Acta Universitatis Apulensis No. 64/2020
ISSN: 1582-5329 pp. 137-170
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.64.10

CERTAIN NEW FORMULAS FOR THE HORN'S HYPERGEOMETRIC FUNCTIONS

M.A. Pathan, A. Shehata, S.I. Moustafa

Abstract. In recent years, several interesting families of recursion formulas for various classes of classical hypergeometric functions are investigated. The aim of this present work is to develop certain interesting new families of recursion formulas, differential recursion formulas, integration formulas, differential operators, integral operators and infinite summation formulas analogue of Horn's hypergeometric functions $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_5$, $\mathbf{H}_6$, $\mathbf{H}_7$, $\mathbf{H}_8$, $\mathbf{H}_9$, $\mathbf{H}_{10}$ and $\mathbf{H}_{11}$ using different techniques including contiguous relations of Horn's hypergeometric series. Some interesting special cases of our main results are also demonstrated.The results to be obtained are presented in a compact way, in order to make the topic accessible to a wider audience.

2010 *Mathematics Subject Classification*: Primary 33C05; 33C20, Secondary 33C15; 11J72.

Keywords: Contiguous relations, recursion formulas, Horn hypergeometric functions.

1. Introduction and Preliminaries

The Gamma function $\Gamma(a)$ is defined as [16]

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt, \Re(a) > 0.$$

The Pochhammer symbol $(a)_n$ is defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)\dots(a+n-1), & n \in \mathbb{N}, a \in \mathbb{C}; \\ 1, & n = 0; a \in \mathbb{C} \setminus \{0\}, \end{cases}$$

where $\mathbb{N}$ and $\mathbb{C}$ are the set of natural numbers and complex numbers, respectively.

For $n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $a \in \mathbb{C}$, we have

$$\begin{aligned} (a)_{n+m} &= (\alpha)_n(\alpha+n)_m = (\alpha)_m(a+m)_n, \\ (a)_{m-n} &= \begin{cases} \frac{(-1)^n (a)_m}{(1-a-m)_n}, & 0 \leq n \leq m; \\ 0, & n > m. \end{cases} \\ (a)_{2m-n} &= \begin{cases} \frac{(-1)^n (a)_{2m}}{(1-a-2m)_n}, & 0 \leq n \leq 2m; \\ 0, & n > 2m. \end{cases} \end{aligned} \tag{1}$$

Suppose that a satisfy the condition (1), the Horn hypergeometric functions $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_5$, $\mathbf{H}_6$, $\mathbf{H}_7$, $\mathbf{H}_8$, $\mathbf{H}_9$, $\mathbf{H}_{10}$ and $\mathbf{H}_{11}$ are defined by [[6], [5], page 226, Equations (29)-(39)]

$$\mathbf{H}_1(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_{m+n}}{(c)_m m! n!} x^m y^n; |x| < 1, |y| < \infty, c \neq 0,-1,-2,\ldots, \tag{2}$$

$$\mathbf{H}_2(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m (c)_n}{(d)_m m! n!} x^m y^n; |x| < 1, |y| < \infty, d \neq 0,-1,-2,\ldots, \tag{3}$$

$$\mathbf{H}_3(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m m! n!} x^m y^n; |x| < 1, |y| < \infty, c \neq 0,-1,-2,\ldots \tag{4}$$

$$\mathbf{H}_4(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_n}{(c)_m m! n!} x^m y^n; |x| < \infty, |y| < \infty, c \neq 0,-1,-2,\ldots, \tag{5}$$

$$\mathbf{H}_5(a;b;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}}{(b)_m m! n!} x^m y^n; |x| < \infty, |y| < \infty, b \neq 0,-1,-2,\ldots, \tag{6}$$

$$\mathbf{H}_6(a;b;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(b)_{m+n} m! n!} x^m y^n; |x| < \frac{1}{4}, |y| < \infty, b \neq 0,-1,-2,\ldots, \tag{7}$$

$$\mathbf{H}_7(a;b,c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(b)_m (c)_n m! n!} x^m y^n; |x| < \frac{1}{4}, |y| < \infty, c \neq 0,-1,-2,\ldots, \tag{8}$$

$$\mathbf{H}_8(a,b;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m-n}(b)_{n-m}}{m!n!}x^m y^n; |x|<\frac{1}{4}, |y|<\infty, b\neq 0,-1,-2,\ldots, \quad (9)$$

$$\mathbf{H}_9(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m-n}(b)_{n}}{(c)_m m!n!}x^m y^n; |x|<\frac{1}{4}, |y|<\infty, c\neq 0,-1,-2,\ldots, \quad (10)$$

$$\mathbf{H}_{10}(a;b;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m-n}}{(b)_m m!n!}x^m y^n; |x|<\frac{1}{4}, |y|<\infty, b\neq 0,-1,-2,\ldots \quad (11)$$

and

$$\mathbf{H}_{11}(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_n(c)_n}{(d)_m m!n!}x^m y^n; |x|<\infty, |y|<1, d\neq 0,-1,-2,\ldots. (12)$$

In order to recall the abbreviated notations, for $n\in\mathbb{N}_0$ and $a,b\in\mathbb{C}$, we need the notations

$$\begin{aligned}
(a)_{n+1} &= a(a+1)_n,\\
(a)_{n-1} &= \frac{1}{a-1}(a-1)_n; a\neq 1,\\
(a+1)_n &= \left(1+\frac{n}{a}\right)(a)_n; a\neq 0,\\
(a-1)_n &= (a-1)(a)_{n-1}=\frac{a-1}{a+n-1}(a)_n=\left(1-\frac{n}{a+n-1}\right)(a)_n; a\neq 1-n,\\
\frac{1}{(b-1)_n} &= \frac{b+n-1}{(b-1)(b)_n}=\frac{1}{(b)_n}+\frac{n}{(b-1)(b)_n}; b\neq 1,0,-1,-2,\ldots.
\end{aligned} \quad (13)$$

Sayyed and Abul-Ez [1, 14] defined an integral operator $\hat{\mathfrak{I}}$ in the form

$$\hat{\mathfrak{I}}=\frac{1}{x}\int_0^x dx+\frac{1}{y}\int_0^y dy. \quad (14)$$

Horn essentially identified 34 distinct convergent series. Amongst them, we have selected eleventh Horn series that occur more frequently in a wide variety of problems in diverse areas of theoretical physics, mathematical physics, mathematical analysis, applied mathematics, numerical analysis, chemistry, potential theory, statistics, probability theory and engineering sciences. Several recursion formulas involving Horn hypergeometric functions which play an important role in several physical problems are developed.

Motivated by the investigations of recursion formulas for special functions, which are mentioned in [2, 3, 4, 7, 8, 9, 10, 11, 12, 15, 13, 17], we provide a new way of obtaining the results of some rather complicated integral formulas and infinite summation formulas. The structure of this work is as follows. In Sections 2-12, we discuss several classes of recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for Horn functions $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_5$, $\mathbf{H}_6$, $\mathbf{H}_7$, $\mathbf{H}_8$, $\mathbf{H}_9$, $\mathbf{H}_{10}$ and $\mathbf{H}_{11}$ with all the parameters.

2. Certain new formulas for $\mathbf{H}_1$

Here we give the recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_1$ with all the parameters.

Theorem 1. *For $c \neq 0, -1, -2, \ldots$ and $n \in \mathbb{N}$, the recursion formulas for Horn function $\mathbf{H}_1$ are as follows*

$$\begin{aligned}&\mathbf{H}_1(a+n,b;c;x,y) = \mathbf{H}_1(a,b;c;x,y) + \frac{bx}{c}\sum_{k=1}^{n}\mathbf{H}_1(a+k,b+1;c+1;x,y)\\&-by\sum_{k=1}^{n}\frac{\mathbf{H}_1(a+k-2,b+1;c;x,y)}{(a+k-1)(a+k-2)}, a \neq 1-k, a \neq 2-k, k \in \mathbb{N},\end{aligned}\tag{15}$$

and

$$\begin{aligned}&\mathbf{H}_1(a-n,b;c;x,y) = \mathbf{H}_1(a,b;c;x,y) - \frac{bx}{c}\sum_{k=1}^{n}\mathbf{H}_1(a-k+1,b+1;c+1;x,y)\\&+by\sum_{k=1}^{n}\frac{\mathbf{H}_1(a-k-1,b+1;c;x,y)}{(a-k)(a-k-1)}, a \neq k, a \neq 1+k, k \in \mathbb{N}.\end{aligned}\tag{16}$$

Proof. By using the definition of the Horn function $\mathbf{H}_1$ (2) and transformation

$$(a+1)_{m-n} = (a)_{m-n}\left(1+\frac{m-n}{a}\right), a \neq 0,$$

we get the relation:

$$\begin{aligned}&\mathbf{H}_1(a+1,b;c;x,y) = \mathbf{H}_1(a,b;c;x,y) + \frac{bx}{c}\mathbf{H}_1(a+1,b+1;c+1;x,y)\\&-\frac{by}{a(a-1)}\mathbf{H}_1(a-1,b+1;c;x,y), c \neq 0, a \neq 0, a \neq 1.\end{aligned}\tag{17}$$

If we compute the Horn function $\mathbf{H}_1$ with the numerator parameter $a+n$ by contiguous relation (17) for n times, we obtain the formula (15).

From the contiguous relation (17) and replacing a by $a-1$, we get

$$\begin{aligned}&\mathbf{H}_1(a-1,b;c;x,y)=\mathbf{H}_1(a,b;c;x,y)-\frac{bx}{c}\mathbf{H}_1(a,b+1;c+1;x,y)\\&+\frac{by}{(a-1)(a-2)}\mathbf{H}_1(a-2,b+1;c;x,y),c\neq 0,a\neq 1,a\neq 2.\end{aligned}\tag{18}$$

If we apply this contiguous relation (18) to the Horn function $\mathbf{H}_1$ with the numerator parameter $a-n$ for n times, we get the recursion formula (16).

Theorem 2. *For $a\neq 1$ and $c\neq 0,-1,-2,\ldots$, the recurrence relations hold true for the Horn hypergeometric function $\mathbf{H}_1$:*

$$\begin{aligned}&\mathbf{H}_1(a,b+n;c;x,y)=\mathbf{H}_1(a,b;c;x,y)+\frac{ax}{c}\sum_{k=1}^{n}\mathbf{H}_1(a+1,b+k;c+1;x,y)\\&+\frac{y}{a-1}\sum_{k=1}^{n}\mathbf{H}_1(a-1,b+k;c;x,y)\end{aligned}\tag{19}$$

and

$$\begin{aligned}&\mathbf{H}_1(a,b-n;c;x,y)=\mathbf{H}_1(a,b;c;x,y)-\frac{ax}{c}\sum_{k=1}^{n}\mathbf{H}_1(a+1,b-k+1;c+1;x,y)\\&-\frac{y}{a-1}\sum_{k=1}^{n}\mathbf{H}_1(a-1,b-k+1;c;x,y).\end{aligned}\tag{20}$$

Proof. Using (2) and the relation

$$(b+1)_{m+n}=(b)_{m+n}\left(1+\frac{m+n}{b}\right),b\neq 0,$$

we get the contiguous function

$$\begin{aligned}&\mathbf{H}_1(a,b+1;c;x,y)=\mathbf{H}_1(a,b;c;x,y)+\frac{ax}{c}\mathbf{H}_1(a+1,b+1;c+1;x,y)\\&+\frac{y}{a-1}\mathbf{H}_1(a-1,b+1;c;x,y),a\neq 1,c\neq 0.\end{aligned}\tag{21}$$

By iterating this method on $\mathbf{H}_1$ with the parameter $b+n$ for n times, we get (19).

Replacing b by $b-1$ in contiguous relation (21), we obtain

$$\begin{aligned}\mathbf{H}_1(a,b-1;c;x,y) =&\mathbf{H}_1(a,b;c;x,y) - \frac{ax}{c}\mathbf{H}_1(a+1,b;c+1;x,y)\\ &- \frac{y}{a-1}\mathbf{H}_1(a-1,b;c;x,y), a \neq 1, c \neq 0.\end{aligned} \tag{22}$$

If we compute the Horn function $\mathbf{H}_1$ with the numerator parameter $b-n$ by the contiguous relation (22) for n times, we obtain the recursion formula (20).

Theorem 3. *The Horn hypergeometric function* $\mathbf{H}_1$ *satisfies the following identity:*

$$\begin{aligned}\mathbf{H}_1(a,b;c-n;x,y) =&\mathbf{H}_1(a,b;c;x,y) + abx\sum_{k=1}^{n}\frac{\mathbf{H}_1(a+1,b+1;c-k+2;x,y)}{(c-k)(c-k+1)},\\ &(c \neq k, c \neq k-1, k \in \mathbb{N})\end{aligned} \tag{23}$$

and

$$\begin{aligned}\mathbf{H}_1(a,b;c+n;x,y) =&\mathbf{H}_1(a,b;c;x,y) - abx\sum_{k=1}^{n}\frac{\mathbf{H}_1(a+1,b+1;c+k+1;x,y)}{(c+k-1)(c+k)},\\ &(c \neq -k, c \neq 1-k, k \in \mathbb{N}).\end{aligned} \tag{24}$$

Proof. From (2) and the relation

$$\frac{1}{(c-1)_m} = \frac{1}{(c)_m} + \frac{m}{(c-1)(c)_m}, c \neq 1,$$

we obtain the contiguous function

$$\mathbf{H}_1(a,b;c-1;x,y) = \mathbf{H}_1(a,b;c;x,y) + \frac{abx}{(c-1)c}\mathbf{H}_1(a+1,b+1;c+1;x,y), c \neq 0, 1. \tag{25}$$

Iterating this method on $\mathbf{H}_1$ with the parameter $c-n$ for n times, we obtain (23).

Replacing c by $c+1$ in contiguous relation (25), we obtain

$$\mathbf{H}_1(a,b;c+1;x,y) = \mathbf{H}_1(a,b;c;x,y) - \frac{abx}{c(c+1)}\mathbf{H}_1(a+1,b+1;c+2;x,y), c \neq 0, -1. \tag{26}$$

If we apply the Horn function $\mathbf{H}_1$ with the denominator parameter $c+n$ by the contiguous relation (26) for n times, we obtain the recursion formula (24).

Now we apply differential operators $\theta_x = x\frac{\partial}{\partial x}$ and $\theta_y = y\frac{\partial}{\partial y}$ and state the following theorem.

Theorem 4. *Differential recursion formulas for the function* $\mathbf{H}_1$ *are as follows*

$$\mathbf{H}_1(a+1,b;c;x,y)=\Big(1+\frac{\theta_x-\theta_y}{a}\Big)\mathbf{H}_1(a,b;c;x,y), a\neq 0 \tag{27}$$

$$\mathbf{H}_1(a,b+1;c;x,y)=\Big(1+\frac{\theta_y+\theta_x}{b}\Big)\mathbf{H}_1(a,b;c;x,y), b\neq 0, \tag{28}$$

and

$$\mathbf{H}_1(a,b;c-1;x,y)=\Big(1+\frac{\theta_x}{c-1}\Big)\mathbf{H}_1(a,b;c;x,y), c\neq 1. \tag{29}$$

Proof. Defining the differential operators

$$\theta_x x^m = x\frac{\partial}{\partial x}x^m = mx^m, \quad \theta_y y^n = y\frac{\partial}{\partial y}y^n = ny^n.$$

By using the above differential operators, we obtain the differential recursion formula for $\mathbf{H}_1(a+1,b;c;x,y)$

$$\mathbf{H}_1(a+1,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{(c)_m m!n!}x^m y^n$$
$$+\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{a(c)_m m!n!}\Big[mx^m y^n - nx^m y^n\Big]$$
$$=\mathbf{H}_1(a,b;c;x,y)+\frac{\theta_x-\theta_y}{a}\mathbf{H}_1(a,b;c;x,y), a\neq 0.$$

Similarly, we can prove (28) and (29).

Theorem 5. *The derivative formulas hold true for Horn hypergeometric function* $\mathbf{H}_1$

$$\frac{\partial^r}{\partial x^r}\mathbf{H}_1(a,b;c;x,y)=\frac{(a)_r(b)_r}{(c)_r}\mathbf{H}_1(a+r,b+r;c+r;x,y), c\neq 0,-1,-2,\ldots, \tag{30}$$

and

$$\frac{\partial^r}{\partial y^r}\mathbf{H}_1(a,b;c;x,y)=\frac{(-1)^r(b)_r}{(1-a)_r}\mathbf{H}_1(a-r,b+r;c;x,y), a\neq 1,2,3,\ldots. \tag{31}$$

Proof. Differentiating (2) with respect to x yields

$$\frac{\partial}{\partial x}\mathbf{H}_1(a,b;c;x,y)=\frac{ab}{c}\mathbf{H}_1(a+1,b+1;c+1;x,y),c\neq 0.$$

Repeating the above process, we eventually arrive at

$$\frac{\partial^r}{\partial x^r}\mathbf{H}_1(a,b;c;x,y)=\frac{a(a+1)\dots(a+r-1)b(b+1)\dots(b+r-1)}{c(c+1)(c+2)\dots(c+r-1)}\mathbf{H}_1(a-r,b+2r;x,y)$$
$$=\frac{(a)_r(b)_r}{(c)_r}\mathbf{H}_1(a+r,b+r;c+r;x,y).$$

In the same way we can prove (31).

We now apply integral operator $\hat{\mathfrak{I}}$ defined in (14) and state the following theorem.

Theorem 6. *For $x,y\neq 0$, $a\neq 1$, $a\neq 2$, $b\neq 1$ and $b\neq 2$. The integration formulas of the Horn hypergeometric function $\mathbf{H}_1$ hold true:*

$$\hat{\mathfrak{I}}^2\mathbf{H}_1(a,b;c;x,y)=\frac{(c-1)(c-2)}{x^2(a-1)(a-2)(b-1)(b-2)}\mathbf{H}_1(a-2,b-2;c-2;x,y)$$
$$+\frac{2(c-1)}{xy(b-1)}\mathbf{H}_1(a,b-1;c-1;x,y)+\frac{a(a+1)}{y^2(b-1)(b-2)}\mathbf{H}_1(a+2,b-2;c;x,y). \tag{32}$$

Proof. Let $\hat{\mathfrak{I}}$ acts on the Horn hypergeometric function $\mathbf{H}_1$, then we have the relation

$$\hat{\mathfrak{I}}\mathbf{H}_1(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{(c)_m(m+1)!n!}x^my^n+\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{(c)_m m!(n+1)!}x^my^n$$
$$=\frac{c-1}{x(a-1)(b-1)}\mathbf{H}_1(a-1,b-1;c-1;x,y)+\frac{a}{y(b-1)}\mathbf{H}_1(a+1,b-1;c;x,y),x,y\neq 0,a,b\neq 1$$

and we can write $\hat{\mathfrak{I}}=\hat{\mathfrak{I}}_x+\hat{\mathfrak{I}}_y$ where $\hat{\mathfrak{I}}_x=\frac{1}{x}\int_0^x dx$ and $\hat{\mathfrak{I}}_y=\frac{1}{y}\int_0^y dy$, then the operator $\hat{\mathfrak{I}}^2$ is such that

$$\hat{\mathfrak{I}}^2=\hat{\mathfrak{I}}\hat{\mathfrak{I}}=(d\hat{\mathfrak{I}}_x)^2+2\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y+(\hat{\mathfrak{I}}_x)^2=\frac{1}{x^2}\int_0^x\int_0^x dxdx+\frac{2}{xy}\int_0^y\int_0^x dxdy+\frac{1}{y^2}\int_0^y\int_0^y dydy.$$

We see that,

$$
\begin{aligned}
&\hat{\mathfrak{I}}^2\mathbf{H}_1(a,b;c;x,y)\\
&=\sum_{m,n=0}^{\infty}\left(\frac{1}{(m+1)(m+2)}+\frac{2}{(n+1)(m+1)}+\frac{1}{(n+1)(n+2)}\right)\frac{(a)_{m-n}(b)_{m+n}}{(c)_m m!n!}x^m y^n\\
=&\sum_{m,n=0}^{\infty}\frac{(a)_{m-n-2}(b)_{m+n-2}}{(c)_{m-2}m!n!}x^{m-2}y^n+2\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n-2}}{(c)_{m-1}m!n!}x^{m-1}y^{n-1}\\
&+\sum_{m,n=0}^{\infty}\frac{(a)_{m-n+2}(b)_{m+n-2}}{(c)_m m!n!}x^m y^{n-2}\\
=&\frac{(c-1)(c-2)}{x^2(a-1)(a-2)(b-1)(b-2)}\mathbf{H}_1(a-2,b-2;c-2;x,y)+\frac{2(c-1)}{xy(b-1)}\mathbf{H}_1(a,b-1;c-1;x,y)\\
&+\frac{a(a+1)}{y^2(b-1)(b-2)}\mathbf{H}_1(a+2,b-2;c;x,y),\ x,y\neq 0,\ a,b\neq 1,2.
\end{aligned}
$$

Theorem 7. *The integration recursion formulas of the Horn hypergeometric function* $\mathbf{H}_1$ *hold true:*

$$
\begin{aligned}
\hat{\mathfrak{I}}^r\mathbf{H}_1(a,b;c;x,y)=&\frac{(1-c)_r}{x^r y^r(1-b)_{2r}}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_1(a,b-2r;c-r;x,y),\\
&(x,y\neq 0,\ b\neq 1,2,3,\ldots).
\end{aligned}
\tag{33}
$$

Proof. Using the operator $\hat{\mathfrak{I}}$, we get

$$
\begin{aligned}
\hat{\mathfrak{I}}\mathbf{H}_1(a,b;c;x,y)&=\sum_{m,n=1}^{\infty}\frac{(m+n+2)(a)_{m-n}(b)_{m+n}}{(c)_m(m+1)!(n+1)!}x^m y^n\\
&=\sum_{m,n=0}^{\infty}\frac{(m+n)(c-1)(a)_{m-n}(b-2)_{m+n}}{m!n!(c-1)_m(b-1)(b-2)}x^{m-1}y^{n-1}\\
&=\frac{(c-1)}{xy(b-1)(b-2)}(\theta_x+\theta_y)\mathbf{H}_1(a,b-2;c-1;x,y),\ x,y\neq 0,\ b\neq 1,2.
\end{aligned}
$$

By iterating for r times, we find (33).

Theorem 8. *For Horn hypergeometric function* $\mathbf{H}_1$*, we have the integral operators* $\hat{I}_x^r$ *and* $\hat{I}_y^r$*:*

$$
\hat{\mathfrak{I}}_x{}^r\ \mathbf{H}_1(a,b;c;x,y)=\frac{(-1)^r(1-c)_r}{x^r(1-a)_r(1-b)_r}\mathbf{H}_1(a-r,b-r;c-r;x,y),\ x\neq 0,\ a,b\neq 1,2,3,\ldots,\tag{34}
$$

$$\hat{\Im_y}^r \, \mathbf{H}_1(a,b;c;x,y) = \frac{(-1)^r (a)_r}{y^r (1-b)_r} \mathbf{H}_1(a+r,b-r;c;x,y), y \neq 0, b \neq 1,2,3,\ldots \tag{35}$$

and

$$\left(\hat{\Im_x}\hat{\Im_y}\right)^r \mathbf{H}_1(a,b;c;x,y) = \frac{(1-c)_r}{x^r y^r (1-b)_{2r}} \mathbf{H}_1(a,b-2r;c-r;x,y), x,y \neq 0, b \neq 1,2,3,\ldots. \tag{36}$$

Proof. Also, we get relations concerning $\hat{\Im_x}$ and $\hat{\Im_y}$ individually,

$$\hat{\Im_x} \, \mathbf{H}_1(a,b;c;x,y) = \frac{c-1}{x(a-1)(b-1)} \mathbf{H}_1(a-1,b-1;c-1;x,y), x \neq 0, a,b \neq 1.$$

Iterating for n times, we find

$$\begin{aligned}
&\hat{\Im_x}^r \, \mathbf{H}_1(a,b;c;x,y)\\
=&\frac{(c-1)(c-2)\ldots(c-r)}{x^r(a-1)(a-2)\ldots(a-r)(b-1)(b-2)\ldots(b-r)} \mathbf{H}_1(a-r,b-r;c-r;x,y)\\
=&\frac{(-1)^r(1-c)(2-c)\ldots(r-c)}{x^r(1-a)(2-a)\ldots(r-a)(1-b)(2-b)\ldots(r-b)} \mathbf{H}_1(a-r,b-r;c-r;x,y)\\
=&\frac{(-1)^r(1-c)_r}{x^r(1-a)_r(1-b)_r} \mathbf{H}_1(a-r,b-r;c-r;x,y), x \neq 0, a,b \neq 1,2,3,\ldots
\end{aligned}$$

Similarly, for $\Im_y$, we have

$$\begin{aligned}
\hat{\Im_y} \, \mathbf{H}_1(a,b;c;x,y) &= \sum_{m=0,n=1}^{\infty} \frac{(a)_{m-n}(b)_{m+n}}{(c)_m (n+1) m! n!} x^m y^n\\
&= \sum_{m,n=0}^{\infty} \frac{a(a+1)_{m-n}(b+1)_{m+n}}{(c)_m m! n! (b-1)} x^m y^{n-1}\\
&= \frac{a}{y(b-1)} \mathbf{H}_1(a+1,b-1;c;x,y), y \neq 0, b \neq 1.
\end{aligned}$$

Iteration of above relation for r, we obtain

$$\begin{aligned}
\hat{\Im_y}^r \, \mathbf{H}_1(a,b;c;x,y) &= \frac{a(a+1)\ldots(a+r-1)}{y^r(b-1)(b-2)\ldots(b-r)} \mathbf{H}_1(a+r,b-r;c;x,y)\\
&= \frac{(-1)^r(a)_r}{y^r(1-b)_r} \mathbf{H}_1(a+r,b-r;c;x,y), y \neq 0, b \neq 1,2,3,\ldots.
\end{aligned}$$

By using the operator $\hat{\Im}_x\hat{\Im}_y$, we get

$$\begin{aligned}\hat{\Im}_x\hat{\Im}_y\ \mathbf{H}_1(a,b;c;x,y)=&\sum_{m=1,n=1}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{(c)_m(m+1)m!(n+1)n!}x^my^n\\=&\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{(c)_m(m+1)!(n+1)!}x^my^n\\=&\frac{(c-1)}{xy(b-1)(b-2)}\mathbf{H}_1(a,b-2;c-1;x,y),x,y\neq 0,b\neq 1,2.\end{aligned}$$

Iteration of above relation for r, we get

$$\begin{aligned}\left(\hat{\Im}_x\hat{\Im}_y\right)^r\mathbf{H}_1(a,b;c;x,y)=&\frac{(1-c)_r}{x^ry^r(b-1)(b-2)(b-3)(b-4)\dots(b-2r)}\mathbf{H}_1(a,b-2r;c;x,y)\\=&\frac{(1-c)_r}{x^ry^r(1-b)_{2r}}\mathbf{H}_1(a,b-2r;c-r;x,y),x,y\neq 0,b\neq 1,2,\dots.\end{aligned}$$

Theorem 9. *For $|t|<1$, the infinite summation formulas for Horn function $\mathbf{H}_1$ hold true:*

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_1(a+k,b;c;x,y)t^k=(1-t)^{-a}\mathbf{H}_1\left(a,b;c;\frac{x}{1-t},y(1-t)\right)\tag{37}$$

and

$$\sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_1(a,b+k;c;x,y)t^k=(1-t)^{-b}\mathbf{H}_1\left(a,b;c;\frac{x}{1-t},\frac{y}{1-t}\right).\tag{38}$$

Proof. Using the fact that

$$(1-t)^{-a}=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k,|t|<1,$$

in (2) and (13), we obtain

$$\begin{aligned}&(1-t)^{-a}\mathbf{H}_1\left(a,b;c;\frac{x}{1-t},y(1-t)\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}}{(c)_mm!n!}x^my^n(1-t)^{-a+n-m}\\&=\sum_{k,m,n=0}^{\infty}\frac{(a)_{m-n+k}(b)_{m+n}}{(c)_mk!m!n!}x^my^nt^k=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k\sum_{m,n=0}^{\infty}\frac{(a+k)_{m-n}(b)_{m+n}}{(c)_mm!n!}x^my^n\\&=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^k\mathbf{H}_1\left(a+k,b;c;x,y\right).\end{aligned}$$

The same technique can prove (38).

3. Certain new formulas for $\mathbf{H}_2$

Here we establish classes of recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_2$ with all the parameters by the same technique as in the above section 2.

Theorem 10. *For $n \in \mathbb{N}$ and $d \neq 0, -1, -2, \ldots$, the Horn function $\mathbf{H}_2$ satisfy recursion formulas*

$$\begin{aligned}&\mathbf{H}_2(a+n,b,c;d;x,y)=\mathbf{H}_2(a,b,c;d;x,y)+\frac{bx}{d}\sum_{k=1}^{n}\mathbf{H}_2(a+k,b+1,c;d+1;x,y)\\&-cy\sum_{k=1}^{n}\frac{\mathbf{H}_2(a+k-2,b,c+1;d;x,y)}{(a+k-1)(a+k-2)}, a\neq 1-k, a\neq 2-k, k\in\mathbb{N},\end{aligned}\tag{39}$$

and

$$\begin{aligned}&\mathbf{H}_2(a-n,b,c;d;x,y)==\mathbf{H}_2(a,b,c;d;x,y)-\frac{bx}{d}\sum_{k=1}^{n}\mathbf{H}_2(a-k+1,b+1,c;d+1;x,y)\\&+cy\sum_{k=1}^{n}\frac{\mathbf{H}_2(a-k-1,b,c+1;d;x,y)}{(a-k)(a-k-1)}, a\neq k, a\neq 1+k, k\in\mathbb{N}.\end{aligned}\tag{40}$$

Theorem 11. *For $d \neq 0, -1, -2, \ldots$, the recurrence relations hold true for the Horn function $\mathbf{H}_2$:*

$$\begin{aligned}&\mathbf{H}_2(a,b+n,c;d;x,y)=\mathbf{H}_2(a,b,c;d;x,y)+\frac{ax}{d}\sum_{k=1}^{n}\mathbf{H}_2(a+1,b+k,c;d+1;x,y),\\&\mathbf{H}_2(a,b-n,c;d;x,y)=\mathbf{H}_2(a,b,c;d;x,y)-\frac{ax}{d}\sum_{k=1}^{n}\mathbf{H}_2(a+1,b-k+1,c;d+1;x,y).\end{aligned}\tag{41}$$

Theorem 12. *For $a \neq 1$, the Horn function $\mathbf{H}_2$ satisfies the identity:*

$$\begin{aligned}&\mathbf{H}_2(a,b,c+n;d;x,y)=\mathbf{H}_2(a,b,c;d;x,y)+\frac{y}{a-1}\sum_{k=1}^{n}\mathbf{H}_2(a-1,b,c+k;d;x,y),\\&\mathbf{H}_2(a,b,c-n;d;x,y)=\mathbf{H}_2(a,b,c;d;x,y)-\frac{y}{a-1}\sum_{k=1}^{n}\mathbf{H}_2(a-1,b,c-k+1;d;x,y).\end{aligned}\tag{42}$$

Theorem 13. *The Horn function* $\mathbf{H}_2$ *satisfy the following identity:*

$$\begin{aligned}\mathbf{H}_2(a,b,c;d-n;x,y) =& \mathbf{H}_2(a,b,c;d;x,y) + abx\sum_{k=1}^{n}\frac{\mathbf{H}_2(a+1,b+1,c;d-k+2;x,y)}{(d-k)(d-k+1)},\\ & (d\neq k-1, k\in\mathbb{N}),\\ \mathbf{H}_2(a,b,c;d+n;x,y) =& \mathbf{H}_2(a,b,c;d;x,y) - abx\sum_{k=1}^{n}\frac{\mathbf{H}_2(a+1,b+1,c;d+k+1;x,y)}{(d+k-1)(d+k)},\\ & (d\neq 1-k, -k\in\mathbb{N}).\end{aligned} \tag{43}$$

Theorem 14. *The differential recursion formulas hold true for the parameters of the Horn function* $\mathbf{H}_2$*:*

$$\begin{aligned}\mathbf{H}_2(a+1,b,c;d;x,y) =& \left(1+\frac{\theta_x-\theta_y}{a}\right)\mathbf{H}_2(a,b,c;d;x,y), a\neq 0,\\ \mathbf{H}_2(a,b+1,c;d;x,y) =& \left(1+\frac{\theta_x}{b}\right)\mathbf{H}_2(a,b,c;d;x,y), b\neq 0,\\ \mathbf{H}_2(a,b,c+1;d;x,y) =& \left(1+\frac{\theta_y}{c}\right)\mathbf{H}_2(a,b,c;d;x,y), c\neq 0,\\ \mathbf{H}_2(a,b,c;d-1;x,y) =& \left(1+\frac{\theta_x}{d-1}\right)\mathbf{H}_2(a,b,c;d;x,y), d\neq 1.\end{aligned} \tag{44}$$

Theorem 15. *The derivative formulas hold true for Horn function* $\mathbf{H}_2$

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_2(a,b,c;d;x,y) =& \frac{(a)_r(b)_r}{(d)_r}\mathbf{H}_2(a+r,b+r,c;d+r;x,y), d\neq 0,-1,-2,\dots,\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_2(a,b,c;d;x,y) =& \frac{(-1)^r(c)_r}{(1-a)_r}\mathbf{H}_2(a-r,b,c+r;d;x,y), a\neq 1,2,3,\dots.\end{aligned} \tag{45}$$

Theorem 16. *For* $x,y\neq 0$*, the integration recursion formulas of the Horn function* $\mathbf{H}_2$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_2(a,b,c;d;x,y) =& \frac{(d-1)(d-2)}{x^2(a-1)(a-2)(b-1)(b-2)}\mathbf{H}_2(a-2,b-2,c;d-2;x,y)\\ &+\frac{2(d-1)}{xy(b-1)(c-1)}\mathbf{H}_1(a,b-1,c-1;d-1;x,y)\\ &+\frac{a(a+1)}{y^2(c-1)(c-2)}\mathbf{H}_2(a+2,b,c-2;d;x,y), a,b,c\neq 1,2,\end{aligned} \tag{46}$$

$$\begin{aligned}\hat{\mathfrak{I}}^r\mathbf{H}_2(a,b,c;d;x,y) =& \frac{(1-d)_r}{x^ry^r(1-b)_r(1-c)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_2(a,b-r,c-r;d-r;x,y),\\ & (b,c\neq 1,2,3,\dots)\end{aligned} \tag{47}$$

and

$$\begin{aligned}\hat{\mathfrak{I}_x}^{\,r}\ \mathbf{H}_2(a,b,c;d;x,y) =&\frac{(-1)^r(1-d)_r}{x^r(1-a)_r(1-b)_r}\mathbf{H}_2(a-r,b-r,c;d-r;x,y), a,b\neq 1,2,3,\dots,\\ \hat{\mathfrak{I}_y}^{\,r}\ \mathbf{H}_2(a,b,c;d;x,y) =&\frac{(-1)^r(a)_r}{y^r(1-c)_r}\mathbf{H}_2(a+r,b,c-r;d;x,y), c\neq 1,2,3,\dots,\\ \left(\hat{\mathfrak{I}_x}\hat{\mathfrak{I}_y}\right)^r\ \mathbf{H}_2(a,b,c;d;x,y) =&\frac{(-1)^r(1-d)_r}{x^ry^r(1-b)_r(1-c)_r}\mathbf{H}_2(a,b-r,c-r;d-r;x,y), b,c\neq 1,2,3,\dots.\end{aligned} \tag{48}$$

Theorem 17. *For $|t|<1$, the infinite summation formulas for Horn function $\mathbf{H}_2$ hold true:*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_2(a+k,b,c;d;x,y)t^k =&(1-t)^{-a}\mathbf{H}_2\left(a,b,c;d;\frac{x}{1-t},y(1-t)\right),\\ \sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_2(a,b+k,c;d;x,y)t^k =&(1-t)^{-b}\mathbf{H}_2\left(a,b,c;d;\frac{x}{1-t},y\right),\\ \sum_{k=0}^{\infty}\frac{(c)_k}{k!}\mathbf{H}_2(a,b,c+k;d;x,y)t^k =&(1-t)^{-c}\mathbf{H}_2\left(a,b,c;d;x,\frac{y}{1-t}\right).\end{aligned} \tag{49}$$

4. Certain new formulas for $\mathbf{H}_3$

In this section, we establish several recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_3$ with all the parameters. In the same way as the technique followed in section 2, we obtain the resulting theorems.

Theorem 18. *For $n\in\mathbb{N}$ and $c\neq 0,-1,-2,\dots$, the recurrence relation hold true for the Horn function $\mathbf{H}_3$:*

$$\begin{aligned}&\mathbf{H}_3(a+n,b;c;x,y) = \mathbf{H}_3(a,b;c;x,y) + \frac{bx}{c}\sum_{k=1}^{n}\mathbf{H}_3(a+k,b+1;c+1;x,y)\\ &-y\sum_{k=1}^{n}\frac{\mathbf{H}_2(a+k-2,b;c;x,y)}{(a+k-1)(a+k-2)}, a\neq 1-k, a\neq 2-k, k\in\mathbb{N},\\ &\mathbf{H}_3(a-n,b;c;x,y) = \mathbf{H}_3(a,b;c;x,y) - \frac{bx}{c}\sum_{k=1}^{n}\mathbf{H}_3(a-k+1,b+1;c+1;x,y)\\ &+y\sum_{k=1}^{n}\frac{\mathbf{H}_2(a-k-1,b;c;x,y)}{(a-k)(a-k-1)}, a\neq k, a\neq 1+k, k\in\mathbb{N}.\end{aligned} \tag{50}$$

and

$$
\begin{aligned}
\mathbf{H}_3(a,b+n;c;x,y) &= \mathbf{H}_3(a,b;c;x,y) + \frac{ax}{c}\sum_{k=1}^{n}\mathbf{H}_3(a+1,b+k;c+1;x,y),\\
\mathbf{H}_3(a,b-n;c;x,y) &= \mathbf{H}_3(a,b;c;x,y) - \frac{ax}{c}\sum_{k=1}^{n}\mathbf{H}_3(a+1,b-k+1;c+1;x,y).
\end{aligned} \tag{51}
$$

Theorem 19. *The Horn function* $\mathbf{H}_3$ *satisfies the following identity:*

$$
\begin{aligned}
\mathbf{H}_3(a,b;c-n;x,y) &= \mathbf{H}_3(a,b;c;x,y) + abx\sum_{k=1}^{n}\frac{\mathbf{H}_3(a+1,b+1;c-k+2;x,y)}{(c-k)(c-k+1)}, c\neq k-1, k\in\mathbb{N},\\
\mathbf{H}_3(a,b;c+n;x,y) &= \mathbf{H}_3(a,b;c;x,y) - abx\sum_{k=1}^{n}\frac{\mathbf{H}_3(a+1,b+1;c+k+1;x,y)}{(c+k-1)(c+k)}, c\neq 1-k, -k\in\mathbb{N}.
\end{aligned} \tag{52}
$$

Theorem 20. *The differential recursion formulas hold true for the parameters of the Horn hypergeometric function* $\mathbf{H}_3$*:*

$$
\begin{aligned}
\mathbf{H}_3(a+1,b;c;x,y) &= \left(1+\frac{\theta_x-\theta_y}{a}\right)\mathbf{H}_3(a,b;c;x,y), a\neq 0,\\
\mathbf{H}_3(a,b+1;c;x,y) &= \left(1+\frac{\theta_x}{b}\right)\mathbf{H}_3(a,b;c;x,y), b\neq 0,\\
\mathbf{H}_3(a,b;c-1;x,y) &= \left(1+\frac{\theta_x}{c-1}\right)\mathbf{H}_2(a,b;c;x,y), c\neq 1.
\end{aligned} \tag{53}
$$

Theorem 21. *The derivative formulas hold true for Horn hypergeometric function* $\mathbf{H}_3$

$$
\begin{aligned}
\frac{\partial^r}{\partial x^r}\mathbf{H}_3(a,b;c;x,y) &= \frac{(a)_r(b)_r}{(c)_r}\mathbf{H}_3(a+r,b+r;c+r;x,y), c\neq 0,-1,-2,\ldots,\\
\frac{\partial^r}{\partial y^r}\mathbf{H}_3(a,b;c;x,y) &= \frac{(-1)^r}{(1-a)_r}\mathbf{H}_3(a-r,b;c;x,y), a\neq 1,2,3,\ldots.
\end{aligned} \tag{54}
$$

Theorem 22. *The integration recursion formulas of the Horn function* $\mathbf{H}_3$ *hold* *true:*

$$\hat{\mathfrak{I}}^2\mathbf{H}_3(a,b;c;x,y) = \frac{(c-1)(c-2)}{x^2(a-1)(a-2)(b-1)(b-2)}\mathbf{H}_3(a-2,b-2;c-2;x,y)$$
$$+\frac{2(c-1)}{xy(b-1)}\mathbf{H}_3(a,b-1;c-1;x,y)+\frac{a(a+1)}{y^2}\mathbf{H}_3(a+2,b;c;x,y), x,y\neq 0, a,b\neq 1,2,$$
$$\hat{\mathfrak{I}}^r\mathbf{H}_3(a,b;c;x,y) = \frac{(1-c)_r}{x^r y^r(1-b)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_1(a,b-r;c-r;x,y),$$
$$(x,y\neq 0, b\neq 1,2,3,\ldots), \tag{55}$$
$$\hat{\mathfrak{I}_x}^r\,\mathbf{H}_3(a,b;c;x,y) = \frac{(-1)^r(1-c)_r}{x^r(1-a)_r(1-b)_r}\mathbf{H}_3(a-r,b-r;c-r;x,y), x\neq 0, a,b\neq 1,2,3,\ldots,$$
$$\hat{\mathfrak{I}_y}^r\,\mathbf{H}_3(a,b;c;x,y) = \frac{(a)_r}{y^r}\mathbf{H}_3(a+r,b;c;x,y), y\neq 0,$$
$$\left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\mathbf{H}_3(a,b;c;x,y) = \frac{(1-c)_r}{x^r y^r(1-b)_r}\mathbf{H}_3(a,b-r;c-r;x,y), x,y\neq 0, b\neq 1,2,3,\ldots.$$

Theorem 23. *For $|t|<1$, the infinite summation formulas for Horn function $\mathbf{H}_3$ hold true:*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_3(a+k,b;c;x,y)t^k =&(1-t)^{-a}\mathbf{H}_3\left(a,b;c;\frac{x}{1-t},y(1-t)\right),\\ \sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_3(a,b+k;c;x,y)t^k =&(1-t)^{-b}\mathbf{H}_3\left(a,b;c;x(1-t),y\right).\end{aligned} \tag{56}$$

5. Certain new formulas for $\mathbf{H}_4$

In this section, we discuss several recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_4$ with all the parameters. The methods of proof given in section 2 can establish the following theorems.

Theorem 24. *For $c\neq 0,-1,-2,\ldots,$ and $n\in\mathbb{N}$. Recursion relations for the Horn function $\mathbf{H}_4$ are as follows*

$$\begin{aligned}&\mathbf{H}_4(a+n,b;c;x,y) = \mathbf{H}_4(a,b;c;x,y)+\frac{x}{c}\sum_{k=1}^{n}\mathbf{H}_4(a+k,b;c+1;x,y)\\ &-by\sum_{k=1}^{n}\frac{\mathbf{H}_4(a+k-2,b+1;c;x,y)}{(a+k-1)(a+k-2)}, a\neq 1-k, a\neq 2-k, k\in\mathbb{N},\end{aligned} \tag{57}$$

and

$$\begin{aligned}\mathbf{H}_4(a-n,b;c;x,y) &= \mathbf{H}_4(a,b;c;x,y) - \frac{x}{c}\sum_{k=1}^{n}\mathbf{H}_4(a-k+1,b;c+1;x,y)\\ &+ by\sum_{k=1}^{n}\frac{\mathbf{H}_4(a-k-1,b+1;c;x,y)}{(a-k)(a-k-1)}, a\neq k, a\neq 1+k, k\in\mathbb{N}.\end{aligned} \tag{58}$$

Theorem 25. *For $a\neq 1$, the recurrence relation hold true for the function $\mathbf{H}_4$:*

$$\begin{aligned}\mathbf{H}_4(a,b+n;c;x,y) &= \mathbf{H}_4(a,b;c;x,y) + \frac{by}{a-1}\sum_{k=1}^{n}\mathbf{H}_4(a-1,b+k;c;x,y),\\ \mathbf{H}_4(a,b-n;c;x,y) &= \mathbf{H}_4(a,b;c;x,y) - \frac{by}{a-1}\sum_{k=1}^{n}\mathbf{H}_4(a-1,b-k+1;c;x,y).\end{aligned} \tag{59}$$

Theorem 26. *The Horn hypergeometric function $\mathbf{H}_4$ satisfies the following identity:*

$$\begin{aligned}\mathbf{H}_4(a,b;c-n;x,y) &= \mathbf{H}_4(a,b;c;x,y) + ax\sum_{k=1}^{n}\frac{\mathbf{H}_4(a+1,b;c-k+2;x,y)}{(c-k)(c-k+1)}, c\neq k-1, k\in\mathbb{N},\\ \mathbf{H}_4(a,b;c+n;x,y) &= \mathbf{H}_4(a,b;c;x,y) - ax\sum_{k=1}^{n}\frac{\mathbf{H}_4(a+1,b;c+k+1;x,y)}{(c+k-1)(c+k)}, c\neq 1-k, -k\in\mathbb{N}.\end{aligned} \tag{60}$$

Theorem 27. *The differential recursion formulas hold true for the parameters of the Horn function $\mathbf{H}_4$:*

$$\begin{aligned}\mathbf{H}_4(a+1,b;c;x,y) &= \left(1+\frac{\theta_x-\theta_y}{a}\right)\mathbf{H}_4(a,b;c;x,y), a\neq 0,\\ \mathbf{H}_4(a,b+1;c;x,y) &= \left(1+\frac{\theta_y}{b}\right)\mathbf{H}_4(a,b;c;x,y), b\neq 0,\\ \mathbf{H}_4(a,b;c-1;x,y) &= \left(1+\frac{\theta_x}{c-1}\right)\mathbf{H}_4(a,b;c;x,y), c\neq 1.\end{aligned} \tag{61}$$

Theorem 28. *The derivative formulas hold true for Horn function $\mathbf{H}_4$*

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_4(a,b;c;x,y) &= \frac{(a)_r}{(c)_r}\mathbf{H}_4(a+r,b;c+r;x,y), c\neq 0,-1,-2,\ldots,\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_4(a,b;c;x,y) &= \frac{(-1)^r(b)_r}{(1-a)_r}\mathbf{H}_4(a-r,b+r;c;x,y), a\neq 1,2,3,\ldots.\end{aligned} \tag{62}$$

Theorem 29. *The integration recursion formulas of the Horn hypergeometric function* $\mathbf{H}_4$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_4(a,b;c;x,y) =&\frac{(c-1)(c-2)}{x^2(a-1)(a-2)}\mathbf{H}_4(a-2,b;c-2;x,y)\\&+\frac{2(c-1)}{xy(b-1)}\mathbf{H}_4(a,b-1;c-1;x,y)\\&+\frac{a(a+1)}{y^2(b-1)(b-2)}\mathbf{H}_4(a+2,b-2;c;x,y), x,y\neq 0, a,b\neq 1,2, \quad (63)\\ \hat{\mathfrak{I}}^r\mathbf{H}_4(a,b;c;x,y) =&\frac{(1-c)_r}{x^r y^r(1-b)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_4(a-r,b-r;c-r;x,y),\\&(x,y\neq 0, b\neq 1,2,3,\ldots),\end{aligned}$$

$$\begin{aligned}\hat{\mathfrak{I}}_x^{\ r}\ \mathbf{H}_4(a,b;c;x,y) &= \frac{(1-c)_r}{x^r(1-a)_r}\mathbf{H}_4(a-r,b;c-r;x,y), x\neq 0, a\neq 1,2,3,\ldots,\\ \hat{\mathfrak{I}}_y^{\ r}\ \mathbf{H}_4(a,b;c;x,y) &= \frac{(-1)^r(a)_r}{y^r(1-b)_r}\mathbf{H}_4(a+r,b-r;c;x,y), y\neq 0, b\neq 1,2,3,\ldots, \quad (64)\\ \left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\ \mathbf{H}_4(a,b;c;x,y) &= \frac{(1-c)_r}{x^r y^r(1-b)_r}\mathbf{H}_4(a,b-r;c-r;x,y), x,y\neq 0, b\neq 1,2,3,\ldots.\end{aligned}$$

Theorem 30. *For* $|t|<1$*, the infinite summation formulas for Horn function* $\mathbf{H}_4$ *hold true:*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_4(a+k,b;c;x,y)t^k =&(1-t)^{-a}\mathbf{H}_4\left(a,b;c;\frac{x}{1-t},y(1-t)\right),\\ \sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_4(a,b+k;c;x,y)t^k =&(1-t)^{-b}\mathbf{H}_4\left(a,b;c;x,\frac{y}{1-t}\right).\end{aligned} \quad (65)$$

6. Certain new formulas for $\mathbf{H}_5$

Here we obtain the recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_5$ with all the parameters by the technique used in section 2.

Theorem 31. *For* $n\in\mathbb{N}$ *and* $b\neq 0,-1,-2,\ldots.$ *Horn function* $\mathbf{H}_5$ *satisfy the* *recursion relations*

$$\begin{aligned}\mathbf{H}_5(a+n;b;x,y) = \mathbf{H}_5(a,b;c;x,y) + \frac{x}{b}\sum_{k=1}^{n}\mathbf{H}_5(a+k;b+1;x,y)\\ - y\sum_{k=1}^{n}\frac{\mathbf{H}_5(a+k-2;b;x,y)}{(a+k-1)(a+k-2)}, a\neq 1-k, a\neq 2-k, k\in\mathbb{N},\end{aligned} \tag{66}$$

and

$$\begin{aligned}\mathbf{H}_5(a-n;b;x,y) = \mathbf{H}_5(a;b;x,y) - \frac{x}{b}\sum_{k=1}^{n}\mathbf{H}_5(a-k+1;b+1;x,y)\\ + y\sum_{k=1}^{n}\frac{\mathbf{H}_5(a-k-1;b;x,y)}{(a-k)(a-k-1)}, a\neq k, a\neq 1+k, k\in\mathbb{N}.\end{aligned} \tag{67}$$

Theorem 32. *The Horn function* $\mathbf{H}_5$ *satisfies the following identity:*

$$\mathbf{H}_5(a;b-n;x,y) = \mathbf{H}_5(a;b;x,y) + ax\sum_{k=1}^{n}\frac{\mathbf{H}_5(a+1;b-k+2;x,y)}{(b-k)(b-k+1)}, b\neq k-1, k\in\mathbb{N}, \tag{68}$$

and

$$\mathbf{H}_5(a;b+n;x,y) = \mathbf{H}_5(a;b;x,y) - ax\sum_{k=1}^{n}\frac{\mathbf{H}_5(a+1;b+k+1;x,y)}{(b+k-1)(b+k)}, b\neq 1-k, -k\in\mathbb{N}. \tag{69}$$

Theorem 33. *The differential recursion formulas hold true for the parameters of the Horn function* $\mathbf{H}_5$*:*

$$\begin{aligned}\mathbf{H}_5(a+1;b;x,y) =&\left(1+\frac{\theta_x-\theta_y}{a}\right)\mathbf{H}_5(a;b;x,y), a\neq 0,\\ \mathbf{H}_5(a;b-1;x,y) =&\left(1+\frac{\theta_x}{b-1}\right)\mathbf{H}_5(a;b;x,y), b\neq 1.\end{aligned} \tag{70}$$

Theorem 34. *The derivative formulas hold true for Horn function* $\mathbf{H}_5$

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_5(a;b;x,y) =&\frac{(a)_r}{(b)_r}\mathbf{H}_5(a+r;b+r;x,y), b\neq 0,-1,-2,\ldots,\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_5(a;b;x,y) =&\frac{(-1)^r}{(1-a)_r}\mathbf{H}_5(a-r;b;x,y), a\neq 1,2,3,\ldots.\end{aligned} \tag{71}$$

Theorem 35. *For $a \neq 1, 2$, the integration recursion formulas of the Horn function* $\mathbf{H}_5$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_5(a;b;x,y) = \frac{(b-1)(b-2)}{x^2(a-1)(a-2)}\mathbf{H}_5(a-2;b-2;x,y)\\ + \frac{2(b-1)}{xy}\mathbf{H}_5(a;b-1;x,y) + \frac{a(a+1)}{y^2}\mathbf{H}_5(a+2;b;x,y), x,y \neq 0, a \neq 1,2,\end{aligned} \tag{72}$$

$$\begin{aligned}\hat{\mathfrak{I}}^r\mathbf{H}_5(a;b;x,y) =& \frac{(-1)^r(1-b)_r}{x^r y^r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_5(a;b-r;x,y), x,y \neq 0,\\ \hat{\mathfrak{I}}_x^{\,r}\ \mathbf{H}_5(a;b;x,y) =& \frac{(1-b)_r}{x^r(1-a)_r}\mathbf{H}_5(a-r;b;x,y), x \neq 0, a \neq 1,2,3,\ldots,\\ \hat{\mathfrak{I}}_y^{\,r}\ \mathbf{H}_5(a;b;x,y) =& \frac{(a)_r}{y^r}\mathbf{H}_5(a+r;b;x,y), y \neq 0,\\ \left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\ \mathbf{H}_5(a;b;x,y) =& \frac{(-1)^r(1-b)_r}{x^r y^r}\mathbf{H}_5(a;b-r;x,y), x,y \neq 0.\end{aligned} \tag{73}$$

Theorem 36. *For $|t| < 1$, the infinite summation formula for Horn's function* $\mathbf{H}_5$ *holds true:*

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_5(a+k;b;x,y)t^k = (1-t)^{-a}\mathbf{H}_5\left(a;b;\frac{x}{1-t}, y(1-t)\right). \tag{74}$$

7. Certain new formulas for $\mathbf{H}_6$

In this section, we derive the recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_6$.

Theorem 37. *For $b \neq 0, -1, -2, \ldots$ and $n \in \mathbb{N}$. the Horn function* $\mathbf{H}_6$ *satisfy recursion formulas*

$$\begin{aligned}\mathbf{H}_6(a+n;b;x,y) =& \mathbf{H}_6(a;b;x,y) + \frac{2x}{b}\sum_{k=1}^{n}(a+k)\mathbf{H}_6(a+k+1;b+1;x,y)\\ &+ \frac{y}{b}\sum_{k=1}^{n}\mathbf{H}_6(a+k;b+1;x,y),\end{aligned} \tag{75}$$

and

$$\begin{aligned}\mathbf{H}_6(a-n;b;x,y) =&\mathbf{H}_6(a;b;x,y) - \frac{2x}{b}\sum_{k=1}^{n}(a-k+1)\mathbf{H}_6(a-k+2;b+1;x,y)\\ &- \frac{y}{b}\sum_{k=1}^{n}\mathbf{H}_6(a-k+1;b+1;x,y).\end{aligned} \tag{76}$$

Theorem 38. *The Horn function* $\mathbf{H}_6$ *satisfies the identity:*

$$\begin{aligned}&\mathbf{H}_6(a;b-n;x,y) = \mathbf{H}_6(a;b;x,y) + a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_6(a+2;b-k+2;x,y)}{(b-k)(b-k+1)}\\ &+ ay\sum_{k=1}^{n}\frac{\mathbf{H}_6(a+1;b-k+2;x,y)}{(b-k)(b-k+1)}, b \neq k-1, b \neq k, k \in \mathbb{N},\end{aligned} \tag{77}$$

and

$$\begin{aligned}&\mathbf{H}_6(a;b+n;x,y) = \mathbf{H}_6(a;b;x,y) - a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_6(a+2;b+k+1;x,y)}{(b+k-1)(b+k)}\\ &- ay\sum_{k=1}^{n}\frac{\mathbf{H}_6(a+1;b+k+1;x,y)}{(b+k-1)(b+k)}, b \neq 1-k, b \neq -k, k \in \mathbb{N}.\end{aligned} \tag{78}$$

Theorem 39. *The differential recursion formulas hold true for the numerator parameters of the Horn function* $\mathbf{H}_6$*:*

$$\begin{aligned}\mathbf{H}_6(a+1;b;x,y) =&\left(1+\frac{2\theta_x+\theta_y}{a}\right)\mathbf{H}_6(a;b;x,y), a \neq 0,\\ \mathbf{H}_6(a;b-1;x,y) =&\left(1+\frac{\theta_x+\theta_y}{b-1}\right)\mathbf{H}_6(a;b;x,y), b \neq 1.\end{aligned} \tag{79}$$

Theorem 40. *For* $b \neq 0, -1, -2, \ldots,$ *the derivative formulas hold true for Horn function* $\mathbf{H}_6$

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_6(a;b;x,y) =& \frac{(a)_{2r}}{(b)_r}\mathbf{H}_6(a+2r;b+r;x,y),\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_6(a;b;x,y) =& \frac{(a)_r}{(b)_r}\mathbf{H}_6(a+r;b+r;x,y).\end{aligned} \tag{80}$$

Theorem 41. *The integration recursion formulas of the Horn function* $\mathbf{H}_6$ *hold* *true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_6(a;b;x,y) =&\frac{(b-1)(b-2)}{x^2(a-1)(a-2)(a-3)(a-4)}\mathbf{H}_6(a-4;b-2;x,y)\\&+\frac{2(b-1)(b-2)}{xy(a-1)(a-2)(a-3)}\mathbf{H}_6(a-3;b-2;x,y)\\&+\frac{(b-1)(b-2)}{y^2(a-1)(a-2)}\mathbf{H}_6(a-2;b-2;x,y), x,y\neq 0, a\neq 1,2,3,4, \quad (81)\\\hat{\mathfrak{I}}^r\mathbf{H}_6(a;b;x,y) =&\frac{(-1)^r(1-b)_{2r}}{x^r y^r(1-a)_{3r}}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_6(a-3r;b-2r;x,y),\\&(x,y\neq 0, a\neq 1,2,3,\ldots),\end{aligned}$$

$$\begin{aligned}\hat{\mathfrak{I}_x}^r\ \mathbf{H}_6(a;b;x,y) &= \frac{(-1)^r(1-b)_r}{x^r(1-a)_{2r}}\mathbf{H}_6(a-2r;b-r;x,y), x\neq 0, a\neq 1,2,3,\ldots,\\\hat{\mathfrak{I}_y}^r\ \mathbf{H}_6(a;b;x,y) &= \frac{(1-b)_r}{y^r(1-a)_r}\mathbf{H}_6(a-r;b-r;x,y), y\neq 0, a\neq 1,2,3,\ldots, \quad (82)\\\left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\ \mathbf{H}_6(a;b;x,y) &= \frac{(-1)^r(1-b)_{2r}}{x^r y^r(1-a)_{3r}}\mathbf{H}_6(a-3r;b-2r;x,y), x,y\neq 0, a\neq 1,2,3,\ldots.\end{aligned}$$

Theorem 42. *For $|t|<1$, the infinite summation formula for Horn function $\mathbf{H}_6$ holds true:*

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_6(a+k;b;x,y)t^k = (1-t)^{-a}\mathbf{H}_6\left(a;b;\frac{x}{(1-t)^2},y(1-t)\right). \quad (83)$$

8. Certain new formulas for $\mathbf{H}_7$

In this section, we list the recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_7$.

Theorem 43. *For $n\in\mathbb{N}$, $b,c\neq 0,-1,-2,\ldots$. Recursion relations for the Horn function $\mathbf{H}_7$ are as follows*

$$\begin{aligned}\mathbf{H}_7(a+n;b,c;x,y) =&\mathbf{H}_7(a;b,c;x,y)+\frac{2x}{b}\sum_{k=1}^{n}(a+k)\mathbf{H}_7(a+k+1;b+1,c;x,y)\\&+\frac{y}{c}\sum_{k=1}^{n}\mathbf{H}_7(a+k;b,c+1;x,y),\end{aligned} \quad (84)$$

and

$$\begin{aligned}\mathbf{H}_7(a-n;b,c;x,y)=&\mathbf{H}_7(a;b,c;x,y)-\frac{2x}{b}\sum_{k=1}^{n}(a-k+1)\mathbf{H}_7(a-k+2;b+1,c;x,y)\\&-\frac{y}{c}\sum_{k=1}^{n}\mathbf{H}_7(a-k+1;b,c+1;x,y).\end{aligned} \tag{85}$$

Theorem 44. *The Horn hypergeometric function* $\mathbf{H}_7$ *satisfies the following identity:*

$$\begin{aligned}\mathbf{H}_7(a;b-n,c;x,y)=&\mathbf{H}_7(a;b,c;x,y)+a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_7(a+2;b-k+2,c;x,y)}{(b-k)(b-k+1)};\\&(b\neq k-1,b\neq k,k\in\mathbb{N}),\\\mathbf{H}_7(a;b+n,c;x,y)=&\mathbf{H}_7(a;b,c;x,y)-a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_7(a+2;b+k+1,c;x,y)}{(b+k-1)(b+k)};\\&(b\neq 1-k,b\neq -k,k\in\mathbb{N}).\end{aligned} \tag{86}$$

Theorem 45. *The Horn function* $\mathbf{H}_7$ *satisfies the following identity:*

$$\begin{aligned}\mathbf{H}_7(a;b,c-n;x,y)=&\mathbf{H}_7(a;b,c;x,y)+ay\sum_{k=1}^{n}\frac{\mathbf{H}_7(a+1;b,c-k+2;x,y)}{(c-k)(c-k+1)};\\&(c\neq k-1,c\neq k,k\in\mathbb{N}),\\\mathbf{H}_7(a;b,c+n;x,y)=&\mathbf{H}_7(a;b,c;x,y)-ay\sum_{k=1}^{n}\frac{\mathbf{H}_7(a+1;b,c+k+1;x,y)}{(c+k-1)(c+k)};\\&(c\neq 1-k,c\neq -k,k\in\mathbb{N}).\end{aligned} \tag{87}$$

Theorem 46. *The differential recursion formulas hold true for the parameters of the Horn function* $\mathbf{H}_7$*:*

$$\begin{aligned}\mathbf{H}_7(a+1;b,c;x,y)=&\left(1+\frac{2\theta_x+\theta_y}{a}\right)\mathbf{H}_7(a;b,c;x,y),a\neq 0,\\\mathbf{H}_7(a;b-1,c;x,y)=&\left(1+\frac{\theta_y}{b-1}\right)\mathbf{H}_7(a;b,c;x,y),b\neq 1,\\\mathbf{H}_7(a;b,c-1;x,y)=&\left(1+\frac{\theta_y}{c-1}\right)\mathbf{H}_7(a;b,c;x,y),c\neq 1.\end{aligned} \tag{88}$$

Theorem 47. *The derivative formulas hold true for Horn hypergeometric function*

$\mathbf{H}_7$

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_7(a;b,c;x,y) =&\frac{(a)_{2r}}{(b)_r}\mathbf{H}_7(a+2r;b+r,c;x,y), b\neq 0,-1,-2,\ldots,\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_7(a;b,c;x,y) =&\frac{(a)_{r}}{(c)_r}\mathbf{H}_7(a+r;b,c+r;x,y), c\neq 0,-1,-2,\ldots.\end{aligned} \tag{89}$$

Theorem 48. *The integration recursion formulas of the Horn function* $\mathbf{H}_7$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_7(a;b,c;x,y) =&\frac{(b-1)(b-2)}{x^2(a-1)(a-2)(a-3)(a-4)}\mathbf{H}_7(a-4;b-2,c;x,y)\\ &+\frac{2(b-1)(c-1)}{xy(a-1)(a-2)(a-3)}\mathbf{H}_7(a-3;b-1,c-1;x,y)\\ &+\frac{(c-1)(c-2)}{y^2(a-1)(a-2)}\mathbf{H}_7(a-2;b,c-2;x,y), x,y\neq 0, a\neq 1,2,3,4,\\ \hat{\mathfrak{I}}^r\mathbf{H}_7(a;b,c;x,y) =&\frac{(-1)^r(1-b)_r(1-c)_r}{x^r y^r(1-a)_{3r}}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_7(a-3r;b-r,c-r;x,y),\\ &(x,y\neq 0, a\neq 1,2,3,\ldots),\end{aligned} \tag{90}$$

$$\begin{aligned}\hat{\mathfrak{I}_x}^r\,\mathbf{H}_7(a;b,c;x,y) =&\frac{(-1)^r(1-b)_r}{x^r(1-a)_{2r}}\mathbf{H}_7(a-2r;b-r,c;x,y), x\neq 0, a\neq 1,2,3,\ldots,\\ \hat{\mathfrak{I}_y}^r\,\mathbf{H}_7(a;b,c;x,y) =&\frac{(1-c)_r}{y^r(1-a)_r}\mathbf{H}_7(a-r;b,c-r;x,y), y\neq 0, a\neq 1,2,3,\ldots,\\ \left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\mathbf{H}_7(a;b,c;x,y) =&\frac{(-1)^r(1-b)_r(1-c)_r}{x^r y^r(1-a)_{3r}}\mathbf{H}_7(a-3r;b-r,c-r;x,y),\\ &(x,y\neq 0, a\neq 1,2,3,\ldots).\end{aligned} \tag{91}$$

Theorem 49. *The infinite summation formula for Horn's hypergeometric function* $\mathbf{H}_7$ *holds true:*

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_7(a+k;b,c;x,y)t^k = (1-t)^{-a}\mathbf{H}_7\left(a;b,c;\frac{x}{(1-t)^2},\frac{y}{1-t}\right), |t|<1, \tag{92}$$

9. Certain new formulas for $\mathbf{H}_8$

In this section, we give some recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_8$.

Theorem 50. *For $b \neq 1$ and $n \in \mathbb{N}$. Horn function $\mathbf{H}_8$ satisfy the Recursion formulas*

$$\begin{aligned}&\mathbf{H}_8(a+n,b;x,y) = \mathbf{H}_8(a;b;x,y) + \frac{2x}{b-1}\sum_{k=1}^{n}(a+k)\mathbf{H}_8(a+k+1,b-1;x,y)\\&- by\sum_{k=1}^{n}\frac{\mathbf{H}_8(a+k-2,b+1;x,y)}{(a+k-1)(a+k-2)}, a \neq 1-k, a \neq 2-k, k \in \mathbb{N},\end{aligned} \tag{93}$$

and

$$\begin{aligned}&\mathbf{H}_8(a-n;b;x,y) = \mathbf{H}_8(a,b;x,y) - \frac{2x}{b-1}\sum_{k=1}^{n}(a-k+1)\mathbf{H}_8(a-k+2,b-1;x,y)\\&+ by\sum_{k=1}^{n}\frac{\mathbf{H}_8(a-k-1,b+1;x,y)}{(a-k)(a-k-1)}, a \neq k, a \neq k+1, k \in \mathbb{N}.\end{aligned} \tag{94}$$

Theorem 51. *For $a \neq 1$, the recurrence relations hold true for the Horn function $\mathbf{H}_8$:*

$$\begin{aligned}&\mathbf{H}_8(a,b+n;x,y) = \mathbf{H}_8(a,b;x,y) + \frac{y}{a-1}\sum_{k=1}^{n}\mathbf{H}_8(a-1,b+k;x,y)\\&- a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_8(a+2,b+k-2;x,y)}{(b+k-1)(b+k-2)}; a \neq 1, b \neq 1-k, b \neq 2-k, k \in \mathbb{N},\end{aligned} \tag{95}$$

and

$$\begin{aligned}&\mathbf{H}_8(a,b-n;x,y) = \mathbf{H}_8(a,b;x,y) - \frac{y}{a-1}\sum_{k=1}^{n}\mathbf{H}_8(a-1,b-k+1;x,y)\\&+ a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_8(a+2,b-k-1;x,y)}{(b-k)(b-k-1)}; a \neq 1, a \neq k, b \neq k+1, k \in \mathbb{N}.\end{aligned} \tag{96}$$

Theorem 52. *The differential recursion formulas hold true for the parameters of the Horn function $\mathbf{H}_8$:*

$$\begin{aligned}&\mathbf{H}_8(a+1,b;x,y) = \left(1 + \frac{2\theta_x - \theta_y}{a}\right)\mathbf{H}_8(a,b;x,y), a \neq 0,\\&\mathbf{H}_8(a,b+1;x,y) = \left(1 + \frac{\theta_y - \theta_x}{b}\right)\mathbf{H}_8(a,b;x,y), b \neq 0.\end{aligned} \tag{97}$$

Theorem 53. *The derivative formulas hold true for Horn hypergeometric function* $\mathbf{H}_8$

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_8(a,b;x,y) =&\frac{(-1)^r(a)_{2r}}{(1-b)_r}\mathbf{H}_8(a+2r,b-r;x,y), b\neq 1,2,3,\ldots,\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_8(a,b;x,y) =&\frac{(-1)^r(b)_r}{(1-a)_r}\mathbf{H}_8(a-r,b+r;x,y), a\neq 1,2,3,\ldots.\end{aligned} \tag{98}$$

Theorem 54. *The integration recursion formulas of the Horn hypergeometric function* $\mathbf{H}_8$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_8(a,b;x,y) =&\frac{b(b+1)}{x^2(a-1)(a-2)}\mathbf{H}_8(a-2,b+2;x,y)+\frac{2}{xy(a-1)}\mathbf{H}_8(a-1,b;x,y)\\ &+\frac{a(a+1)}{y^2(b-1)(b-2)}\mathbf{H}_8(a+2,b-2;x,y), x,y\neq 0, a,b\neq 1,2,\\ \hat{\mathfrak{I}}^r\mathbf{H}_8(a,b;x,y) =&\frac{(-1)^r}{x^r y^r(1-a)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_8(a-r,b;x,y),\\ &(x,y\neq 0, a\neq 1,2,3,\ldots),\end{aligned} \tag{99}$$

$$\begin{aligned}\hat{\mathfrak{I}}_x^{\ r}\ \mathbf{H}_8(a,b;x,y) =& \frac{(-1)^r(b)_r}{x^r(1-a)_r}\mathbf{H}_8(a-r,b+r;x,y), x\neq 0, a\neq 1,2,3,\ldots,\\ \hat{\mathfrak{I}}_y^{\ r}\ \mathbf{H}_8(a,b;x,y) =& \frac{(-1)^r(a)_r}{y^r(1-b)_r}\mathbf{H}_8(a+r,b-r;x,y), y\neq 0, b\neq 1,2,3,\ldots,\\ \left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\mathbf{H}_8(a,b;x,y) =& \frac{(-1)^r}{x^r y^r(1-a)_r}\mathbf{H}_8(a-r,b;x,y), x,y\neq 0, a\neq 1,2,3,\ldots.\end{aligned} \tag{100}$$

Theorem 55. *For* $|t|<1$*, the infinite summation formulas for Horn function* $\mathbf{H}_8$ *hold true:*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_8(a+k,b;x,y)t^k &= (1-t)^{-a}\mathbf{H}_8\left(a,b;\frac{x}{(1-t)^2},y(1-t)\right),\\ \sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_8(a,b+k;x,y)t^k &= (1-t)^{-b}\mathbf{H}_8\left(a,b;x(1-t),\frac{y}{1-t}\right).\end{aligned} \tag{101}$$

10. Certain new formulas for $\mathbf{H}_9$

In this section, we discuss the recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for Horn function $\mathbf{H}_9$ with all the parameters.

Theorem 56. *For $n \in \mathbb{N}$ and $c \neq 0, -1, -2, \ldots$. Recursion formulas for Horn function $\mathbf{H}_9$ are as follows*

$$\begin{aligned}\mathbf{H}_9(a+n, b; c; x, y) = \mathbf{H}_9(a; b; c; x, y) + \frac{2x}{c}\sum_{k=1}^{n}(a+k)\mathbf{H}_9(a+k+1, b; c+1; x, y)\\ - by\sum_{k=1}^{n}\frac{\mathbf{H}_9(a+k-2, b+1; c; x, y)}{(a+k-1)(a+k-2)}, a \neq 1-k, a \neq 2-k, k \in \mathbb{N}\end{aligned} \tag{102}$$

and

$$\begin{aligned}\mathbf{H}_9(a-n, b; c; x, y) = \mathbf{H}_9(a; b; c; x, y) - \frac{2x}{c}\sum_{k=1}^{n}(a-k+1)\mathbf{H}_9(a-k+2, b; c+1; x, y)\\ - by\sum_{k=1}^{n}\frac{\mathbf{H}_9(a-k-1, b+1; c; x, y)}{(a-k)(a-k-1)}, a \neq k, a \neq k+1, k \in \mathbb{N}.\end{aligned} \tag{103}$$

Theorem 57. *For $a \neq 1$, the recurrence relations hold true for the Horn function $\mathbf{H}_9$:*

$$\begin{aligned}\mathbf{H}_9(a, b+n; c; x, y) =& \mathbf{H}_9(a; b; c; x, y) + \frac{y}{a-1}\sum_{k=1}^{n}(b+k-1)\mathbf{H}_9(a-1, b+k; c; x, y),\\ &(b \neq 1-k, b \neq 2-k, k \in \mathbb{N}),\\ \mathbf{H}_9(a, b-n; c; x, y) =& \mathbf{H}_9(a; b; c; x, y) - \frac{y}{a-1}\sum_{k=1}^{n}(b-k)\mathbf{H}_9(a-1, b-k+1; c; x, y),\\ &(a \neq k, b \neq k+1, k \in \mathbb{N}).\end{aligned} \tag{104}$$

Theorem 58. *The recurrence relations hold true for the Horn function $\mathbf{H}_9$:*

$$\begin{aligned}\mathbf{H}_9(a, b; c-n; x, y) =& \mathbf{H}_9(a; b; c; x, y) + a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_9(a+2, b; c-k+2; x, y)}{(c-k)(c-k+1)},\\ &(c \neq k-1, c \neq k, k \in \mathbb{N}),\\ \mathbf{H}_9(a, b; c+n; x, y) =& \mathbf{H}_9(a; b; c; x, y) - a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_9(a+2, b; c+k+1; x, y)}{(c+k)(c+k-1)},\\ &(a \neq -k, c \neq 1-k, k \in \mathbb{N}).\end{aligned} \tag{105}$$

Theorem 59. *The differential recursion formulas hold true for the parameters of* *the Horn hypergeometric function* $\mathbf{H}_9$*:*

$$\begin{aligned}\mathbf{H}_9(a+1,b;c;x,y)=&\left(1+\frac{2\theta_x-\theta_y}{a}\right)\mathbf{H}_9(a,b;c;x,y), a\neq 0,\\ \mathbf{H}_9(a,b+1;c;x,y)=&\left(1+\frac{\theta_y}{b}\right)\mathbf{H}_9(a,b;c;x,y), b\neq 0,\\ \mathbf{H}_9(a,b;c-1;x,y)=&\left(1+\frac{\theta_x}{c-1}\right)\mathbf{H}_9(a,b;c;x,y), c\neq 1.\end{aligned}\tag{106}$$

Theorem 60. *The derivative formulas hold true for Horn hypergeometric function* $\mathbf{H}_9$

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_9(a,b;c;x,y)=&\frac{(a)_{2r}}{(c)_r}\mathbf{H}_9(a+2r,b;c+r;x,y), c\neq 0,-1,-2,\dots,\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_9(a,b;c;x,y)=&\frac{(-1)^r(b)_r}{(1-a)_r}\mathbf{H}_9(a-r,b+r;c;x,y), a\neq 1,2,3,\dots.\end{aligned}\tag{107}$$

Theorem 61. *The integration recursion formulas of the Horn hypergeometric function* $\mathbf{H}_9$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_9(a,b;c;x,y)=&\frac{(c-1)(c-2)}{x^2(a-1)(a-2)(a-3)(a-4)}\mathbf{H}_9(a-4,b;c-2;x,y)\\ &+\frac{2(c-1)}{xy(a-1)(b-1)}\mathbf{H}_9(a-1,b-1;c-1;x,y)\\ &+\frac{a(a+1)}{y^2(b-1)(b-2)}\mathbf{H}_9(a+2,b-2;c;x,y), x,y\neq 0, a\neq 1,2,3,4, b\neq 1,2,\\ \hat{\mathfrak{I}}^r\mathbf{H}_9(a,b;c;x,y)=&\frac{(-1)^r(1-c)_r}{x^ry^r(1-a)_r(1-b)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_9(a-r,b-r;c-r;x,y),\\ &(x,y\neq 0, a,b\neq 1,2,3,\dots),\\ \hat{\mathfrak{I}_x}^r\ \mathbf{H}_9(a,b;c;x,y)=&\frac{(-1)^r(1-c)_r}{x^r(1-a)_{2r}}\mathbf{H}_9(a-2r,b;c-r;x,y), x\neq 0, a\neq 1,2,3,\dots,\\ \hat{\mathfrak{I}_y}^r\ \mathbf{H}_9(a,b;c;x,y)=&\frac{(-1)^r(a)_r}{y^r(1-b)_r}\mathbf{H}_9(a+r,b-r;c;x,y), y\neq 0, b\neq 1,2,3,\dots,\\ \left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\ \mathbf{H}_9(a,b;c;x,y)=&\frac{(-1)^r(1-c)_r}{x^ry^r(1-a)_r(1-b)_r}\mathbf{H}_9(a-r,b-r;c-r;x,y), x,y\neq 0, a,b\neq 1,2,3,\dots.\end{aligned}\tag{108}$$

Theorem 62. *For* $|t|<1$*, the infinite summation formulas for Horn's function* $\mathbf{H}_9$ *hold true:*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_9(a+k,b;c;x,y)t^k=&(1-t)^{-a}\mathbf{H}_9\left(a,b;c;\frac{x}{(1-t)^2},y(1-t)\right),\\ \sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_8(a,b+k;c;x,y)t^k=&(1-t)^{-b}\mathbf{H}_8\left(a,b;c;x,\frac{y}{1-t}\right).\end{aligned}\tag{109}$$

11. Certain new formulas for $\mathbf{H}_{10}$

In this section, we obtain several recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_{10}$.

Theorem 63. *For $b \neq 0, -1, -2, \ldots$ and $n \in \mathbb{N}$. Recursion formulas for the function $\mathbf{H}_{10}$ are as follows*

$$\begin{aligned}&\mathbf{H}_{10}(a+n;b;x,y) = \mathbf{H}_{10}(a;b;x,y) + \frac{2x}{b}\sum_{k=1}^{n}(a+k)\mathbf{H}_{10}(a+k+1;b+1;x,y)\\&- y\sum_{k=1}^{n}\frac{\mathbf{H}_{10}(a+k-2;b;x,y)}{(a+k-1)(a+k-2)}, a \neq 1-k, a \neq 2-k, k \in \mathbb{N}\end{aligned} \tag{110}$$

and

$$\begin{aligned}&\mathbf{H}_{10}(a-n;b;x,y) = \mathbf{H}_{10}(a;b;x,y) - \frac{2x}{b}\sum_{k=1}^{n}(a-k+1)\mathbf{H}_{10}(a-k+2;b+1;x,y)\\&- y\sum_{k=1}^{n}\frac{\mathbf{H}_{10}(a-k-1;b;x,y)}{(a-k)(a-k-1)}, a \neq k, a \neq k+1, k \in \mathbb{N}.\end{aligned} \tag{111}$$

Theorem 64. *The following recurrence relations hold true for the Horn function $\mathbf{H}_{10}$:*

$$\begin{aligned}&\mathbf{H}_{10}(a;b-n;x,y) =\mathbf{H}_{10}(a;b;x,y) - a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_{10}(a+2;b-k+2;x,y)}{(b-k)(b-k+1)}, \quad (b \neq k, k-1; k \in \mathbb{N}),\\&\mathbf{H}_{10}(a;b+n;x,y) =\mathbf{H}_{10}(a;b;x,y) + a(a+1)x\sum_{k=1}^{n}\frac{\mathbf{H}_{10}(a+2;b+k+1;x,y)}{(b+k)(b+k-1)},\\&\qquad (b \neq -k, 1-k; k \in \mathbb{N}).\end{aligned} \tag{112}$$

Theorem 65. *The differential recursion formulas hold true for Horn function $\mathbf{H}_{10}$:*

$$\begin{aligned}\mathbf{H}_{10}(a+1;b;x,y) &= \left(1+\frac{2\theta_x-\theta_y}{a}\right)\mathbf{H}_{10}(a;b;x,y), a \neq 0,\\ \mathbf{H}_{10}(a;b-1;x,y) &= \left(1+\frac{\theta_x}{b-1}\right)\mathbf{H}_{10}(a;b;x,y), b \neq 1.\end{aligned} \tag{113}$$

Theorem 66. *The derivative formulas hold true for Horn hypergeometric function $\mathbf{H}_{10}$*

$$\begin{aligned}\frac{\partial^r}{\partial x^r}\mathbf{H}_{10}(a;b;x,y) &= \frac{(a)_{2r}}{(b)_r}\mathbf{H}_{10}(a+2r;b+r;x,y), b \neq 0,-1,-2,\ldots\\ \frac{\partial^r}{\partial y^r}\mathbf{H}_{10}(a,b;c;x,y) &= \frac{(-1)^r}{(1-a)_r}\mathbf{H}_{10}(a-r,b;c;x,y), a \neq 1,2,3,\ldots.\end{aligned} \tag{114}$$

Theorem 67. *The integration recursion formulas of the Horn function* $\mathbf{H}_{10}$ *hold true:*

$$\begin{aligned}\hat{\mathfrak{I}}^2\mathbf{H}_{10}(a;b;x,y) =&\frac{(b-1)(b-2)}{x^2(a-1)(a-2)(a-3)(a-4)}\mathbf{H}_{10}(a-4;b-2;x,y)\\&+\frac{2(b-1)}{xy(a-1)}\mathbf{H}_{10}(a-1;b-1;x,y)+\frac{a(a+1)}{y^2}\mathbf{H}_{10}(a+2;b;x,y),\\&(x,y\neq 0, a\neq 1,2,3,4),\\\hat{\mathfrak{I}}^r\mathbf{H}_{10}(a;b;x,y) =&\frac{(1-b)_r}{x^ry^r(1-a)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_{10}(a-r;b-r;x,y),\\&(x,y\neq 0, a\neq 1,2,3,\ldots),\end{aligned} \tag{115}$$

$$\begin{aligned}\hat{\mathfrak{I}_x}^r\;\mathbf{H}_{10}(a;b;x,y) =&\frac{(-1)^r(1-b)_r}{x^r(1-a)_{2r}}\mathbf{H}_{10}(a-2r;b-r;x,y), x\neq 0, a\neq 1,2,3,\ldots,\\\hat{\mathfrak{I}_y}^r\;\mathbf{H}_{10}(a;b;x,y) =&\frac{(a)_r}{y^r}\mathbf{H}_{10}(a+r;b;x,y), y\neq 0,\\\left(\hat{\mathfrak{I}}_x\hat{\mathfrak{I}}_y\right)^r\mathbf{H}_{10}(a;b;x,y) =&\frac{(1-b)_r}{x^ry^r(1-a)_r}\mathbf{H}_{10}(a-r;b-r;x,y), x,y\neq 0, a\neq 1,2,3,\ldots.\end{aligned} \tag{116}$$

Theorem 68. *For* $|t|<1$*, the infinite summation formula for Horn function* $\mathbf{H}_{10}$ *holds true:*

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_{10}(a+k;b;x,y)t^k=(1-t)^{-a}\mathbf{H}_{10}\left(a;b;\frac{x}{(1-t)^2},y(1-t)\right). \tag{117}$$

12. Certain new formulas for $\mathbf{H}_{11}$

In this section, we establish the recursion formulas, differential recursion formulas, integral formulas and infinite summation formulas for the Horn function $\mathbf{H}_{11}$ with all the parameters.

Theorem 69. *For* $n\in\mathbb{N}$ *and* $d\neq 0,-1,-2,\ldots$. *Recursion formulas for the function* $\mathbf{H}_{11}$ *are as follows*

$$\begin{aligned}&\mathbf{H}_{11}(a+n,b,c;d;x,y)=\mathbf{H}_{11}(a,b,c;d;x,y)+\frac{x}{d}\sum_{k=1}^{n}\mathbf{H}_{11}(a+k,b,c;d+1;x,y)\\&-bcy\sum_{k=1}^{n}\frac{\mathbf{H}_{11}(a+k-2,b+1,c+1;d;x,y)}{(a+k-1)(a+k-2)}, a\neq 1-k, a\neq 2-k, k\in\mathbb{N}\end{aligned} \tag{118}$$

and

$$\mathbf{H}_{11}(a-n,b,c;d;x,y) == \mathbf{H}_{11}(a,b,c;d;x,y) - \frac{x}{d}\sum_{k=1}^{n}\mathbf{H}_{11}(a-k+1,b+1,c;d+1;x,y)$$
$$+ bcy\sum_{k=1}^{n}\frac{\mathbf{H}_{11}(a-k-1,b,c+1;d;x,y)}{(a-k)(a-k-1)}, a\neq k, a\neq 1+k, k\in\mathbb{N}. \tag{119}$$

Theorem 70. *For $a\neq 1$, the recurrence relation hold true for the Horn function $\mathbf{H}_{11}$:*

$$\begin{aligned}
\mathbf{H}_{11}(a,b+n,c;d;x,y) &= \mathbf{H}_{11}(a,b,c;d;x,y) + \frac{cy}{a-1}\sum_{k=1}^{n}\mathbf{H}_{11}(a-1,b+k,c+1;d;x,y),\\
\mathbf{H}_{11}(a,b-n,c;d;x,y) &= \mathbf{H}_{11}(a,b,c;d;x,y) - \frac{cy}{a-1}\sum_{k=1}^{n}\mathbf{H}_{11}(a-1,b-k+1,c+1;d;x,y),\\
\mathbf{H}_{11}(a,b,c+n;d;x,y) &= \mathbf{H}_{11}(a,b,c;d;x,y) + \frac{by}{a-1}\sum_{k=1}^{n}\mathbf{H}_{11}(a-1,b+1,c+k;d;x,y),\\
\mathbf{H}_{11}(a,b,c-n;d;x,y) &= \mathbf{H}_{11}(a,b,c;d;x,y) - \frac{by}{a-1}\sum_{k=1}^{n}\mathbf{H}_{11}(a-1,b+1,c-k+1;d;x,y).
\end{aligned} \tag{120}$$

Theorem 71. *The Horn function $\mathbf{H}_{11}$ satisfies the following identity:*

$$\begin{aligned}
\mathbf{H}_{11}(a,b,c;d-n;x,y) =&\mathbf{H}_{11}(a,b,c;d;x,y) + ax\sum_{k=1}^{n}\frac{\mathbf{H}_{11}(a+1,b,c;d-k+2;x,y)}{(d-k)(d-k+1)},\\
&(d\neq k-1, k\in\mathbb{N}),\\
\mathbf{H}_{11}(a,b,c;d+n;x,y) =&\mathbf{H}_{11}(a,b,c;d;x,y) - ax\sum_{k=1}^{n}\frac{\mathbf{H}_{11}(a+1,b,c;d+k+1;x,y)}{(d+k-1)(d+k)},\\
&(d\neq 1-k, -k\in\mathbb{N}).
\end{aligned} \tag{121}$$

Theorem 72. *The differential recursion formulas hold true for the parameters of the Horn function $\mathbf{H}_{11}$:*

$$\begin{aligned}
\mathbf{H}_{11}(a+1,b,c;d;x,y) =&\left(1+\frac{\theta_x-\theta_y}{a}\right)\mathbf{H}_{11}(a,b,c;d;x,y), a\neq 0,\\
\mathbf{H}_{11}(a,b+1,c;d;x,y) =&\left(1+\frac{\theta_y}{b}\right)\mathbf{H}_{11}(a,b,c;d;x,y), b\neq 0,\\
\mathbf{H}_{11}(a,b,c+1;d;x,y) =&\left(1+\frac{\theta_y}{c}\right)\mathbf{H}_{11}(a,b,c;d;x,y), c\neq 0,\\
\mathbf{H}_{11}(a,b,c;d-1;x,y) =&\left(1+\frac{\theta_x}{d-1}\right)\mathbf{H}_{11}(a,b,c;d;x,y), d\neq 1.
\end{aligned} \tag{122}$$

Theorem 73. *The derivative formulas hold true for Horn function* $\mathbf{H}_{11}$

$$\begin{aligned}
&\frac{\partial^r}{\partial x^r}\mathbf{H}_{11}(a,b,c;d;x,y)=\frac{(a)_r}{(d)_r}\mathbf{H}_{11}(a+r,b,c;d+r;x,y), d\neq 0,-1,-2,\dots,\\
&\frac{\partial^r}{\partial y^r}\mathbf{H}_{11}(a,b,c;d;x,y)=\frac{(-1)^r(b)_r(c)_r}{(1-a)_r}\mathbf{H}_{11}(a-r,b+r,c+r;d;x,y), a\neq 1,2,3,\dots.
\end{aligned}\tag{123}$$

Theorem 74. *The integration recursion formulas of the Horn hypergeometric function* $\mathbf{H}_{11}$ *hold true:*

$$\begin{aligned}
\hat{\mathfrak{I}}^2\mathbf{H}_{11}(a,b,c;d;x,y)=&\frac{(d-1)(d-2)}{x^2(a-1)(a-2)}\mathbf{H}_{11}(a-2,b,c;d-2;x,y)\\
&+\frac{2(d-1)}{xy(b-1)(c-1)}\mathbf{H}_1(a,b-1,c-1;d-1;x,y)\\
&+\frac{a(a+1)}{y^2(b-1)(b-2)(c-1)(c-2)}\mathbf{H}_{11}(a+2,b-2,c-2;d;x,y),\\
&(x,y\neq 0,a,b,c\neq 1,2),\\
\hat{\mathfrak{I}}^r\mathbf{H}_{11}(a,b,c;d;x,y)=&\frac{(-1)^r(1-d)_r}{x^ry^r(1-b)_r(1-c)_r}\prod_{k=1}^{r}(\theta_x+\theta_y-k+1)\mathbf{H}_{11}(a,b-r,c-r;d-r;x,y),\\
&(x,y\neq 0,b,c\neq 1,2,3,\dots),
\end{aligned}\tag{124}$$

$$\begin{aligned}
\hat{\mathfrak{I}_x}^r\mathbf{H}_{11}(a,b,c;d;x,y)=&\frac{(1-d)_r}{x^r(1-a)_r}\mathbf{H}_{11}(a-r,b,c;d-r;x,y), x\neq 0, a\neq 1,2,3,\dots,\\
\hat{\mathfrak{I}_y}^r\mathbf{H}_{11}(a,b,c;d;x,y)=&\frac{(-1)^r(a)_r}{y^r(1-b)_r(1-c)_r}\mathbf{H}_{11}(a+r,b-r,c-r;d;x,y),\\
&(y\neq 0,b,c\neq 1,2,3,\dots),\\
\left(\hat{\mathfrak{I}_x}\hat{\mathfrak{I}_y}\right)^r\mathbf{H}_{11}(a,b,c;d;x,y)=&\frac{(-1)^r(1-d)_r}{x^ry^r(1-b)_r(1-c)_r}\mathbf{H}_{11}(a,b-r,c-r;d-r;x,y),\\
&(x,y\neq 0,b,c\neq 1,2,3,\dots).
\end{aligned}\tag{125}$$

Theorem 75. *For* $|t|<1$*, the infinite summation formulas for Horn function* $\mathbf{H}_{11}$ *hold true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(a)_k}{k!}\mathbf{H}_{11}(a+k,b,c;d;x,y)t^k=(1-t)^{-a}\mathbf{H}_{11}\left(a,b,c;d;\frac{x}{1-t},y(1-t)\right),\\
&\sum_{k=0}^{\infty}\frac{(b)_k}{k!}\mathbf{H}_{11}(a,b+k,c;d;x,y)t^k=(1-t)^{-b}\mathbf{H}_{11}\left(a,b,c;d;x,\frac{y}{1-t})\right),\\
&\sum_{k=0}^{\infty}\frac{(c)_k}{k!}\mathbf{H}_{11}(a,b,c+k;d;x,y)t^k=(1-t)^{-c}\mathbf{H}_{11}\left(a,b,c;d;x,\frac{y}{1-t})\right).
\end{aligned}\tag{126}$$

13. Concluding remarks

The several new recursion formulas, differential recursion formulas, differential operators, integral operators, infinite summation formulas and interesting results for Horn hypergeometric functions thus derived in this work are general in character and likely to find various applications in the theory of special functions, which are involved in several diverse fields of mathematical, statistical, physical, and engineering sciences. In conclusion, therefore, we remark that the results derived in this work are sufficiently significant and sufficiently general in nature and are capable of yielding numerous other recursion formulas, differential recursion formulas, differential operators, integral operators and infinite summation formulas involving various special functions by some appropriate choices of the essentially arbitrary parameters which are involved in these results.

References

[1] M.A. Abul-Ez,, and K.A.M. Sayyed, *On integral operator sets of polynomials of two complex variables*, Quart. J. pure Appl. Math. 64 (1990), 157-167.

[2] Yu.A. Brychkov, *Reduction formulas for the Appell and Humbert functions*, Integral Transforms Spec. Funct. 28, 1 (2017), 22-38.

[3] Yu.A, Brychkov, and N. Saad, *On some formulas for the Appell function* $F_2(a, b, b'; c, c'; w, z)$, Integral Transforms Spec Funct. 25, 2 (2014), 111-123.

[4] Yu.A. Brychkov, and N. Saad, *On some formulas for the Appell function* $F_4(a, b; c, c'; w, z)$, Integral Transforms Spec Funct. 28, 9 (2015), 629-644.

[5] A. Erdélyi, W. Mangus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[6] J. Horn, *Hypergeometrische Funktionen zweier Vernäderlichen*, Math. Ann. 105, (1931), 381-407.

[7] S.O. Opps, N. Saad, and H.M. Srivastava, *Some recursion and transformation formulas for the Appell's hypergeometric function* F_2, J. Math. Anal. Appl. 302 (2005), 180-195.

[8] S.O. Opps, N. Saad and H.M. Srivastava, *Recursion formulas for Appell's hypergeometric function with some applications to radiation field problems*, Appl. Math. Comput. 207 (2009), 545-558

[9] R. Sahin, *Recursion formulas for Srivastava's hypergeometric functions*, Math. Slovaca 65, 6 (2015), 1345-1360.

[10] R. Sahin and S.R.S. Agha, *Recursion formulas for G_1 and G_2 horn hypergeometric functions*, Miskolc Math. Notes 16, 2 (2015), 1153-1162.

[11] V. Sahai and A. Verma, *Recursion formulas for Srivastava's general triple hypergeometric functions*, Asian-Eur. J. Math. 9, 3 (2016) 1650063 (17 pages).

[12] V. Sahai and A. Verma, *Recursion formulas for the Srivastava-Daoust and related multivariable hypergeometric functions*, Asian-Eur. J. Math. 9, 4 (2016), 1650081, (35 pages).

[13] B.L. Sharma, *Some formulae for Appell functions*, Proc. Camb. Phil. Soc. 67 (1970), 613-618.

[14] K.A.M. Sayyed, *Basic Sets of Polynomials of two Complex Variables and Convergence Propertiess*, Ph. D. Thesis, Assiut University, Egypt, 1975.

[15] A. Shehata and Shimaa I. Moustafa, *Some new results for Horn's hypergeometric functions Γ_1 and Γ_2*, J. Math. Comput. Sci. in press .

[16] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series.* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[17] X. Wang, *Recursion formulas for Appell functions*, Integral Transforms Spec. Funct. 23, 6 (2012), 421-433.

M. A. Pathan
Department of Mathematics, Aligarh Muslim University,
Aligarh-202002, India
email: *mapathan@gmail.com*

Ayman Shehata
Department of Mathematics, Faculty of Science, Assiut University,
Assiut 71516, Egypt.
Department of Mathematics, College of Science and Arts, Unaizah,
Qassim University, Qassim, Saudi Arabia.
email: *drshehata2006@yahoo.com, aymanshehata@science.aun.edu.eg,*
A.Ahmed@qu.edu.sa

Shimaa I. Moustafa
Department of Mathematics, Faculty of Science,
Assiut University, Assiut 71516, Egypt,
email: *shimaa1362011@yahoo.com, shimaa_m.@science.aun.edu.eg*