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A WEAKENED VERSION OF STRICT SCHUR NUMBERS

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Abstract. In this paper, we introduce a generalization of Schur numbers, analogous to the weakened generalization of Ramsey numbers. Specifically, define $WS^k_\ell(n)$ to be the least natural number N such that every n-coloring of $\{1,2,\dots,N\}$ (using all n colors) contains a solution to the equation

$$x_1 + x_2 + \cdots + x_{k-1} = x_k, \quad \text{where } x_1 < x_2 < \cdots < x_k,$$

that uses at most ℓ of the n colors. We provide explicit evaluations of $WS^3_2(n)$, $WS^k_{k-1}(k)$, $WS^4_2(3)$, and $WS^5_3(4)$, and offer some directions for future inquiry.

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1. Introduction

While investigating the modular version of Fermat's Last Theorem in 1916, Schur [17] proved the finiteness of Schur numbers. Since that time, Schur numbers have seen many generalizations, many of which have sought to determine how large the set $[1,N] := \{1,2,\dots,N\}$ must be in order to guarantee that every n-coloring of $[1,N]$ contains a certain monochromatic equation. Although Schur's work predates Ramsey's foundational paper [15], the finiteness of Schur numbers follows from Ramsey's theorem. Our focus in this paper is to weaken Schur numbers in an analogous way to that of weakened Ramsey numbers (e.g., [8], [9], [10], [12], and [13]). That is, rather than guaranteeing the existence of a certain monochromatic equation, we seek to guarantee an equation that uses at most ℓ of the n colors.

To be precise, begin by defining an exact n-coloring of $[1,N]$ to be a surjective map $c : [1,N] \longrightarrow [1,n]$. Given $k \geq 3$, a solution to the equation

$$x_1 + x_2 + \cdots + x_{k-1} = x_k, \qquad \text{where } x_1, x_2, \dots, x_k \in [1,N],$$

is called a k-Schur solution. We say that such a solution is monochromatic if $|c(\{x_1, x_2, \dots, x_k\})| = 1$ and is rainbow if $|c(\{x_1, x_2, \dots, x_k\})| = k$. The k-Schur number $S^k(n)$ is defined to be the least natural number N such that every exact n-coloring of $[1, N]$ contains a monochromatic k-Schur solution. Such numbers were the focus of [2], [4], and [16].

For $1 \leq \ell \leq \min\{n-1, k-1\}$, we weaken the problem of Schur by seeking k-Schur solutions such that $|c(\{x_1, x_2, \ldots, x_k\})| \leq \ell$. Whenever $\ell \geq 2$, we run into a complication since $x_1, x_2, \ldots, x_{k-1}$ are not assumed to be distinct. For example, when $\ell = 2$, the equation $1 + 1 = 2$ uses at most 2 colors regardless of the value of n. We resolve this issue by only considering strict k-Schur solutions:

$$x_1 + x_2 + \cdots + x_{k-1} = x_k, \quad \text{where } x_1 < x_2 < \cdots < x_k$$

(see [1] and [14]). Such numbers are sometimes called weak Schur numbers (e.g., see [5] and [11]), but we will use the term "strict" to avoid two distinct uses of the word "weak" in this paper. Define the ℓ-weakened strict k-Schur number $WS_\ell^k(n)$ to be the least natural number N such that every exact n-coloring of $[1, N]$ contains a strict k-Schur solution that uses at most ℓ colors.

To obtain a simple lower bound for $WS_\ell^k(n)$, partition the set $[1, jn]$ into n sets (color classes) of cardinality j. Then every strict k-Schur solution

$$x_1 + x_2 + \cdots + x_{k-1} = x_k, \quad \text{where } x_1 < x_2 < \cdots < x_k,$$

uses at least $\lceil \frac{k}{j} \rceil$ colors. It follows that

$$WS^k_{\lceil \frac{k}{j} \rceil - 1}(n) > jn.$$

In Section 2, we offer two general theorems, giving explicit evaluations of $WS_2^3(n)$ and $WS_{k-1}^k(k)$. In Section 3, we turn our attention to the evaluation of numbers of the form $WS_{k-1}^{k+1}(k)$. Here, we show that $WS_2^4(3) = 11$ and $WS_3^5(4) = 14$. We conclude with a conjecture about the values of $WS_{k-1}^{k+1}(k)$ for all k and offer a few directions for future inquiry.

2. Some General Results

We begin our investigation with the evaluation of $WS_2^3(n)$. The following lemma will serve us in determining the necessary upper bounds.

Lemma 1. *If a coloring of the set $[1, N]$ $(N \geq 3)$ does not contain a strict 3-Schur solution that uses at most 2 colors, then at least $N - \lfloor \frac{N}{2} \rfloor + \lfloor \frac{N}{3} \rfloor$ colors must be used.*

Proof. Consider an n-coloring of $[1, N]$ in which every strict 3-Schur solution is rainbow. Such a coloring can be obtained by successively coloring each i from 1 to N. Since $1 + 2 = 3$, 1, 2, and 3 must receive distinct colors. For any $i > 3$ that is even, the equations

$$1 + (i-1) = i, \quad 2 + (i-2) = i, \quad \ldots, \quad \frac{i-2}{2} + \frac{i+2}{2} = i$$

force i to receive a color different from all $1 \leq j < i$, except possibly $\frac{i}{2}$. From $\frac{i}{2} + i = \frac{3i}{2}$, it follows that i requires a new color if and only if $\frac{3i}{2} \leq N$. For any $i > 3$ that is odd, the equations

$$1 + (i-1) = i, \quad 2 + (i-2) = i, \quad \ldots, \quad \frac{i-1}{2} + \frac{i+1}{2} = i$$

force i to receive a color different from all $1 \leq j < i$. Overall, we find that the only numbers that are allowed to repeat colors are those in the set

$$S := \{i \in [1, N] \mid 2i \in [1, N] \text{ and } 3i \notin [1, N]\},$$

which has cardinality $\lfloor \frac{N}{2} \rfloor - \lfloor \frac{N}{3} \rfloor$. Since all other colors must be distinct, we obtain the statement of the theorem.

Using the above lemma, we now give an explicit evaluation of $WS_2^3(n)$ for all $n \geq 3$. Interestingly, we find that the evaluation depends upon the value of n modulo 5.

Theorem 2. *For all $n \geq 3$,*

$$WS_2^3(n) = \begin{cases} \frac{6n+5}{5} & \text{if} \quad n \equiv 0 \pmod 5 \\ \frac{6n+9}{5} & \text{if} \quad n \equiv 1 \pmod 5 \\ \frac{6n+3}{5} & \text{if} \quad n \equiv 2 \pmod 5 \\ \frac{6n+7}{5} & \text{if} \quad n \equiv 3 \pmod 5 \\ \frac{6n+6}{5} & \text{if} \quad n \equiv 4 \pmod 5. \end{cases}$$

Proof. We break the proof into cases, based on divisibility by 5.
Case 1 Let $n \equiv 0 \pmod 5$ and write $n = 5m$. First, we provide an n-coloring of $[1, n+m] = [1, 6m]$ that lacks a strict 3-Schur solution that uses at most two colors. For each $i \in [1, 6m]$, whose double $2i \in [1, 6m]$, but whose triple $3i \notin [1, 6m]$, both

i and $2i$ may receive the same color. This is true for all $2m < i \leq 3m$. It follows that m colors can be repeated without forcing a strict 3-Schur solution that is not rainbow. Hence,

$$WS_2^3(n) \geq n + m + 1 = \frac{6n+5}{5}.$$

To prove the other direction, we must argue that every n-coloring of $[1, n+m+1]$ contains a strict 3-Schur solution that uses at most two colors. If we let $N = n + m + 1 = 6m + 1$, then

$$N - \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{3} \right\rfloor = 5m + 1 = n + 1 > n.$$

The inequality

$$WS_2^3(n) \leq n + m + 1 = \frac{6n+5}{5}$$

then follows from the contrapositive statement to Lemma 1.

<u>Case 2</u> Let $n \equiv 1 \pmod 5$ and write $n = 5m + 1$. First, we provide an n-coloring of $[1, n+m+1] = [1, 6m+2]$ that lacks a strict 3-Schur solution that uses at most two colors. For each $i \in [1, 6m+2]$, whose double $2i \in [1, 6m+2]$, but whose triple $3i \notin [1, 6m+2]$, both i and $2i$ may receive the same color. This is true for all $2m < i \leq 3m+1$. It follows that $m+1$ colors can be repeated without forcing a strict 3-Schur solution that is not rainbow. Hence,

$$WS_2^3(n) \geq n + m + 2 = \frac{6n+9}{5}.$$

To prove the other direction, we must argue that every n-coloring of $[1, n+m+2]$ contains a strict 3-Schur solution that uses at most two colors. If we let $N = n + m + 2 = 6m + 3$, then

$$N - \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{3} \right\rfloor = 5m + 3 > n.$$

The inequality

$$WS_2^3(n) \leq n + m + 2 = \frac{6n+9}{5}$$

then follows from the contrapositive statement to Lemma 1.

<u>Case 3</u> Let $n \equiv 2 \pmod 5$ and write $n = 5m + 2$. First, we provide an n-coloring of $[1, n+m] = [1, 6m+3]$ that lacks a strict 3-Schur solution that uses at most two colors. For each $i \in [1, 6m+3]$, whose double $2i \in [1, 6m+3]$, but whose triple $3i \notin [1, 6m+3]$, both i and $2i$ may receive the same color. This is true for all

$2m < i \leq 3m+1$. It follows that $m+1$ colors can be repeated without forcing a strict 3-Schur solution that is not rainbow. Hence,

$$WS_2^3(n) \geq n+m+1 = \frac{6n+3}{5}.$$

To prove the other direction, we must argue that every n-coloring of $[1, n+m+1]$ contains a strict 3-Schur solution that uses at most two colors. If we let $N = n+m+1 = 6m+3$, then

$$N - \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{3} \right\rfloor = 5m+3 > n.$$

The inequality

$$WS_2^3(n) \leq n+m+1 = \frac{6n+3}{5}$$

then follows from the contrapositive statement to Lemma 1.

<u>Case 4</u> Let $n \equiv 3 \pmod 5$ and write $n = 5m+3$. First, we provide an n-coloring of $[1, n+m+1] = [1, 6m+4]$ that lacks a strict 3-Schur solution that uses at most two colors. For each $i \in [1, 6m+4]$, whose double $2i \in [1, 6m+4]$, but whose triple $3i \notin [1, 6m+4]$, both i and $2i$ may receive the same color. This is true for all $2m+1 < i \leq 3m+2$. It follows that $m+1$ colors can be repeated without forcing a strict 3-Schur solution that is not rainbow. Hence,

$$WS_2^3(n) \geq n+m+2 = \frac{6n+7}{5}.$$

To prove the other direction, we must argue that every n-coloring of $[1, n+m+2]$ contains a strict 3-Schur solution that uses at most two colors. If we let $N = n+m+2 = 6m+5$, then

$$N - \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{3} \right\rfloor = 5m+5 > n.$$

The inequality

$$WS_2^3(n) \leq n+m+2 = \frac{6n+7}{5}$$

then follows from the contrapositive statement to Lemma 1.

<u>Case 5</u> Let $n \equiv 4 \pmod 5$ and write $n = 5m+4$. First, we provide an n-coloring

of $[1, n+m+1] = [1, 6m+5]$ that lacks a strict 3-Schur solution that uses at most two colors. For each $i \in [1, 6m+5]$, whose double $2i \in [1, 6m+5]$, but whose triple $3i \notin [1, 6m+5]$, both i and $2i$ may receive the same color. This is true for all $2m+1 < i \leq 3m+2$. It follows that $m+1$ colors can be repeated without forcing a strict 3-Schur solution that is not rainbow. Hence,

$$WS_2^3(n) \geq n+m+2 = \frac{6n+6}{5}.$$

To prove the other direction, we must argue that every n-coloring of $[1, n+m+2]$ contains a strict 3-Schur solution that uses at most two colors. If we let $N = n+m+2 = 6m+6$, then

$$N - \left\lfloor \frac{N}{2} \right\rfloor + \left\lfloor \frac{N}{3} \right\rfloor = 5m+5 > n.$$

The inequality

$$WS_2^3(n) \leq n+m+2 = \frac{6n+6}{5}$$

then follows from the contrapositive statement to Lemma 1.

From the previous theorem, it follows that $WS_2^3(3) = 5$. We further investigate numbers of the form $WS_{k-1}^k(k)$ in the following theorem.

Theorem 3. *For all $k \geq 3$,*

$$WS_{k-1}^k(k) = \frac{k(k-1)}{2} + 2.$$

Proof. First, we give a k-coloring of $[1, \frac{k(k-1)}{2}+1]$ that lacks a strict k-Schur solution using at most $k-1$ colors. The only strict k-Schur solutions that apply are

$$1+2+\cdots+(k-2)+(k-1) = \frac{k(k-1)}{2} \quad (1)$$

and

$$1+2+\cdots+(k-2)+k = \frac{k(k-1)}{2}+1. \quad (2)$$

Color $1, 2, \ldots, k$ distinct, with $\frac{k(k-1)}{2}$ the same as k, and $\frac{k(k-1)}{2}+1$ the same as $k-1$. All other numbers can receive any colors. From this coloring, it follows that

$$WS_{k-1}^k(k) \geq \frac{k(k-1)}{2} + 2.$$

Now we must show that every k-coloring of $[1, \frac{k(k-1)}{2}+2]$ contains a strict k-Schur solution using at most $k-1$ colors. In addition to Equations (1) and (2), we also have

$$1+2+\cdots+(k-2)+(k+1) = \frac{k(k-1)}{2}+2 \tag{3}$$

and

$$1+2+\cdots+(k-3)+(k-1)+k = \frac{k(k-1)}{2}+2. \tag{4}$$

From (1), it follows that $1, 2, \ldots, k-1$ are assigned distinct colors. By (2) and (4), k is distinct from the colors assigned to $1, 2 \ldots, k-1$. Equations (3) and (4) then force $k-2$ and $\frac{k(k-1)}{2}+2$ to receive the same color, from which it follows that

$$WS^k_{k-1}(k) \leq \frac{k(k-1)}{2}+2,$$

completing the proof.

3. A Couple of Specific Values of $WS^{k+1}_{k-1}(k)$

A full evaluation of numbers of the form $WS^{k+1}_{k-1}(k)$ eludes us at the present time, but we offer a few special values in this section. In particular, it is known that $WS^3_1(2) = 9$ (see [11]) as this is a standard strict Schur number. In the following two theorems, we prove that $WS^4_2(3) = 11$ and $WS^5_3(4) = 14$.

Theorem 4. *$WS^4_2(3) = 11$.*

Proof. Consider the following 3-coloring of $[1, 10]$:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

It follows that $WS^4_2(3) \geq 11$. To prove the other direction, we must argue that every 3-coloring of $[1, 11]$ contains a strict 4-Schur solution using at most 2 colors. If false, then there exists some 3-coloring of $[1, 11]$ in which every strict 4-Schur solution uses all 3 colors. Consider such a coloring. Since $1+2+3 = 6$ is 3-colored, then we break the proof down into cases, based on which two numbers receive the same color.

Case 1 Suppose that 1 and 2 receive the same color (without loss of generality, suppose they are red). Then 3 and 6 must receive the other two colors (without loss of generality, suppose that 3 is blue and 6 is green). From $1+2+6 = 9$, it follows that 9 is blue. From $1+3+5 = 9$, it follows that 5 is red. From $2+3+5 = 10$, it follows that 10 is green. Regardless of the color assigned to 8, we find that $1+2+5 = 8$ uses at most 2 colors.

Case 2 Suppose that 1 and 3 receive the same color. Without loss of generality, suppose that 1 and 3 are red, 2 is blue, and 6 is green. Then $1+3+6=10$ implies that 10 must be blue. From $1+2+7=10$ and $2+3+5=10$, it follows that 5 and 7 are both green. From $1+3+7=11$, it follows that 11 is blue. From $1+2+8=11$, it follows that 8 is green. From $1+3+4=8$, it follows that 4 is blue. Then $2+4+5=11$ is 2-colored.
Case 3 Suppose that 2 and 3 receive the same color. Without loss of generality, suppose that 2 and 3 are red, 1 is blue, and 6 is green. Then $2+3+6=11$ implies that 11 is blue. Then $1+2+8=11$ and $1+3+7=11$ implies that 7 and 8 are green. Then $1+4+6=11$ implies that 4 is red. Then $2+3+4=9$ uses at most 3 colors.
Case 4 Suppose that 1 and 6 receive the same color. Without loss of generality, suppose that 1 and 6 are red, 2 is blue, and 3 is green. Then $1+2+6=9$ implies that 9 is green and $1+3+6=10$ implies that 10 is blue. Then $2+3+4=9$ and $2+3+5=10$ imply that 4 and 5 are red. Then $1+4+5=10$ uses 2 colors.
Case 5 Suppose that 2 and 6 receive the same color. Without loss of generality, suppose that 2 and 6 are red, 1 is blue, and 3 is green. Then $1+2+6=9$ implies that 9 is green and $2+3+6=11$ implies that 11 is blue. Then $2+3+4=9$ implies that 4 is blue. Then $1+4+6=11$ uses 2 colors.
Case 6 Suppose that 3 and 6 receive the same color. Without loss of generality, suppose that 3 and 6 are red, 1 is blue, and 2 is green. Then $1+3+5=9$ implies that 9 is green and $2+3+5=10$ implies that 10 is blue. Then $2+3+4=9$ implies that 4 is blue and $1+4+5=10$ is 2 colored.
In all six cases, we find that a 3-coloring must contain a strict 4-Schur solution using at most 2 colors, implying the inequality $WS_2^4(3) \leq 11$, and completing the proof.

Theorem 5. $WS_3^5(4) = 14$.

Proof. Consider the following 4-coloring of $[1, 13]$:

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}.$$

It follows that $WS_3^5(4) \geq 14$. To prove the other direction, we must show that every 4-coloring of $[1, 14]$ contains a strict 5-Schur solution using at most 3 colors. We have numerous cases to consider, based on which two numbers in $1+2+3+4=10$ receive the same color.
Case 1 Suppose that 1 and 2 receive the same color (without loss of generality, suppose they are red). Then 3, 4, and 10 must receive the other three colors (without loss of generality, suppose that 3 is blue, 4 is green, and 10 is orange). Then $1+2+3+5=11$ implies that $\{5, 11\}$ is colored green and orange. However, $1+2+4+5=12$ prevents 5 from being green if a 3-colored strict 5-Schur solution is to be avoided.

Thus, 5 is orange, 11 is green, and 12 is blue. Now $1+2+3+6=12$ uses at most 3 colors.
Case 2 Suppose that 1 and 3 are the same color. Without loss of generality, suppose that 1 and 3 are red, 2 is blue, 4 is green, and 10 is orange. Then $1+2+3+5=11$ implies that $\{5,11\}$ is colored green and orange. However, $1+3+4+5=13$ prevents 5 from being colored green. So, 5 is orange, 11 is green, and 13 is blue. Now $1+2+3+7=13$ uses at most 3 colors.
Case 3 Suppose that 1 and 4 are the same color. Without loss of generality, suppose that 1 and 4 are red, 2 is blue, 3 is green, and 10 is orange. Then $1+2+4+5=12$ implies that $\{5,12\}$ is colored green and orange. However, $1+3+4+5=13$ prevents 5 from being colored green. So, 5 is orange, 12 is green, and 13 is blue. Now $1+2+4+6=13$ uses at most 3 colors.
Case 4 Suppose that 1 and 10 are the same color. Without loss of generality, suppose that 1 and 10 are red, 2 is blue, 3 is green, and 4 is orange. Then $2+3+4+5=14$ implies that $\{5,14\}$ includes the color red, giving us two subcases.
Subcase 4.1 Suppose that 5 is red. Then $1+2+4+5=12$, $1+2+3+6=12$, and $1+3+4+6=14$ imply that 12 is green, 6 is orange, and 14 is blue. Now $1+2+5+6=14$ uses at most 3 colors.
Subcase 4.2 Suppose that 14 is red. Then $1+3+4+6=14$ implies that 6 is blue. Now $1+2+5+6=14$ uses at most 3 colors.
Case 5 Suppose that 2 and 3 are the same color. Without loss of generality, suppose that 2 and 3 are red, 1 is blue, 4 is green, and 10 is orange. Then $1+2+3+5=11$ implies that $\{5,11\}$ is colored green and orange. However, $2+3+4+5=14$ implies that 5 is not green. Thus, 5 is orange, 11 is green, and 14 is blue. Now $1+2+3+8=14$ uses at most 3 colors.
Case 6 Suppose that 2 and 4 are the same color. Without loss of generality, suppose that 2 and 4 are red, 1 is blue, 3 is green, and 10 is orange. Then $1+2+4+5=12$ implies that $\{5,12\}$ is colored green and orange. However, $2+3+4+5=14$ prevents 5 from being green. Thus, 5 is orange, 12 is green, and 14 is blue. Now $1+2+4+7=14$ uses at most 3 colors.
Case 7 Suppose that 2 and 10 are the same color. Without loss of generality, suppose that 2 and 10 are red, 1 is blue, 3 is green, and 4 is orange. Then $2+3+4+5=14$ implies that $\{5,14\}$ includes the color blue, giving us two subcases.
Subcase 7.1 Suppose that 5 is blue. Then $1+2+4+5=12$ implies that 12 is green, $1+2+3+6=12$ implies that 6 is orange, and $1+3+4+5=13$ implies that 13 is red. Now $1+2+4+6=13$ uses at most 3 colors.
Subcase 7.2 Suppose that 14 is blue. Then $1+3+4+6=14$ implies that 6 is red. Now $1+2+5+6=14$ uses at most 3 colors.
Case 8 Suppose that 3 and 4 are the same color. Without loss of generality, suppose

that 3 an 4 are red, 1 is blue, 2 is green, and 10 is orange. Then $2+3+4+5=14$ implies that $\{5,14\}$ is colored blue and orange. However, $1+3+4+5=13$ implies that 5 is not blue. Thus, 5 is orange, 14 is blue, and 13 is green. Now $1+3+4+6=14$ uses at most 3 colors.

Case 9 Suppose that 3 and 10 are the same color. Without loss of generality, suppose that 3 and 10 are red, 1 is blue, 2 is green, and 4 is orange. Then $2+3+4+5=14$ implies that $\{5,14\}$ includes the color blue, giving us two subcases.

Subcase 9.1 Suppose that 5 is blue. Then $1+3+4+5=13$, $1+2+4+6=13$, and $1+2+3+6=12$ imply that 13 is green, 6 is red, and 12 is orange. Then $1+2+4+5=12$ uses at most 3 colors.

Subcase 9.2 Suppose that 14 is blue. Then $1+3+4+6=14$ implies that 6 is green. Now $1+2+5+6=14$ uses at most 3 colors.

Case 10 Suppose that 4 and 10 are the same color. Without loss of generality, suppose that 4 and 10 are red, 1 is blue, 2 is green, and 3 is orange. Then $2+3+4+5=14$ implies that $\{5,14\}$ includes the color blue, giving us two subcases.

Subcase10.1 Suppose that 5 is blue. then $1+3+4+5=13$, $1+2+4+6=13$, $1+2+3+7=13$, and $1+3+4+6=14$ imply that 13 is green, 6 is orange, 7 is red, and 14 is green. Now $1+2+4+7=14$ uses 3 at most colors.

Subcase 10.2 Suppose that 14 is blue. Then $1+3+4+6=14$ implies that 6 is green. Now $1+2+5+6=14$ uses at most 3 colors.

In all cases, we find that there exists a strict 5-Schur solution that uses at most 3 colors, implying $WS_3^5(4) \leq 14$, and completing the proof.

4. Conclusion

The straight-forward methods used in Section 3 to evaluate $WS_{k-1}^{k+1}(k)$ when $k=3,4$ quickly become unreasonable for larger values of k due to the number of cases and subcases one must consider. Combining these values with the known strict Schur number $WS_1^3(2)=9$, we conjecture that

$$WS_{k-1}^{k+1}(k) = \frac{k^2-k+16}{2}, \quad \text{for all } k \geq 2,$$

but we are unable to offer a complete proof at this time. Besides this conjecture, we conclude by mentioning two specific variations of Schur numbers worthy of inquiry.

1. Weakened Ramsey numbers have been considered for colorings that avoid certain rainbow subgraphs (e.g., see [7] and [3]). In this sense, one could define the weakened Gallai-Schur number $GS_\ell^{k_1,k_2}(n)$ to be the least natural number N such that every n-coloring of $[1,N]$ that avoids rainbow k_1-Schur solutions contains a k_2-Schur solution that uses at most ℓ colors.

2. Consider a weakened version of the Rado numbers introduced in [6]. Specifically, instead of determining the presence of k-Schur solutions using at most ℓ colors, one can search for solutions to

$$x_1 + x_2 + \cdots + x_{k-1} < x_k, \quad \text{where } x_1 < x_2 < \cdots < x_k,$$

that use at most ℓ colors.

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