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BI-INVARIANT $(\alpha, \beta)-$METRICS ON LIE GROUPS

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Abstract. In this paper, we study the geometry of Lie groups with bi-invariant $(\alpha, \beta)-$metrics. We first show by an elementary proof that bi-invariant $(\alpha, \beta)-$metrics are of Berwald type. We give an explicit formula for the flag curvature of bi-invariant $(\alpha, \beta)-$metrics which improves the flag curvature formula of bi-invariant Randers metrics given in [10]. A necessary and sufficient condition that left invariant $(\alpha, \beta)-$metrics on Lie groups are bi-invariant is given.

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1. Introduction

The study of invariant structures on homogeneous spaces and Lie groups is an essential problem in differential geometry. Lie groups are, in a sense, the nicest examples of manifolds and are good spaces on which to test conjectures. Therefore it is important to study invariant Finsler metrics. S. Deng and Z. Hou studied invariant Finsler structure on homogeneous spaces and gave some descriptions of these metrics [5]. Basic notions and tools for invariant Finsler metrics on homogeneous manifolds were introduced by S. Deng and collaborators e.g. in [1, 5]. Also, in [9] the author have studied homogeneous geodesics in homogeneous Finsler spaces.

An important class of Finsler metrics is the family of $(\alpha, \beta)-$ metrics. An $(\alpha, \beta)-$ metric is a Finsler metric of the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx_i \otimes dx_j$ on a connected smooth $n-$dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M.

Bi-invariant metrics on Lie groups are among the simplest left invariant metrics. They have nice and simple geometric properties, but still form a large enough class to be of interest. In [5] S. Deng and Z. Hou gave an algebraic description of bi-invariant Finsler metrics and in [9] the author proved that the geodesics of a Lie group endowed with a bi-invariant Finsler metric, are the cosets of the one-parameter subgroups. Also in [9, 10, 11] the author have studied bi-invariant Rander metric and bi-invariant Finsler metrics.

The arrangement of this paper is as the following. In sec. 2, we present some preliminaries on invariant $(\alpha, \beta)-$metrics. In sec. 3, we study the geometric properties of bi-invariant $(\alpha, \beta)-$metrics on Lie groups. A necessary and sufficient condition that left-invariant $(\alpha, \beta)-$metrics are of Berwald type is given then with an elementary proof we prove that bi-invariant $(\alpha, \beta)-$metrics are of Berwald type. We give an explicit formula for the flag curvature of bi-invariant $(\alpha, \beta)-$metrics. A necessary and sufficient condition that left invariant $(\alpha, \beta)-$metrics on Lie groups are bi-invariant is given. This paper provided a convenient method to compute the flag curvature of bi-invariant $(\alpha, \beta)-$metrics on Lie groups.

2. Left invariant $(\alpha, \beta)-$metrics on Lie groups

A Finsler metric on a manifold M is a continuous function, $F : TM \longrightarrow [0, \infty)$ differentiable on $TM - \{0\}$ and satisfying three conditions:

(a) $F(y) = 0$ if and only if $y = 0$;

(b) $F(\lambda y) = \lambda F(y)$ for any $y \in T_xM$ and $\lambda > 0$;

(c) For any non-zero $y \in T_xM$, the symmetric bilinear form $g_y : T_xM \times T_xM \longrightarrow R$ given by

$$\mathrm{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]|_{s=t=0}$$

is positive definite.

For each $y \in T_xM - \{0\}$, define

$$C_y(u, v, w) = \frac{1}{4}\frac{\partial^3}{\partial s \partial t \partial r}[F^2(y + su + tv + rw)]\,|_{s=t=r=0}\ .$$

C is called the Cartan torsion.

Let π^*TM be the pull-back bundle of the of the tangent bundle TM by $\pi : TM - \{0\} \longrightarrow M$. The pull-back bundle π^*TM admits a unique linear connection, called the Chern connection which is torsion free and almost $g-$compatible. (see [2, 3, 7]). It is easy to see that torsion freeness is equivalent to the absence of dy^k terms in connection 1-forms,

$$\omega^i_j = \Gamma^i_{jk} dx^k,$$

together with the symmetry

$$\Gamma^i_{jk} = \Gamma^i_{kj}.$$

A Finsler metric F on a manifold M is called a Berwald metric if in any standard local coordinate system (x^i, y^i) in $TM - \{0\}$, the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are functions of $x \in M$ only, in which case, $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x$.

Let $V = V^i \frac{\partial}{\partial x^i}$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric g_V and a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as following:

$$g_V(X,Y) = X^i Y^j g_{ij}(x,V), \quad \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i}$$

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x,V) \frac{\partial}{\partial x^k}.$$

From the torsion freeness and $g-$compatibility of Chern connection we have

$$\nabla^V_X Y - \nabla^V_Y X = [X,Y],$$

$$X g_V(Y,Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2C_V(\nabla^V_X V, Y, Z),$$

where C denotes the Cartan tensor. The curvature tensor $R^V(X,Y)Z$ for vector fields X, Y, Z on $\mathcal{U}$ is defined by

$$R^V(X,Y)Z = \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.$$

For a Finsler manifold (M,F) and a flag $(X;P)$ consisting of a nonzero tangent vector $X \in T_xM$ and a plane $P \subset T_xM$ spanned by the tangent vector X, Y, the flag curvature is defined as

$$K(X;P) = \frac{g_X(R^X(Y,X)X,Y)}{g_X(X,X)g_X(Y,Y) - g_X(X,Y)}. \tag{1}$$

Definition 1. *Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta(x,y) = b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold M. Let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{2}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{3}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \le b < b_0. \tag{4}$$

Then by Lemma 1.1.2 of [3], F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (3) is called an $(\alpha,\beta)-$metric.

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T_x^*M such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T_x^*M induce a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{5}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$ (for more details see [4] and [17]). Therefore we can write $(\alpha, \beta)-$metrics as follows:

$$F(x, y) = \alpha(x, y)\phi(\frac{\tilde{a}(\tilde{X}(x), y)}{\alpha(x, y)}), \tag{6}$$

where for any $x \in M$, $\sqrt{\tilde{a}(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < b_0$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g} = T_eG$. A Finsler function $F : TG \longrightarrow R_+$ on Lie group G is called left-invariant if

$$F(x, y) = F(L_a(x), (L_a)_{*x}(y)),$$

for all $a, x \in G$ and $y \in T_xG$. Similarly, a Finsler metric is right-invariant if each $R_a : G \longrightarrow G$ is an isometry. A Finsler metric on G that is both left-invariant and right-invariant is called bi-invariant. About the existence of bi-invariant Finsler metrics on compact Lie groups we refer to [6]

Let G be a Lie group. Then there is a one-to-one correspondence between the left invariant Finsler metric on G and the Minkowski norm on $\mathfrak{g}$ [5].

In [1] An and Deng studied invariant $(\alpha, \beta)-$metrics on homogeneous manifolds. Suppose $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ is an left invariant Riemannian metric on G and

$$X \in V_\alpha = \{X \in \mathfrak{g} | \langle X, X\rangle < b_0^2, \forall h \in H\},$$

then the invariant 1-form $\beta = \beta(x, y)$ on G corresponding to X satisfies $\|\beta\|_\alpha = \sqrt{\tilde{a}^{ij}b_ib_j} < b_0$. If $\phi = \phi(s)$ is a C^∞ positive function on the symmetric open interval $I = (-b_0, b_0)$ satisfies

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad ,$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$, then $F = \alpha\phi(\frac{\beta}{\alpha})$ is an invariant $(\alpha, \beta)-$metric on G.

3. Bi-invariant $(\alpha,\beta)-$metrics on Lie groups

The following theorem is a simple reformation of the well-known characterization of $(\alpha,\beta)-$metrics being Berwald.

Proposition 1. *Let G be a Lie group with a left invariant (α,β)-metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X. Then the (α,β)-metric F is of Berwald type if and only if ad_X is skew-adjoint with respect to $\tilde{a}$ and $\tilde{a}(X,[\mathfrak{g},\mathfrak{g}]) = 0$.*

Proof: The proof is similar to the Randers space case [10].□

Corollary 1. *Let G be a Lie group with a left invariant (α,β)-metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X. If the (α,β)-metric F is of Berwald type then the one-parameter subgroup $t \longrightarrow \exp(tX)$ is a geodesic of F.*

Proof: The corollary is a consequence of Proposition 1 and Theorem 3.1 of [9].□

A geodesic $\gamma(t)$ through the origin o of $M = G/H$ is called homogeneous if it is an orbit of one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(o), \qquad t \in R,$$

where Z is a nonzero vector in the Lie algebra $\mathfrak{g}$ of G. For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [9].

Some Lie groups may posses a $(\alpha,\beta)-$metrics which is invariant not only under left translation but also under right translation. In the following theorem with an elementary proof we show that bi-invariant $(\alpha,\beta)-$metrics are of Berwald type.

Theorem 2. *Let G be a Lie group with a bi-invariant $(\alpha,\beta)-$metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X such that $\phi'(r) \neq 0$. Then the $(\alpha,\beta)-$metric F is of Berwald type.*

Proof: Let $F(x,y) = \alpha(x,y)\phi\left(\frac{\tilde{a}(X(x),y)}{\alpha(x,y)}\right)$, by using the formula

$$\mathrm{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}F^2(y+su+tv)|_{s=t=0},$$

and some computation for the $(\alpha,\beta)-$metric F we get

$$\begin{aligned} g_y(u,v) &= \tilde{a}(u,v)\phi^2(r)+\tilde{a}(y,u)\phi(r)\phi'(r)\Big(\frac{\tilde{a}(X,v)}{\sqrt{\tilde{a}(y,y)}}-\frac{\tilde{a}(X,y)\tilde{a}(y,v)}{(\tilde{a}(y,y))^{\frac{3}{2}}}\Big) \\ &+\Big((\phi'(r))^2+\phi(r)\phi''(r)\Big)\Big(\frac{\tilde{a}(X,v)}{\sqrt{\tilde{a}(y,y)}}-\frac{\tilde{a}(X,y)\tilde{a}(y,v)}{(\tilde{a}(y,y))^{\frac{3}{2}}}\Big) \\ &\times\Big(\tilde{a}(X,u)\sqrt{\tilde{a}(y,y)}-\frac{\tilde{a}(y,u)\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}\Big) \\ &+\frac{\phi(r)\phi'(r)}{\sqrt{\tilde{a}(y,y)}}\Big(\tilde{a}(X,u)\tilde{a}(y,v)-\tilde{a}(u,v)\tilde{a}(X,y)\Big), \end{aligned} \tag{7}$$

where $r=\frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}$. So for any $y,z\in\mathfrak{g}$ we have

$$\begin{aligned} g_y(y,[y,z]) &= \tilde{a}(y,[y,z])\Big(\phi^2(r)-\phi(r)\phi'(r)r\Big) \\ &+\tilde{a}(X,[y,z])\Big(\phi'(r)F(y)\Big). \end{aligned} \tag{8}$$

Since $\tilde{a}$ is bi-invariant $\tilde{a}(y,[y,z])=0$ and $ad(x)$ is skew adjoint for every $x\in\mathfrak{g}$. Since F is bi-invariant $g_y(y,[y,z])=0$ [9]. So from (8) we get $\tilde{a}(X,[y,z])=0$ for all $y,z\in\mathfrak{g}$. Therefore by Proposition 1 we see that (G,F) is of Berwald type.□

Corollary 3. *Let G be a Lie group with a bi-invariant (α,β)-metric F defined by the Riemannian metric $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$ and the vector field X such that $\phi^{'}(r)\neq 0$, then*
(a) The Chern connection of (G,F) is given by $\nabla_Y Z=\frac{1}{2}[Y,Z]$ for all $Y,Z\in\mathfrak{g}$.
(b) The geodesics of (G,F) starting at e are the one-parameter subgroup of G.

Theorem 4. *Let G be a Lie group with a bi-invariant $(\alpha,\beta)-$metric F defined by the Riemannian metric $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$ and the vector field X such that $\phi^{'}(r)\neq 0$. Let (P,y) be a flag in $\mathfrak{g}$ such that $\{y,u\}$ is an orthonormal basis of P with respect to $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$. Then the flag curvature of the flag (P,y) in $\mathfrak{g}$ is given by*

$$K(P,y)=\Big(\frac{\phi(r)-\phi^{'}(r)\tilde{a}(X,y)}{4\phi^2(r)\psi}\Big)\parallel[u,y]\parallel^2$$

where $\parallel[u,y]\parallel$ denotes the norm of $[u,y]$ with respect to $\tilde{a}=\tilde{a}_{ij}dx^i\otimes dx^j$, and $r=\tilde{a}(X,y)$ and $\psi=\phi(r)+\phi^{''}(r)r^2-\phi^{'}(r)r$.

Proof: By Corollary 3, $\nabla_Y Z=\frac{1}{2}[Y,Z]$, hence by the definition of the curvature we obtain

$$R(v,w)z=-\frac{1}{4}[[v,w],z], \tag{9}$$

for any $v, w, z \in \mathfrak{g}$. Also for the flag curvature we have

$$K(P, y) = \frac{g_y(R(u, y)y, u)}{g_y(y, y)g_y(u, u) - g_y^2(y, u)}. \tag{10}$$

According to the formula (7) we have

$$g_y(y, y) = \phi^2(r) = F^2(y) \tag{11}$$

$$\begin{aligned} g_y(u, u) &= \phi^2(r) + \left((\phi'(r))^2 + \phi(r)\phi''(r)\right) \tilde{a}^2(X, u) \\ &\quad -\phi(r)\phi'(r)\tilde{a}(X, y) \end{aligned} \tag{12}$$

$$g_y(y, u) = \phi(r)\phi'(r)\tilde{a}(X, u) \tag{13}$$

$$\begin{aligned} g_y(R(u, y)y, u) &= \tilde{a}(R(u, y)y, u)\phi^2(r) + \tilde{a}(y, R(u, y)y)\tilde{a}(X, u)\phi(r)\phi'(r) \\ &+ \left((\phi'(r))^2 + \phi(r)\phi''(r)\right) \tilde{a}(X, u)(\tilde{a}(X, R(v, w)z) - \tilde{a}(y, R(v, w)z)\tilde{a}(X, y)) \\ &-\phi(r)\phi'(r)\tilde{a}(R(v, w)z, u)\tilde{a}(X, y). \end{aligned} \tag{14}$$

Substituting (9), (11), (12), (13), (14) in equation (10) completes the proof.□

For results on flag curvature of invariant $(\alpha, \beta)-$metrics on homogeneous manifolds we refer to [15, 17]. A Finsler space with Finsler function:

$$F(x, y) = \alpha(x, y) + \beta(x, y)$$

is called a Randers space [8, 12, 16]. In [10] the author gives an explicit formula for the flag curvature of bi-invariant Rander metrics. In the following corollary we improve this formula.

Remark 1. *Let G be a Lie group with a bi-invariant Randers metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X. Let (P, y) be a flag in $\mathfrak{g}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$. Then the flag curvature of the flag (P, y) in $\mathfrak{g}$ is given by*

$$K(P, y) = \frac{1}{(1 + a(X, y))^2} \frac{1}{4} \parallel [u, y] \parallel^2$$

where $\parallel [u, y] \parallel$ denotes the norm of $[u, y]$ with respect to $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$.

We note that if the Randers space (G, F) is Riemannian, i.e. $X = 0$, then the above formula for the flag curvature is just the formula for the sectional curvature of a bi-invariant Riemannian metric [14]

$$K(u, y) = \frac{1}{4} \parallel [u, y] \parallel^2$$

A Finsler space having the Finsler function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)}$$

is called a Kropina space [1, 3].

Remark 2. *Let G be a Lie group with a bi-invariant Kropina metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field X. Let (P, y) be a flag in $\mathfrak{g}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$. Then the flag curvature of the flag (P, y) in $\mathfrak{g}$ is given by*

$$K(P, y) = \frac{1}{8} \tilde{a}^2(X, y) \parallel [u, y] \parallel^2$$

where $\parallel [u, y] \parallel$ denotes the norm of $[u, y]$ with respect to $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$.

A Finsler space having the Finsler function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)}$$

is called a Matsumoto space [1, 13]

Remark 3. *Let G be a Lie group with a bi-invariant Matsumoto metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field X. Let (P, y) be a flag in $\mathfrak{g}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$. Then the flag curvature of the flag (P, y) in $\mathfrak{g}$ is given by*

$$K(P, y) = \frac{(1 - \tilde{a}(X, y))^3 (1 - 2\tilde{a}(X, y))}{16\tilde{a}^2(X, y) - 12\tilde{a}(X, y) + 4} \parallel [u, y] \parallel^2$$

where $\parallel [u, y] \parallel$ denotes the norm of $[u, y]$ with respect to $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$.

In the following a necessary and sufficient condition that left invariant $(\alpha, \beta)-$metrics on Lie groups are bi-invariant is given.

Theorem 5. *Let G be a connected Lie group with a left invariant $(\alpha, \beta)-$metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X such that $\phi^{'}(r) \neq 0$. Then the $(\alpha, \beta)-$metric F is right invariant, hence bi-invariant if and only if $ad(x)$ is skew-adjoint with respect to the $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ for every $x \in \mathfrak{g}$ and $\tilde{a}(X, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

Proof: Suppose that $ad(x)$ is skew-adjoint for every $x \in \mathfrak{g}$ and $\tilde{a}(X, [\mathfrak{g}, \mathfrak{g}]) = 0$. We show that the left invariant $(\alpha, \beta)-$metric F is also right invariant. We note, by using Theorem 2 that (G, F) is of Berwald type. According to the formula (7) we get

$$\begin{aligned} g_y([z,u],v) &= \left(\phi^2(r) - \phi(r)\phi^{'}(r)r\right)\tilde{a}([z,u],v) \\ &+ \left(\phi(r)\phi^{'}(r) - (\phi^{'}(r)^2 + \phi(r)\phi^{''}(r))r\right)\left(\frac{\tilde{a}(X,v)}{\sqrt{\tilde{a}(y,y)}} - \frac{\tilde{a}(y,v)}{\tilde{a}(y,y)}r\right)\tilde{a}([z,u],y), \end{aligned} \tag{15}$$

and

$$\begin{aligned} g_y([z,v],u) &= \left(\phi^2(r) - \phi(r)\phi^{'}(r)r\right)\tilde{a}([z,v],u) \\ &+ \left(\phi(r)\phi^{'}(r) - (\phi^{'}(r)^2 + \phi(r)\phi^{''}(r))r\right)\left(\frac{\tilde{a}(X,u)}{\sqrt{\tilde{a}(y,y)}} - \frac{\tilde{a}(y,u)}{\tilde{a}(y,y)}r\right)\tilde{a}([z,v],y). \end{aligned} \tag{16}$$

Now by using the definition

$$C_y(z,u,v) = \frac{1}{2}\frac{d}{dt}[g_{y+tv}(z,u)]|_{t=0},$$

for the Cartan tensor we get

$$\begin{aligned} 2C_y([z,y],u,v) &= \left(\phi(r)\phi^{'}(r) - (\phi^{'}(r)^2 + \phi(r)\phi^{''}(r))r\right)\left(\frac{\tilde{a}(X,v)}{\sqrt{\tilde{a}(y,y)}} - \frac{\tilde{a}(y,v)}{\tilde{a}(y,y)}r\right)\tilde{a}([z,y],u) \\ &+ \left(\phi(r)\phi^{'}(r) - (\phi^{'}(r)^2 + \phi(r)\phi^{''}(r))r\right)\left(\frac{\tilde{a}(X,u)}{\sqrt{\tilde{a}(y,y)}} - \frac{\tilde{a}(y,u)}{\tilde{a}(y,y)}r\right)\tilde{a}([z,y],v). \end{aligned}$$

Therefore

$$g_y([z,u],v) + g_y(u,[z,v]) + 2C_y([z,y],u,v) =$$

$$(a([z,u],y)+a([z,y],u))\left(\phi(r)\phi^{'}(r)-(\phi^{'}(r)^2+\phi(r)\phi^{''}(r))r\right)\left(\frac{a(X,v)}{\sqrt{a(y,y)}}-\frac{a(v,y)}{a(y,y)}r\right)$$
$$+(a([z,v],y)+a([z,y],v))\left(\phi(r)\phi^{'}(r)-(\phi^{'}(r)^2+\phi(r)\phi^{''}(r))r\right)\left(\frac{a(X,u)}{\sqrt{a(y,y)}}-\frac{a(u,y)}{a(y,y)}r\right)$$
$$+(a([z,u],v)+a(u,[z,v]))\left(\phi^2(r)-\phi(r)\phi^{'}(r)r\right).$$

Since $ad(x)$ is skew-adjoint for every $x\in\mathfrak{g}$ we have

$$g_y([z,u],v)+g_y(u,[z,v])+2C_y([z,y],u,v)=0.$$

Now, we consider the function

$$\psi(t)=\mathrm{g}_{Ad(\exp(tz))y}(Ad(\exp(tz))u,Ad(\exp(tz)v)).$$

Taking the derivative with respect to t, we see that $\psi'(t)=0$ therefor $\psi(t)=\psi(0)$, $\forall t\in R$. Since G is connected we have

$$\mathrm{g}_y(u,v)=\mathrm{g}_{Ad(g)y}(Ad(g)u,Ad(g)v),\qquad \forall g\in G.$$

Using the formula $F(y)=\sqrt{g_y(y,y)}$, we have $F(Ad(g)u)=F(u)$ for all $g\in G$, $u\in\mathfrak{g}$. Therefore

$$F((R_{g^{-1}})_*u)=F(u),\qquad \forall g\in G,u\in\mathfrak{g}.$$

□

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