d}) = \sum_{d|n} S_{k-1}(d),$$

and we are done. □

From the theorem above it follows that $S_2(n) = \sum_{d|n} S_1(d) = \sum_{d|n} \mathbf{1}(d) = \sum_{d|n} 1 = \tau(n)$, following that $S_2 = \tau$, the well-known number of divisors function.

Theorem 3. *If p is a prime and α is a positive integer, then*

$$S_k(p^\alpha) = \binom{\alpha + k - 1}{k - 1}. \tag{4}$$

Proof. We proceed by induction on k. Clearly, we have $S_1(p^\alpha) = 1$. From the previously proved equality (3) we obtain

$$S_2(p^\alpha) = \sum_{d|p^\alpha} S_1(d) = 1 + \cdots + 1 = \alpha + 1 = \binom{\alpha+1}{1},$$

so the desired property holds.

Assume that $S_k(p^\alpha) = \binom{\alpha+k-1}{k-1}$. Using the same relation, it follows

$$S_{k+1}(p^\alpha) = \sum_{j=0}^{\alpha} S_k(p^j) = \binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{\alpha+k-1}{k} = \binom{\alpha+k}{k},$$

where we have used the well-known combinatorial identity

$$\binom{s}{s} + \binom{s+1}{s} + \cdots + \binom{s+l}{s} = \binom{s+l+1}{s+1}.$$

□

We present two proofs for the following corollary.

Corollary 4. *Assume that $n = p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ is the prime factorization of the positive integer n. Then*

$$S_k(n) = \binom{\alpha_1+k-1}{k-1}\cdots\binom{\alpha_s+k-1}{k-1}. \tag{5}$$

First proof. Taking into account that the function S_k is multiplicative it follows

$$S_k(n) = S_k(p_1^{\alpha_1}\cdots p_s^{\alpha_s}) = S_k(p_1^{\alpha_1})\cdots S_k(p_s^{\alpha_s}) = \binom{\alpha_1+k-1}{k-1}\cdots\binom{\alpha_s+k-1}{k-1},$$

hence the desired conclusion.

□

Second proof. Alternatively, we can prove this by using the summation formula in Theorem 2 and together with Euler's product formula. We have

$$S_k(n) = \sum_{d|n} S_{k-1}(d) = \prod_{i=1}^{s}(1 + S_{k-1}(p_i) + \cdots + S_{k-1}(p_i^{\alpha_i})) =$$

$$\prod_{i=1}^{s}\left(\binom{k-1}{0} + \binom{k-1}{1} + \binom{k}{2}\cdots + \binom{\alpha_i+k-3}{\alpha_i-1} + \binom{\alpha_i+k-2}{\alpha_i}\right) =$$

$$\prod_{i=1}^{s}\left(\binom{k}{1} + \binom{k}{2} + \cdots + \binom{\alpha_i+k-3}{\alpha_i-1} + \binom{\alpha_i+k-2}{\alpha_i}\right) =$$

$$\prod_{i=1}^{s}\left(\binom{k+1}{2} + \cdots + \binom{\alpha_i+k-3}{\alpha_i-1} + \binom{\alpha_i+k-2}{\alpha_i}\right) = \cdots =$$

$$\prod_{i=1}^{s}\left(\binom{\alpha_i+k-2}{\alpha_i-1} + \binom{\alpha_i+k-2}{\alpha_i}\right) = \prod_{i=1}^{s}\binom{\alpha_i+k-2}{\alpha_i}.$$

□

Remark. Assume that $n = p_1^{\alpha_1}\cdots p_s^{\alpha_s}$. From Theorem 2 and the corollary above we have

$$S_{k+1}(n) = \sum_{d|n} S_k(d) = \sum_{0\le r_i\le \alpha_i} S_k(p_1^{r_1}\cdots p_s^{r_s}) = \sum_{0\le r_i\le\alpha_i}\binom{r_1+k-1}{k-1}\cdots\binom{r_s+k-1}{k-1},$$

hence we derived the following combinatorial identity involving the decomposition of a product of binomial coefficients as a sum of terms of the same form:

$$\binom{\alpha_1+k-1}{k-1}\cdots\binom{\alpha_s+k-1}{k-1} = \sum_{0\le r_i\le\alpha_i}\binom{r_1+k-1}{k-1}\cdots\binom{r_s+k-1}{k-1}. \tag{6}$$

3. Some upper bounds for $S_k(n)$

The asymptotic behavior of the function $S_k(n)$ when $n \to \infty$ is difficult to establish. One reason for this is that is very hard to estimate the asymptotics of the function $\omega(n)$, which counts the number of distinct prime divisors of n. To see the connection between these two functions recall that Corollary 4 asserts that

$$S_k(n) = \binom{\alpha_1 + k - 1}{k-1} \cdots \binom{\alpha_s + k - 1}{k-1},$$

where $s = \omega(n)$ and $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$. It is therefore obvious that for k fixed and $n \to \infty$ that asymptotics of $S_k(n)$ is influenced by $\omega(n)$ and by the exponents α_i, where $i \in \{1, \dots, s\}$.

Lemma 1 proved in the appendix of [8] asserts that for fixed k, $\lim_{n\to\infty} \frac{S_k(n)}{n} = 0$. If correct, the proof given in loc. cit. would generalize mutatis mutandis to a proof of the following fact: Given fixed $k \in \mathbb{N} \setminus \{0\}$ and $\epsilon > 0$, the limit $\lim_{n\to\infty} \frac{S_k(n)}{n^\epsilon} = 0$ holds. In our opinion, this is not the case for the proof given in the aforementioned article. We would like to mention that Lemma 1 is not a central result in [8].

From Proposition 7.10 of [4], it follows that there exists a positive integer n_0 such that for any $n \geq n_0$ we have

$$\omega(n) < \frac{2 \ln n}{\ln \ln n}. \tag{7}$$

It can be easily observed that the exponents α_i are bounded above by $\log_2 n$ for any $i \in \{1, \dots, n\}$. At the same time, we will make use of the following upper bound for binomial coefficients $\binom{m}{j}$. First, one notices that

$$\binom{m}{j} \leq \frac{m^j}{j!} = \frac{m^j}{j^j} \cdot \frac{j^j}{j!}.$$

For any positive integer integer j, from the Taylor expansion of the exponential function we can deduce that $\frac{j^j}{j!} < e^j$. Using this in the bound above, we obtain that for any positive integers $1 \leq j \leq m$, we have

$$\binom{m}{j} < \left(\frac{m \cdot e}{j}\right)^j.$$

Applying the latter bound, we find that the inequality

$$\binom{\alpha_i + k - 1}{k-1} < \left(\frac{(\alpha_i + k - 1) \cdot e}{k-1}\right)^{k-1} \tag{8}$$

holds for $k \geq 2$ and for all $i \in \{1, \ldots, n\}$.

Taking into the inequalities (7) and (8), we deduce that for k fixed, there exists an absolute constant $C > 0$ such that for n large enough we have

$$S_k(n) < (C \cdot \ln n)^{\frac{2(k-1)\ln n}{\ln\ln n}} = n^{2(k-1)+\frac{2(k-1)\ln C}{\ln\ln n}}.$$

Indeed, the last equality can be explained by the intermediary

$$(C \cdot \ln n)^{\frac{\ln n}{\ln\ln n}} = e^{\ln n \cdot \frac{\ln C}{\ln\ln n}} \cdot e^{\ln\ln n \cdot \frac{\ln n}{\ln\ln n}} = n^{\frac{\ln C}{\ln\ln n}} \cdot n$$

which has to be raised at the power $2(k-1)$.

The upper bound deduced above is not strong enough to imply that $\lim_{n\to\infty} \frac{S_k(n)}{n} = 0$. However, we remark that when $k = 2$, from Proposition 7.12 of [4] it follows that the number of divisors function $S_2(n) = n^{O(1/\ln\ln n)}$. This implies that for any fixed $\epsilon > 0$, the limit $\lim_{n\to\infty} \frac{S_2(n)}{n^\epsilon} = 0$ holds. The last equality can be proved using the Sandwich Theorem, following the observation that for any fixed constants $C, \epsilon > 0$ we have $\lim_{n\to\infty} e^{\frac{C\ln n}{\ln\ln n} - \epsilon\ln n} = 0$.

In what follows, we will prove that for any fixed k and any $\epsilon > 0$, the quantity $S_k(n)$ is asymptotically smaller than n^ϵ, for almost all n. To be precise, let us consider the following definition.

Definition 1. *We say that a set of positive integers has asymptotic density λ if*

$$\lambda = \lim_{x\to\infty} \frac{|A \cap [1, x]|}{x}.$$

We will make use of the following result.

Lemma 5 (Lemma 7.18 in [4])**.** *Let $\delta > 0$ and write*

$$A_\delta = \left\{ n : \left| \frac{\omega(n)}{\ln\ln n} - 1 \right| > \delta \right\}.$$

Then A_δ is of asymptotic density zero.

Setting $\delta = 1$, we see that the inequality

$$\omega(n) \leq 2\ln\ln n \tag{9}$$

holds for all $n \in \mathbb{N} \setminus A_1$, where A_1 is a set of asymptotic density zero.

Theorem 6. *For a fixed positive integer k, there is an absolute constant $C > 0$ such that for all $n \in \mathbb{N} \setminus A_1$, the following inequality holds*

$$S_k(n) \leq C^{(\ln\ln n)^2 + \ln\ln n}.$$

Proof. Let $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $s = \omega(n)$ and recall that

$$S_k(n) = \binom{\alpha_1 + k - 1}{k-1} \cdots \binom{\alpha_s + k - 1}{k-1}.$$

Denote by $\alpha = \max\{\alpha_1, \dots, \alpha_s\}$ and recall that $\alpha \le \log_2 n$. We have seen that for every $k \ge 2$, we have

$$\binom{\alpha + k - 1}{k-1} \le \left(\frac{(\alpha + k - 1)\cdot e}{k-1}\right)^{k-1}.$$

As k is fixed, there is a constant $K > 0$ such that

$$\binom{\alpha + k - 1}{k-1} \le K^{k-1} \cdot \ln n^{k-1}.$$

We now have that $S_k(n) \le \left(K^{k-1} \cdot \ln^{k-1} n\right)^{\omega(n)}$ which together with the inequality (9) implies that for $n \in \mathbb{N} \setminus A_1$ the following holds

$$S_k(n) \le K^{(k-1)\ln\ln n} (\ln n)^{(k-1)\ln\ln n} = \left(e^{(k-1)\ln K}\right)^{\ln\ln n} \cdot e^{(k-1)\cdot(\ln\ln n)^2}.$$

The conclusion now follows by setting $C = \max\left(e^{(k-1)\ln K}, e^{(k-1)}\right)$.

□

Remark. We remark that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{C^{(\ln\ln n)^2 + \ln\ln)}}{n^\epsilon} = \frac{C^{(\ln\ln n)^2 + \ln\ln n}}{e^{\epsilon \ln n}} = 0,$$

hence we can conclude that given a fixed k, for almost all n, the value of the function $S_k(n)$ is indeed asymptotically smaller than any positive power of n.

4. A new multiplicative function related to S_k

Let us recall that for positive integers n and k, the multiplicative function $S_k(n)$ was defined as the number of solutions in positive integers of the equation

$$x_1x_2\cdots x_k = n. \tag{10}$$

It is natural to study the number of solutions in positive integers to (10) that are subject to various conditions, such as the $\gcd(x_1, x_2, \dots, x_k) = d$ for some fixed value of d.

It is easy to see that if the k-tuple $(x_1, x_2, \dots, x_k) \in (\mathbb{N} \setminus \{0\})^k$ is a solution to (10) such that $\gcd(x_1, x_2, \dots, x_k) = d$, then $\gcd(x_1/d, x_2/d \dots, x_k/d) = 1$ and $d^k \mid n$. Moreover, the k-tuple $(x_1/d, x_2/d, \dots, x_k/d) \in (\mathbb{N} \setminus \{0\})^k$ is also a solution to an equation of the type (10), where the right hand side is equal to n/d^k.

For the values of d for which there is a solution to (10), the previous statement gives a bijection between the set of solutions to (10) satisfying $\gcd(x_1, x_2, \dots, x_k) = d$ and the set of solutions to

$$x_1x_2\cdots x_k = \frac{n}{d^k}$$

subject to $\gcd(x_1, x_2, \dots, x_k) = 1$.

We define $M_k(n)$ as the number of k-tuples $(x_1, x_2, \dots, x_k) \in (\mathbb{N}\setminus\{0\})^k$ satisfying

$$x_1x_2\cdots x_k = n$$

and $\gcd(x_1, x_2, \dots, x_k) = 1$. Remark that $M_1(n) = 1$, if $n = 1$ and $M_1(n) = 0$ otherwise.

In what follows, we write $\omega(n)$ for the number of distinct prime divisors and $\tau(n)$ for the number of distinct positive divisors of n. Both ω and τ are well-studied arithmetic functions. It is worth mentioning that ω is additive and τ is multiplicative.

The following theorem gives a link between the function $n \mapsto M_3(n)$ and the more familiar arithmetic functions $\omega, \tau : \mathbb{N} \setminus \{0\} \to \mathbb{N}$.

Theorem 7. *For every positive integer $n \geq 2$, we have that $M_3(n) = 3^{\omega(n)} \cdot \tau(n)$.*

Proof. Let us write $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$ for the factorisation of n into distinct prime factors. For each i, there is at least one $j \in \{1, 2, 3\}$ such that $p_i \nmid x_j$. Fixing j, there are $\alpha_i + 1$ ways in which one can distribute the powers of p_i to the remaining two terms. As for every $i \in \{1, \dots, \omega(n)\}$ the choices described above are independent, we have

$$M_3(n) = \prod_{i=1}^{\omega(n)} 3(\alpha_i + 1) = 3^{\omega(n)} \cdot \tau(n).$$

□

More generally, we have the following result which gives a relation between the functions M_k, S_{k-1} and the number of prime divisors function ω.

Theorem 8. *For any positive integers $n, k \geq 2$, we have $M_k(n) = k^{\omega(n)} S_{k-1}(n)$.*

Proof. Write $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$ for the factorisation of n into distinct prime factors. For every i, there is at least one $j \in \{1,2,\dots,k\}$ such that $p_i \nmid x_j$. Choosing such an index j, we must distribute the remaining powers of p_i into $\{x_1,x_2,\dots,x_k\} \setminus \{x_j\}$. The number of ways in which we can do this is $S_{k-1}(p_i^{\alpha_i})$. As for every $i \in \{1,\dots,\omega(n)\}$ the choices we make are independent, we have

$$M_k(n) = \prod_{i=1}^{\omega(n)} kS_{k-1}(p_i^{\alpha_i}) = k^{\omega(n)}S_{k-1}(n).$$

In the last step above we have used that $S_{k-1} : \mathbb{N} \to \mathbb{N}$ is multiplicative, result that was proved in Section 2. □

An immediate corollary which follows easily from the preceding theorem and from the additive property of ω is the following

Corollary 9. *For any $k \in \mathbb{N} \setminus \{0\}$, the function $M_k : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ is multiplicative.*

The next corollary can be proved with argument very similar to the one given in the previous section.

Corollary 10. *For a fixed positive integer k, there is an absolute constant $C > 0$ such that the inequality*

$$M_k(n) \le C^{(\ln\ln n)^2 + \ln\ln n}$$

holds for every $n \in \mathbb{N}$ except a set of asymptotic density zero.

5. The associated Dirichlet series

Let f and g be two arithmetic functions. Their *convolution product* is defined as

$$(f * g)(n) := \sum_{d|n} f(d)g(\frac{n}{d})$$

The convolution product has interesting algebraic properties, for instance it is commutative and associative (see [1, pp. 108–111]).

Given an arithmetic function f, the series

$$F(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z}, \tag{11}$$

is called the *Dirichlet series* associate with f. A Dirichlet series can be regarded as a purely formal infinite series, or as a function of the complex variable z, defined in the region in which the series converges.

When the function f is multiplicative we have the following formula involving the associated Euler product

$$F(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} = \prod_p \left(1 + \frac{f(p)}{p^z} + \frac{f(p^2)}{p^{2z}} + \frac{f(p^3)}{p^{3z}} + \cdots\right) \tag{12}$$

where the product is over all primes.

Let f and g be arithmetic functions with associated Dirichlet series $F(z)$ and $G(z)$. Let $h = f * g$ be the convolution product of f and g, and let $H(z)$ be its associated Dirichlet series. If $F(z)$ and $G(z)$ converge absolutely at some point z, then so does $H(z)$, and we have $H(z) = F(z)G(z)$. Indeed, we have

$$F(z)G(z) = (\sum_{l=1}^{\infty} \frac{f(l)}{l^z})(\sum_{m=1}^{\infty} \frac{g(m)}{m^z}) =$$

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \frac{f(l)g(m)}{l^z m^z} = \sum_{n=1}^{\infty} \frac{1}{n^z}\left(\sum_{lm=n} f(l)g(m)\right) = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^z},$$

where the rearranging of the terms in the double sum is justified by the absolute convergence of the series $F(z)$ and $G(z)$.

The most famous Dirichlet series is the *Riemann zeta function* $\zeta(z)$, defined as the Dirichlet series associated with constant function $\mathbf{1}$, that is $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, converging absolutely in the half-plane $Re(z) > 1$.

For the rest of this section k will denote a positive integer. The next theorem concerns the Dirichlet series of the multiplicative function S_k.

Theorem 11. *The following relations hold:*

1. $S_k = \mathbf{1} * \mathbf{1} * \cdots * \mathbf{1}$, *where there are k factors appearing in the convolution product.*
2. $\sum_{n=1}^{\infty} \frac{S_k(n)}{n^z} = (\zeta(z))^k, Re(z) > 1$, *where ζ is the Riemann zeta function.*

Proof. 1. Using the assertion of Theorem 2, we obtain

$$S_k(n) = \sum_{d|n} S_{k-1}(d) = \sum_{d|n} S_{k-1}(d)\mathbf{1}(\frac{n}{d}) = (S_{k-1} * \mathbf{1})(n),$$

hence $S_k = S_{k-1} * \mathbf{1}$. Since $S_1 = \mathbf{1}$, from the associativity property of the convolution product, it follows $S_k = \mathbf{1} * \mathbf{1} * \cdots * \mathbf{1}$, where in the convolution product there are k factors, and we are done.

2. The second part follows easily from the first. Indeed, using the general result concerning the Dirichlet series of a convolution product described above, we have

$$\sum_{n=1}^{\infty} \frac{S_k(n)}{n^z} = \sum_{n=1}^{\infty} \frac{(\mathbf{1} * \mathbf{1} * \cdots * \mathbf{1})(n)}{n^z} = (\zeta(z))^k.$$

□

Regarding the Dirichlet series $F_{M_k}(z)$ of the multiplicative function M_k, we present the following result.

Theorem 12. *Let $F_{M_k}(z)$ be the Dirichlet series of M_k. The following equality holds*

$$F_{M_k}(z) = \zeta(z)^{k-1} \prod_p Q_k\left(1 - \frac{1}{p^z}\right),$$

where $\zeta(z)$ is the Riemann zeta function and $Q_k(z) = k - (k-1)z^{k-1}$.

Proof. In the previous section we proved that M_k is multiplicative. It follows that we can apply the Euler product formula (12) and obtain

$$F_{M_k}(z) = \prod_p \left(1 + \frac{M_k(p)}{p^z} + \frac{M_k(p^2)}{p^{2z}} + \cdots\right) = \prod_p \left(1 + k\left(\frac{S_{k-1}(p)}{p^z} + \frac{S_{k-1}(p^2)}{p^{2z}} + \cdots\right)\right) =$$

$$= \prod_p \left(1 + k\left(\frac{\binom{k-1}{k-2}}{p^z} + \frac{\binom{k}{k-2}}{p^{2z}} + \cdots\right)\right).$$

Using the well-known relation

$$\sum_{\alpha=0}^{\infty} \binom{\alpha + k - 2}{k-2} z^{\alpha} = \frac{1}{(1-z)^{k-1}},$$

we obtain

$$F_{M_k}(z) = \prod_p \left(1 + k\left(\frac{1}{(1 - \frac{1}{p^z})^{k-1}} - 1\right)\right) = \zeta(z)^{k-1} \prod_p Q_k(1 - \frac{1}{p^z}).$$

□

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