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ON STAR COLORING OF MODULAR PRODUCT OF CERTAIN GRAPHS

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Abstract. A star coloring of a graph G is a proper vertex coloring in which every path on four vertices in G is not bicolored. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G. In this paper, we obtain the star chromatic number of modular product of two graphs G and H, denoted by $G \diamond H$. We consider the graph $G \diamond H$, where G and H be the path graphs, the cycle graphs, the star graphs and Petersen graphs.

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1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph and we follow [2, 3, 9] for terminology and notation that are not defined here. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively.

The concept of star chromatic number was introduced by Branko Grünbaum in 1973.

Definition 1. *A star coloring [1, 5, 8, 11, 12] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.*

Star coloring also arises naturally in combinatorial computing. As one would imagine, finding an optimal star- coloring of a general graph is NP-hard. Coleman and Moré showed that star-coloring remains an NP-hard problem even on bipartite graphs [4]. Coloring variants (like acyclic or star coloring) have been used to compute sparse Hessian and Jacobian matrices with techniques like finite differences and automatic differentiation. Gebremedhin, Tarafdar, Manne, and Pothen provided algorithms for finding heuristic solutions to star coloring and acyclic coloring problems [7]. Their techniques utilize the structure of subgraphs induced by color classes and their findings have applications to efficient computation of Hessian matrices. Because the problems of computing these matrices can be recast as graph coloring problems, employing graph coloring as a model for computation can yield particularly effective algorithms. See [6] for a detailed survey of using graph coloring to compute derivatives.

During the years star coloring of graphs has been studied extensively by several authors, for instance see [1, 4, 5].

2. Preliminaries

Definition 2. *The modular product [10] $G \diamond H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which a vertex (v, w) is adjacent to a vertex (v', w') if and only if either*

- *v s adjacent to v' and w is adjacent to w' or*
- *v is not adjacent to v' and w is not adjacent to w'.*

For any graph G, we denote the number of vertices of G by $\nu(G)$. By the definition of the modular product $G \diamond H$ of G and H, if $\nu(G) = 1$ then $G \diamond H = H$ and if $\nu(H) = 1$ then $G \diamond H = G$, thus $\lambda(G \diamond H) = \lambda(H)$ or $\lambda(G \diamond H) = \lambda(G)$. Therefore in the following, we assume $\nu_G \geq 2$ and $\nu_H \geq 2$.

Kozen studied the complexity of finding cliques in the modular product of two graphs. By observing that $G \diamond H$ has a clique of size (G) and $G \diamond H$ (both G and H have n vertices) has a clique of size n if and only if G and H are isomorphic, Kozen [10] proved that the problem of finding a clique of size n in $G \diamond H$ is equivalent to the isomorphism problem and that the problem of determining whether $G \diamond H$ has a clique of size $(1 - -\epsilon)$ is NP-complete.

Consider the modular product of two graphs G and H. Graphs G and H are called the factors of the modular product. An important fact about the modular product is that the projections into its factors are not weak homomorphisms. The modular product has some interesting properties, for example the modular product of two graphs is disconnected if and only if one factor is complete and the other disconnected or if both factors are complete and the modular product in general does not have a unique prime factorization. Imrich [10] proved a theorem regarding a situation in which the prime factorization of the modular product is unique.

3. Main Results

In this section, we find the exact values of the star chromatic number of modular product of path graph with path graph $P_m \diamond P_n$, cycle graph with cycle graph $C_m \diamond C_n$, star graph with star graph $K_{1,m} \diamond K_{1,n}$ and Petersen graphs.

3.1. Star Coloring of Modular Product of Path Graphs and Cycle Graphs

Now we consider the graph G and H be the path graphs or cycle graphs of order $m \leq n$. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(H) = \{v_j : 1 \leq j \leq n\}$. By the definition of the modular product, the vertices of $G \diamond H$ is denoted as follows:

$$V(G \diamond H) = \bigcup_{i=1}^{m} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Theorem 1. *Let G and H be the path graphs of order $m \geq 3$ and $n \geq 3$, then*

$$\chi_s(G \diamond H) = \begin{cases} n, & \text{if} \quad m = 3, \\ mn - 2(m+n) + 6, & \text{if} \ \ m = 4, 5, \ n \geq 4 \\ mn - 2m - n + 3, & \text{if} \ \ m = 6, 6 \leq n \leq 8 \\ mn - 2n, & \text{otherwise} \ . \end{cases}$$

Proof. **Case (i):** When $m = 3$.

Let $\{c_1, c_2, \ldots, c_n$ be the set of n distinct colors. For $1 \leq j \leq n$ and $1 \leq i \leq 3$ the vertices (u_i, v_j) can be colored with color j.

Case (ii): When $m = 4, 5$ and $n \geq 4$.

Subcase (i): When $m = 4$, and $n \geq 4$.

- For $1 \leq i \leq m$, $1 \leq j \leq 3$ the vertices (u_i, v_j) can be colored with color i.
- For $i = 1, 3$, $2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ the vertices (u_i, v_{2j}) can be colored with color $4j - 3$.
- For $i = 2, 4$, $2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ the vertices (u_i, v_{2j}) can be colored with color $4j - 2$.
- For $i = 1, 4$, $2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ the vertices (u_i, v_{2j+1}) can be colored with color $4j - 1$.
- For $i = 2, 3$, $2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$ the vertices (u_i, v_{2j+1}) can be colored with color $4j$.

Subcase (ii): When $m = 5$, and $n \geq 4$.

- For $1 \leq i \leq m$, $1 \leq j \leq 3$ the vertices (u_i, v_j) can be colored with color i.
- For $i = 1, 4$, $1 \leq j \leq n-3$ the vertices (u_i, v_{j+3}) can be colored with color $3j + 3$.
- For $i = 2, 5$, $1 \leq j \leq n-3$ the vertices (u_i, v_{j+3}) can be colored with color $3j + 4$.
- For $i = 3$, $1 \leq j \leq n-3$ the vertices (u_i, v_{j+3}) can be colored with color $3j + 5$.

Suppose $\chi_s(G \diamond H) \leq mn - 2(m+n) + 6$, for $m = 4, 5$, $n \geq 4$. Now we colored the vertices (u_i, v_j) where $1 \leq j \leq n$, $1 \leq i \leq 5$ has to be colored with one of the colors $\{1, 2, \ldots, mn - 2(m+n) + 5\}$ which results in bicolored paths on four vertices and so contradicts the definition of star coloring. So we need one more color, hence $\chi_s(G \diamond H) \geq mn - 2(m+n) + 6$. Therefore $\chi_s(G \diamond H) = mn - 2(m+n) + 6$.

Case (iii): When $m = 6$, and $6 \leq n \leq 8$.

- For $1 \leq i \leq m$, $1 \leq j \leq 3$ the vertices (u_i, v_j) can be colored with color i.
- For $1 \leq i \leq m-2$, $1 \leq j \leq n-3$ the vertices (u_i, v_{j+3}) can be colored with color $5j + i + 1$.
- For $i = m-1$, $1 \leq j \leq n-3$ the vertices (u_i, v_{j+3}) can be colored with color $5j + 3$.
- For $i = m$, $1 \leq j \leq n-3$ the vertices (u_i, v_{j+3}) can be colored with color $5j + m$.

Suppose $\chi_s(G \diamond H) \leq mn - 2m - n + 3$, for $m = 6$, $6 \leq n \leq 8$. Now we colored the vertices (u_i, v_j) where $1 \leq j \leq n$, $1 \leq i \leq m$ has to be colored with one of the colors $\{1, 2, \ldots, mn - 2m - n + 2\}$ which results in bicolored paths on four vertices and so contradicts the definition of star coloring. So we need one more color, hence $\chi_s(G \diamond H) \geq mn - 2m - n + 3$. Therefore $\chi_s(G \diamond H) = mn - 2m - n + 3$.

Case (iv): When $m > 6$.

- For $1 \leq i \leq 3$, $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color j.
- For $4 \leq i \leq m$, $1 \leq j \leq n$, the vertices (u_i, v_j) can be colored with color $(i-3)n + j$.

Suppose $\chi_s(G \diamond H) \leq mn - 2n$, $m > 6$. Now we colored the vertices (u_i, v_j) where $1 \leq j \leq n$, $\quad 1 \leq i \leq m$ has to be colored with one of the colors $\{1, 2, \ldots, mn-2n-1\}$ which results in bicolored paths on four vertices and so contradicts the definition of star coloring. So we need one more color, hence $\chi_s(G \diamond H) \geq mn - 2n$. Therefore $\chi_s(G \diamond H) = mn - 2n$.

Theorem 2. *Let G and H be the cycle graphs of order $m > 6$ and $n > 6$, then*

$$\chi_s(G \diamond H) = \begin{cases} n\,(m-3)+3, & \text{if } n \equiv 0, 1, 3 \mod 4, \\ n\,(m-3)+4, & \text{if } n \equiv 2 \mod 4. \end{cases}$$

Proof. **Case (i):** When $n \equiv 0, 1, 3 \mod 4$.

Let $\{c_1, c_2, \ldots, c_{n(m-3)+3}$ be the set of $n\,(m-3)+3$ distinct colors. For $1 \leq j \leq n$ the vertices (u_1, v_j) can be colored with color 1.

Subcase (i): When $n \equiv 0 \mod 4$.

- For $j \equiv 1, 2 \mod 4$, $1 \leq j \leq n$ the vertices (u_2, v_j) can be colored with color 2.
- For $j \equiv 0, 3 \mod 4$, $1 \leq j \leq n$ the vertices (u_2, v_j) can be colored with color 3.
- For $1 \leq i \leq m-3$ and $1 \leq j \leq n$ the vertices (u_{i+2}, v_j) can be colored with color $n(i-1)+j+3$.
- For $1 \leq j \leq n$ the vertices (u_m, v_j) can be colored with color $j+3$.

Subcase (ii): When $n \equiv 1 \mod 4$.

- For $j \equiv 1, 2 \mod 4$, $1 \leq j \leq n-1$, the vertices (u_2, v_j) can be colored with color 2.
- For $j \equiv 0, 3 \mod 4$, $1 \leq j \leq n-1$ the vertices (u_2, v_j) can be colored with color 3.
- For $1 \leq i \leq 2$ and $1 \leq j \leq n-1$ the vertices (u_{i+2}, v_j) can be colored with color $n(i-1)+j+4$.
- For $3 \leq i \leq m-3$ and $1 \leq j \leq n$ the vertices (u_{i+2}, v_j) can be colored with color $n(i-1)+j+3$.
- For $1 \leq j \leq n$ the vertices (u_m, v_j) can be colored with color $j+4$.
- For the vertices (u_i, v_n), $i = 2, 4$ and (u_3, v_n) can be colored with color 4 and $n+4$.

Subcase (iii): When $n \equiv 3 \mod 4$.

- For $j \equiv 1, 2 \mod 4$, $1 \leq j \leq n-2$, the vertices (u_2, v_j) can be colored with color 2.

- For $j \equiv 0, 3 \mod 4$, $1 \le j \le n-2$ the vertices (u_2, v_j) can be colored with color 3.
- For $1 \le i \le 2$ and $1 \le j \le n-2$ the vertices (u_{i+2}, v_j) can be colored with color $n(i-1)+j+4$.
- For $3 \le i \le m-3$ and $1 \le j \le n$ the vertices (u_{i+2}, v_j) can be colored with color $n(i-1)+j+3$.
- For $1 \le j \le n$ the vertices (u_m, v_j) can be colored with color $j+4$.
- For the vertices (u_i, v_{n-1}), $i = 2, 4$ can be colored with color 4.
- For the vertices (u_3, v_{n-1}) and (u_3, v_n) can be colored with color $n+3$ and $n+4$.
- For the vertices (u_2, v_n) and (u_4, v_n) can be colored with color 3 and $2n+3$.

Suppose $\chi_s(G \diamond H) \le n(m-3)+3$, for $n \equiv 0, 1, 3 \mod 4$,. Now we colored the vertices (u_i, v_j) where $j \equiv 0, 1, 3 \pmod 4), 1 \le i \le m$ has to be colored with one of the colors $\{1, 2, \dots, n(m-3)+2\}$ which results in bicolored paths on four vertices and so contradicts the definition of star coloring. So we need one more color, hence $\chi_s(G \diamond H) \ge n(m-3)+3$. Therefore $\chi_s(G \diamond H) = n(m-3)+3$.

Case (ii): When $n \equiv 2 \mod 4$.
Let $\{c_1, c_2, \dots, c_{n(m-3)+4}$ be the set of $n(m-3)+4$ distinct colors. For $1 \le j \le n$ the vertices (u_1, v_j) can be colored with color 1.

- For $j \equiv 1, 2 \mod 4$, $1 \le j \le n-2$ the vertices (u_2, v_j) can be colored with color 2.
- For $j \equiv 0, 3 \mod 4$, $1 \le j \le n-2$ the vertices (u_2, v_j) can be colored with color 3.
- For $1 \le i \le m-3$ and $1 \le j \le n$ the vertices (u_{i+2}, v_j) can be colored with color $n(i-1)+j+4$.
- For $1 \le j \le n$ the vertices (u_m, v_j) can be colored with color $j+4$.
- For the vertices (u_2, v_{n-1}) and (u_2, v_n) can be colored with color 4.

Suppose $\chi_s(G \diamond H) \le n(m-3)+4$, for $n \equiv 2 \mod 4$,. Now we colored the vertices (u_i, v_j) where $j \equiv 2 \pmod 4), 1 \le i \le m$ has to be colored with one of the colors $\{1, 2, \dots, n(m-3)+3\}$ which results in bicolored paths on four vertices and so contradicts the definition of star coloring. So we need one more color, hence $\chi_s(G \diamond H) \ge n(m-3)+4$. Therefore $\chi_s(G \diamond H) = n(m-3)+4$.

3.2. Star Coloring of Modular Product of Star Graphs

Now we consider the graph G and H be the star graphs of order $m \leq n$. Let $V(G) = \{u_1\} \cup \{u_i : 2 \leq i \leq m+1\}$ and $V(H) = \{v_1\} \cup \{v_j : 2 \leq j \leq n+1\}$. By the definition of the modular product, the vertices of $G \diamond H$ is denoted as follows:

$$V(G \diamond H) = \bigcup_{i=1}^{m+1} \{(u_i, v_j) : 1 \leq j \leq n+1\}.$$

Theorem 3. *Let G and H be the star graphs of order $m \geq 3$ and $n \geq 3$, then $\chi_s(G \diamond H) = n\,(m-1)+1$.*

Proof. Let $\{c_1, c_2, \ldots, c_{n(m-1)} + 1\}$ be the set of $n(m-1)+1$ distinct colors.

- For $1 \leq i \leq m+1$, the vertices (u_i, v_1) can be colored with color 1.
- For $1 \leq i \leq 3$ and $2 \leq j \leq n+1$ the vertices (u_i, v_j) can be colored with color j.
- For $4 \leq i \leq m+1$ and $2 \leq j \leq n+1$ the vertices (u_i, v_j) can be colored with color $n(i-3)+j$.

Suppose $\chi_s(G \diamond H) \leq n(m-1)+1$. Now we colored the vertices (u_i, v_j) where $1 \leq i \leq m, \quad 1 \leq j \leq n$ has to be colored with one of the colors $\{1, 2, \ldots, n(m-1)\}$ which results in bicolored paths on four vertices and so contradicts the definition of star chromatic number. So we need one more color, hence $\chi_s(G \diamond H) \geq n(m-1)+1$. Therefore $\chi_s(G \diamond H) = n(m-1)+1$.

3.3. Star Coloring of Modular Product of Petersen Graphs

Theorem 4. *Let G and H be the Petersen graph of order m and n, then the star chromatic number of modular product of G and H is 67.*

Proof. Let $V(G) = \{u_i : 1 \leq i \leq m\}$ and $V(H) = \{v_j : 1 \leq j \leq n\}$, where $m = n = 10$. By the definition of the modular product, the vertices of $G \diamond H$ is denoted as follows:

$$V(G \diamond H) = \bigcup_{i=1}^{m} \{(u_i, v_j) : 1 \leq j \leq n\}.$$

Let the colors $\{1, 2, 3, \ldots, 67\}$ be the set of 67 distinct colors to appear in the vertices (u_i, v_j) as in Table 1.

Suppose $\chi_s(G \diamond H) \leq 67$. Now we colored the vertices (u_i, v_j) where $1 \leq i \leq m, \quad 1 \leq j \leq n$ has to be colored with one of the colors $\{1, 2, \ldots, 66\}$ which results

Table 1: Values of the star chromatic number of modular $G \diamond H$ is 67.

(u_i, v_j)	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
$i=1$	1	2	8	16	16	8	28	38	48	58
$i=2$	1	2	9	17	17	9	29	39	49	59
$i=3$	1	3	10	18	18	10	30	40	50	60
$i=4$	1	3	11	19	19	11	31	41	51	61
$i=5$	1	4	4	20	20	26	32	42	52	62
$i=6$	1	5	12	21	21	12	33	43	53	63
$i=7$	1	5	13	22	22	13	34	44	54	64
$i=8$	1	6	14	23	23	14	35	45	55	65
$i=9$	1	6	15	24	24	15	36	46	56	66
$i=10$	1	7	7	25	25	27	37	47	57	67

in bicolored paths on four vertices and so contradicts the definition of star chromatic number. So we need one more color, hence $\chi_s(G \diamond H) \geq 67$. Therefore $\chi_s(G \diamond H) = 67$.

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