Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 64/2020
pp. 37-42
doi: 10.17114/j.aua.2020.64.04

POMPEIU'S THEOREM IN AN INNER PRODUCT SPACES

D. Ş. Marinescu and M. Monea

Abstract. The aim of this paper to present a version for an inner product space of the classical theorem of Pompeiu.

2010 *Mathematics Subject Classification*: 46C05.

Keywords: inner product space, Pompeiu's Theorem.

1. Introduction

Pompeiu's theorem is considered one of the most "*elegant triangle theorem*"[3]. It is said that the segements MA, MB and MC could be the sides lenght of a triangle, for any point M not situated on the circumcircle of the equilateral triangle ABC. Today, we find more references about this theorem. (*e.g.* [2], [5])

Recently [1], it was presented and proved a n-dimensional generalization of Pompeiu's theorem.

Proposition 1.1. ([1], Theorem 4) *Let $n \geq 2$, and let $S = [A_1, A_2, ..., A_{n+1}]$ be a regular n-simplex of edge length a. Let B be a point in the afine hull of S, and let $b_1, b_2, ..., b_{n+1}$ be the distances from B to the vertices of S.*

If $n = 2$, i.e., if S is a triangle, then b_1, b_2, b_3 can serve as the side lengths of a triangle if and only if B does not lie on the circumcircle of S.

If $n \geq 3$, then $b_1, b_2, ..., b_{n+1}$ can serve as the facet contents of an n-simplex if and only if B is not a vertex of S.

Motivated by this result and the relation from [6], we intend to present a new viewpoint related to Pompeiu's Theorem. The main result are included in third section and contains a generalization for an inner product space of this theorem. For the proof we need some useful results. These are some relations that hold for any inner product space, some of these being generalizations of similar relations from [1].

To the end of this section, we recall a tool that helps us to complete our proof.

Proposition 1.2. ([4], Theorem 3.1) *Let X a real inner product space. Then*

$$\left\| x-\sum_{k=1}^{n}a_k x_k\right\|^2=\sum_{k=1}^{n}a_k\left\| x-x_k\right\|^2-\sum_{1\leq k<l\leq n}a_k a_l\left\| x_l-x_k\right\|^2, \tag{1}$$

for all integer n, $n\geq 2$, for all $a_1,a_2,...,a_n\in\mathbb{R}$ with $\sum\limits_{k=1}^{n}a_k=1$, and for any $x,x_1,x_2,...,x_n\in X$.

2. Some useful results

In this section X represents a real inner product spaces. Let $n\in\mathbb{N}$, $n\geq 2$ and $x_1,x_2,...,x_n\in X$, distingues, and $t_0\in(0,\infty)$ such that $\|x_i-x_j\|=t_0$, for any $i,j\in\{1,2,...,n\}$, $i\neq j$. Let affin$[x_1,x_2,...,x_n]$ the set $\left\{\sum\limits_{k=1}^{n}a_k x_k \,\middle|\, a_1,a_2,...,a_n\in\mathbb{R},\ \sum\limits_{k=1}^{n}a_k=1\right\}$.

First result is a generalization of Theorem 2 from [1].

Proposition 2.1. *For any $x\in$affin$[x_1,x_2,...,x_n]$ we denote $t_k=\|x-x_k\|$, $k\in\{1,2,...,n\}$. Then*

$$\left(\sum_{k=0}^{n}t_k^2\right)^2=n\left(\sum_{k=0}^{n}t_k^4\right). \tag{2}$$

Proof. If $x\in$affin$[x_1,x_2,...,x_n]$ there exists $a_1,a_2,...,a_n\in\mathbb{R}$ with $\sum\limits_{k=1}^{n}a_k=1$ such that $x=\sum\limits_{i=1}^{n}a_i x_i$. Then, for any $k\in\{1,2,...,n\}$, we have

$$\begin{aligned} t_k^2 &= \|x_k-x\|^2=\left\|x_k-\sum_{i=1}^{n}a_i x_i\right\|^2 \\ &= \sum_{i=1}^{n}a_i\|x_k-x_i\|^2-\sum_{1\leq i<l\leq n}a_i a_l\|x_l-x_i\|^2 \\ &= (1-a_k)\,t_0^2-\frac{\left(\sum_{i=1}^{n}a_i\right)^2-\sum_{i=1}^{n}a_i^2}{2}t_0^2 \\ &= \left(1-a_k-\frac{1-\sum_{i=1}^{n}a_i^2}{2}\right)t_0^2=\frac{1+\sum_{i=1}^{n}a_i^2-2a_k}{2}t_0^2. \end{aligned}$$

With the notation $A=\frac{1+\sum_{i=1}^{n}a_i^2}{2}$, we obtain $t_k^2=(A-a_k)\,t_0^2$. Then

$$\left(\sum_{k=0}^{n}t_k^2\right)^2=t_0^4\left(1+\sum_{k=1}^{n}(A-a_k)\right)^2=t_0^4\left(1+nA-\sum_{k=1}^{n}a_k\right)^2=n^2A^2t_0^4$$

and

$$\begin{aligned} n\left(\sum_{k=0}^{n} t_k^4\right) &= n\left(t_0^4+\sum_{k=1}^{n}(A-a_k)^2 t_0^4\right) = nt_0^4\left(1+nA^2-2A\sum_{k=1}^{n} a_k+\sum_{k=1}^{n} a_k^2\right) \\ &= n\left(1+nA^2-2A+2A-1\right) = n^2A^2t_0^k, \end{aligned}$$

that concludes the proof.

Proposition 2.2. *For any* $x \in$*affin*$[x_1, x_2, ..., x_n]$ *we denote* $t_k = \|x - x_k\|$, $k \in \{1, 2, ..., n\}$. *Then*

$$\left(\sum_{k=1}^{n} t_k^2\right)^2 - (n-1)\left(\sum_{k=1}^{n} t_k^4\right) = \frac{(n-1)^2}{n}\left(t_0^2 - \frac{1}{n-1}\sum_{k=1}^{n} t_k^2\right)^2. \tag{3}$$

Proof. Let $U = \sum_{k=1}^{n} t_k^2$ and $V = \sum_{k=1}^{n} t_k^4$. From previous proposition we have $\left(t_0^2 + U\right)^2 = n\left(t_0^4 + V\right)$. Then $t_0^4 + 2Ut_0^2 + U^2 = nt_0^4 - nV$, also $U^2 - nV = (n-1)\,t_0^4 - 2Ut_0^2$.

Further

$$U^2 + \frac{1}{n-1}U^2 - nV = (n-1)\,t_0^4 - 2Ut_0^2 + \frac{1}{n-1}U^2 - nV,$$

so

$$\frac{n}{n-1}U^2 - nV = (n-1)\left(t_0^2 - \frac{1}{n-1}U\right)^2.$$

Now, we obtain the conclusion if we multiply with $\frac{n-1}{n}$.

Further, let $g = \frac{1}{n}\sum_{k=1}^{n} x_k$ and $R = t_0\sqrt{\frac{n-1}{2n}}$. The next two propositions present some properties of g.

Proposition 2.3. *For any* $k \in \{1, 2, ..., n\}$ *we have* $\|g - x_k\| = R$.

Proof. We apply Proposition 1.2 for $a_1 = a_2 = ... = a_n = \frac{1}{n}$. We obtain

$$\begin{aligned} \|g - x_k\|^2 &= \left\|x_k - \frac{1}{n}\sum_{i=1}^{n} x_i\right\|^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\|x_k - x_i\|^2 - \frac{1}{n^2}\sum_{1\le i<l\le n} a_i a_l \|x_l - x_i\|^2 \\ &= \frac{n-1}{n}t_0^2 - \frac{n(n-1)}{2n^2}t_0^2 = \frac{n-1}{2n}t_0^2. \end{aligned}$$

We obtain $\|g - x_k\|^2 = R^2$ and the conclusion follows now.

Proposition 2.4. *For any $x \in X$ we denote $t_k = \|x - x_k\|$, $k \in \{1, 2, ..., n\}$. Then $\|x - g\| = R$ if and only if*

$$(n-1)\, t_0^2 = t_1^2 + t_2^2 + ... + t_n^2.$$

Proof. The equality $\|x - g\| = R$ is equivalent with $\|x - g\|^2 = \frac{n-1}{2n} t_0^2$. By other hand, (1) lead us to

$$\begin{aligned}\|x-g\|^2 &= \left\| x - \frac{1}{n}\sum_{k=1}^{n} x_k \right\|^2 \\ &= \frac{1}{n}\sum_{k=1}^{n}\|x - x_k\|^2 - \frac{1}{n^2}\sum_{1\le i<l\le n} a_i a_l \|x_l - x_i\|^2 \\ &= \frac{1}{n}\sum_{k=1}^{n} t_k^2 - \frac{n-1}{2n} t_0^2.\end{aligned}$$

We obtain $\frac{n-1}{2n} t_0^2 = \frac{1}{n}\sum_{k=1}^{n} t_k^2 - \frac{n-1}{2n} t_0^2$ and the conclusion follows now.

3. The Pompeiu's theorem

This section is reserved to the generalization of Pompeiu's theorem. Let $x_1, x_2, x_3 \in X$ distingues such that $\|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_1\| = t_0$. Let $g = \frac{x_1+x_2+x_3}{3}$. From Proposition 2.3 we have $\|x_k - g\| = \frac{1}{\sqrt{3}} t_0 = R$. We denote $\mathcal{C}(g, R)$ the set $\{y \in X \mid \|y - g\| = R\}$ .This set represent the analogues of the circumcircle from triangle geometry.

Theorem 3.1. *For any $x \in$affin$[x_1, x_2, x_3]$, denote $t_1 = \|x - x_1\|$, $t_2 = \|x - x_2\|$ and $t_3 = \|x - x_3\|$. Then t_1, t_2, t_3 could be the side lengths of a triangle if only if $x \notin C(g, R)$.*

Proof. We will use the following identity that holds for any $a, b, c \in \mathbb{R}$:

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = \left(a^2+b^2+c^2\right)^2 - 2\left(a^4+b^4+c^4\right). \tag{4}$$

If t_1, t_2, t_3 are the side lengths of a triangle then

$$(t_1+t_2+t_3)(-t_1+t_2+t_3)(t_1-t_2+t_3)(t_1+t_2-t_3) > 0. \tag{5}$$

From (4) we obtain $\left(t_1^2+t_2^2+t_3^2\right)^2 > 2\left(t_1^4+t_2^4+t_3^4\right)$. For $n = 3$, the identity (3) led us to conclusion $t_0^2 \neq \frac{1}{2}\left(t_1^2+t_2^2+t_3^2\right)$ and, with Proposition 2.4, we obtain $x \notin C(g, R)$.

For the only if part, we consider $x \notin C(g,R)$. By reversing the previous calculus we obtain (5), also

$$(-t_1+t_2+t_3)(t_1-t_2+t_3)(t_1+t_2-t_3)>0.$$

We assume that two factors are negative. Let $t_1-t_2+t_3<0$ and $t_1+t_2-t_3<0$. Then $t_1<t_2-t_3$ and $t_1<t_3-t_2$ that is false. Then all three factors are positive and t_1,t_2,t_3 could be the side lengths of a triangle.

We conclude this paper with the next proposition.

Proposition 3.2. *Let $n\in\mathbb{N}$, $n\geq 4$ and $x_1,x_2,...,x_n\in X$, distingues, and $t_0\in(0,\infty)$ such that $\|x_i-x_j\|=t_0$, for any $i,j\in\{1,2,...,n\}$, $i\neq j$. For any $x\in$affin$[x_1,x_2,...,x_n]$ we denote $t_k=\|x-x_k\|$, $k\in\{1,2,...,n\}$. Then $t_1,t_2,...,t_n$ could be the side lengths of a n-side poligon if and only if $x\notin\{x_1,x_2,...,x_n\}$.*

Proof. If $t_1,t_2,...,t_n$ are the side lengths of a n-side poligon, then $t_k>0$, for any $k\in\{1,2,...,n\}$. Then $\|x-x_k\|>0$, so $x\neq x_k$.We obtain $x\notin\{x_1,x_2,...,x_n\}$.

Now, if $x\notin\{x_1,x_2,...,x_n\}$ then $\left(\sum_{k=1}^n t_k^2\right)^2-(n-1)\left(\sum_{k=1}^n t_k^4\right)\geq 0$ from (2). Now thw conclusion is consequence of Theorem 2 from [1].

References

[1] G. Al-Afifi, M. Hajja & A. Hamdan, *Another n-dimensional generalization of Pompeiu's theorem*, Amer. Math. Monthly, 125(2019), 7, 612-622, DOI: 10.1080/00029890.2018.1466609.

[2] M. Fiedler, *Isodynamic systems in Euclidean spaces and an n-dimensional analogue of a theorem by Pompeiu.* Časopis Pro Pěstování Matematiky, 102 (1977), 370–381.

[3] M. Gardner, *Elegant triangle theorems not to be found in Euclid*, Mathematical Games. Sci. Amer, 222 (1979), 6,132–141.

[4] D. Ş. Marinescu, M. Monea, M. Opincariu, M. Stroe, *Some equivalent characterizations of inner product spaces and their consequences*, Filomat, 29 (2015), 1587-1599.

[5] G. R. Veldkamp, *Still another generalisation of Pompeiu's theorem*, Nieuw Tijdschr. Wisk, 45 (1957/1958), 197–225.

[6] F. Zachary, *Problem 12139*, Amer. Math. Monthly, 126 (2019), 9, 850-858, DOI: 10.1080/00029890.2019.1647063.

Dan Ştefan Marinescu
National College "IANCU DE HUNEDOARA",

Hunedoara, Romania
email: *marinescuds@gmail.com*

Mihai Monea
National College "DECEBAL",
Deva, Romania
email: *mihaimonea@yahoo.com*