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Protecting Regular Polygons

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Abstract. The minimum number of mutually non-overlapping congruent copies of a convex body K so that they can touch K and prevent any other congruent copy of K from touching K without overlapping each other is called the protecting number of K. In this paper we prove that the protecting number of any regular polygon is three or four, and both values are indeed attained.

1. Introduction

The Newton number $N(K)$ of a convex body K is the maximum number of mutually non-overlapping congruent copies of K that can touch K. The problem of finding the Newton number of the d-dimensional ball B^d is a challenging open problem in discrete geometry for dimensions $d \geq 4$, $d \neq 8, 24$. Though the answer is trivial for the plane, $N(B^2) = 6$, it is well known that once the exact value of $N(B^3)$ was a point of controversy between Sir Isaac Newton and David Gregory. Newton conjectured that the answer is 12 while Gregory thought 13 is also possible. It took almost two hundred years before Hoppe [3] proved that Newton's conjecture, the '12', was correct. But even in the plane the question for $N(K)$ becomes nontrivial if $K \neq B^2$. The exact value of $N(K)$ is known only for some special

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convex discs K, e.g., for regular n-gons [1, 5, 7, 8, 10, 11, 12], for Reuleaux triangles [2], for certain isosceles triangles [2, 6, 10], and for certain rectangles [5]. Also, some estimates are given for more general classes of convex discs involving different parameters of the discs, e.g. diameter, width, etc. [2, 4, 9, 10].

In this paper we consider the counterpart of the Newton number problem, i.e., the problem of finding the minimum number of mutually non-overlapping congruent copies of a convex body K so that they can touch K and prevent any other congruent copy of K from touching K without overlapping each other. This quantity will be called the protecting number of K.

We note the existence of a more restrictive variant of the above problem, the so called blocking number problem introduced in [13]. The blocking number of a convex body K is the minimum number of mutually non-overlapping translates of K so that they can touch K and prevent any other translates of K from touching K without overlapping each other. It is proved in [13] that the blocking number of any plane convex body is four.

Turning back to the protecting number it is clear that the protecting number of B^2 is four. However, the protecting number problem is less trivial for convex bodies different from B^2. It is easy to see that the protecting number of any plane convex body K is at least three. Indeed, if K_1 and K_2 are two non-overlapping copies of K which touch K then the convex cone or strip containing K and surrounded by the lines separating K from K_1 and K_2, respectively, can always contain a third copy of K which touches K. In this paper we concentrate on regular polygons and first we prove

Theorem 1. *The protecting number of any regular polygon is not greater than four.*

This theorem together with the above observation shows that the protecting number of any regular polygon is three or four. We also prove that these values are indeed attained and, in addition, the value three is attained infinitely many times.

Theorem 2. *The protecting number of the square is four.*

Theorem 3. *The protecting number of the regular $6n$-gon is three for all positive integers n.*

Finally, we mention a rather surprising consequence of the last theorem. There exists a convergent sequence of convex discs (with respect to the usual Hausdorff metric) with protecting numbers three such that the protecting number of the limit disc is four. We note, however, that there do not exist convergent sequences of convex discs with protecting numbers four such that the protecting number of the limit disc is three. The proof of this latter statement is easy and is left to the reader.

2. Proof of Theorem 1

First, let K be a regular n-gon, where $n \geq 8$ and $n \neq 9, 13$, and let K_1, K_2, K_3, K_4 be the images of K under the reflections with respect to the sides of K with indices $\lfloor \frac{n}{4} \rfloor$, $\lfloor \frac{n}{2} \rfloor$, $\lfloor \frac{3n}{4} \rfloor$, n, respectively. Then K_1, K_2, K_3, K_4 are mutually non-overlapping. Consider a further congruent copy K' of K which does not overlap K, K_1, K_2, K_3, K_4. Then the incircle of K' also does not overlap K, K_1, K_2, K_3, K_4, of course. The angle between any two consecutive pairs of the halflines emanating at the center of K and going through the centers of K_1, K_2, K_3, K_4

is at most $\lceil \frac{n}{4} \rceil \frac{2\pi}{n}$ which is not greater than $\frac{3\pi}{5}$. Easy calculation shows that the distance between the centers of K and K' is greater than the diameter of the circumcircle of K which implies that K and K' are disjoint. Thus K_1, K_2, K_3, K_4 form a protecting system for K.

In Section 4 we will prove that the protecting number of the regular hexagon is three so it remains to show that the protecting number is not greater than four for $n = 3, 4, 5, 7, 9, 13$. Figure 2.1 shows protecting systems consisting of four elements in these cases. The proof of the fact that these systems are indeed protecting systems is similar to the previous argument and is left to the reader.

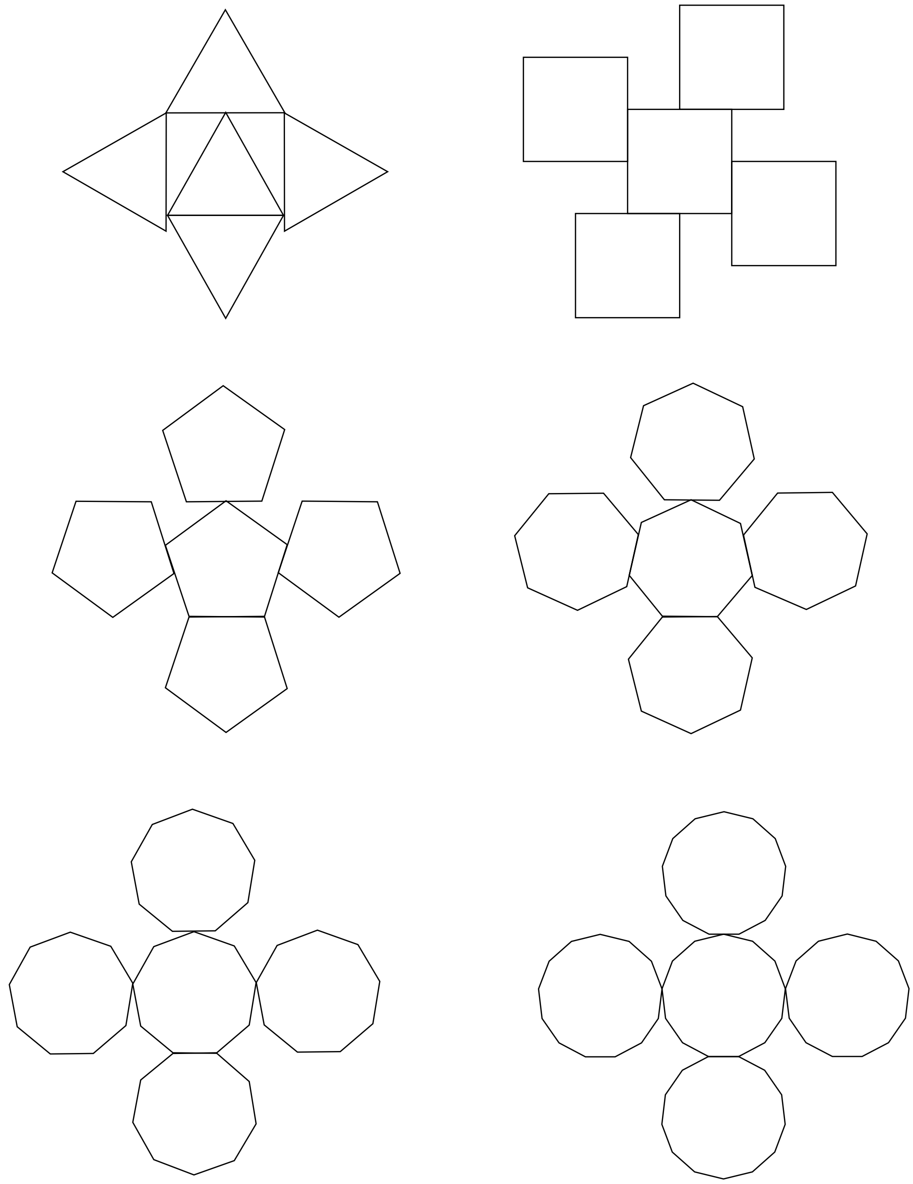

Figure 2.1 Protecting systems for regular 3-, 4-, 5-, 7-, 9-, 13-gons

3. Proof of Theorem 2

According to Theorem 1 it is enough to show that three unit squares cannot protect a unit square. Let S be the unit square with vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$ and suppose, for contradiction, that a collection $\mathcal{S} = \{S_1, S_2, S_3\}$ of three unit squares is a protecting system for S. Consider the four images of S under the reflections with respect to the points A, B, C, D. Each of these four squares is overlapped by the union of $\mathcal{S}$ since $\mathcal{S}$ is a protecting system for S. Furthermore there are two squares among the above four squares which are overlapped by the same member of $\mathcal{S}$. Without loss of generality we may assume that S_1 overlaps the images of S under the reflections with respect to the points C and D. Let E, F, G, H denote the vertices of S_1 in the counterclockwise order such that E lies in the relative interior of the side $\overline{CD}$.

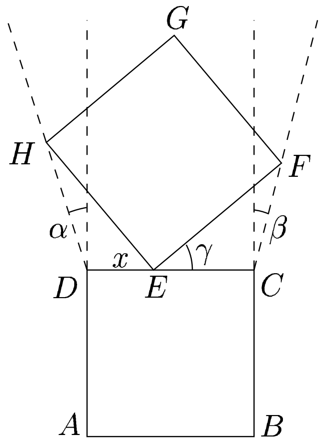

Figure 3.1

We claim that $\measuredangle HDC + \measuredangle DCF < \frac{5\pi}{4}$. For convenience, introduce the angles $\alpha = \measuredangle HDC - \frac{\pi}{2}$, $\beta = \measuredangle DCF - \frac{\pi}{2}$, $\gamma = \measuredangle FEC$ and let x denote the length of the segment $\overline{DE}$ (see Figure 3.1). Then

$$\tan\alpha = \frac{\sin\gamma - x}{\cos\gamma} \quad \text{and} \quad \tan\beta = \frac{\cos\gamma + x - 1}{\sin\gamma}$$

and thus

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
&= \frac{\sin\gamma(\sin\gamma - x) + \cos\gamma(\cos\gamma + x - 1)}{\sin\gamma\cos\gamma - (\sin\gamma - x)(\cos\gamma + x - 1)} \\
&= \frac{1 + x(\cos\gamma - \sin\gamma) - \cos\gamma}{x^2 - x + x(\cos\gamma - \sin\gamma) + \sin\gamma}\ .
\end{aligned}$$

It is enough to show that $\tan(\alpha+\beta) < 1$, i.e.,

$$1 + x(\cos\gamma - \sin\gamma) - \cos\gamma < x^2 - x + x(\cos\gamma - \sin\gamma) + \sin\gamma$$

or, equivalently,

$$x(1-x) < \sin\gamma + \cos\gamma - 1. \tag{1}$$

If we translate S_1 parallel to CD towards the midpoint of $\overline{CD}$ then the right hand side of inequality (1) does not change while the left hand side increases. If α or β becomes zero during this translation then we stop in that position. Otherwise we translate S_1 until E reaches the midpoint of $\overline{CD}$ and then we rotate S_1 around E clockwise if $\gamma \leq \frac{\pi}{4}$ and counterclockwise if $\gamma > \frac{\pi}{4}$ until α or β becomes zero. During this rotation the right hand side of inequality (1) decreases while the left hand side does not change. Thus it is enough to prove inequality (1) when α or β is zero. By symmetry, we may assume that $\alpha = 0$. Then inequality (1) is equivalent with the inequality

$$\sin\gamma(1-\sin\gamma) < \sin\gamma + \cos\gamma - 1,$$

i.e., $\cos^2\gamma < \cos\gamma$ which holds for $0 < \gamma < \frac{\pi}{2}$ trivially. This proves the claim.

Next consider the square S' with vertices $(0,1)$, $(-\sin\alpha, 1+\cos\alpha)$, $(-\sin\alpha-\cos\alpha, 1+\cos\alpha-\sin\alpha)$, $(-\cos\alpha, 1-\sin\alpha)$. Since S_1 does not overlap this square, S_2 or S_3, say S_2, overlaps it. Let J be that point of $\overline{AB}$ whose distance from A is $\sin\alpha(\sin\alpha+\cos\alpha)$. We show that the half line emanating at J and having direction vector $(-1,-1)$ does not intersect S_2. This trivially holds if S_2 has points on $\overline{CD}$ thus we have to deal with two cases only,

(1) a vertex of S_2 lies in the relative interior of $\overline{AD}$,
(2) A lies on the boundary of S_2.

Let K, L, M, N denote the vertices of S_2 in the counterclockwise order.

Case 1. Without loss of generality we may assume that K lies on $\overline{AD}$. For convenience, introduce the angle $\varphi = \measuredangle AKN$. If $\varphi > \frac{\pi}{4}$ then we are done. On the other hand, if $\varphi \leq \frac{\pi}{4}$ then $L = (-\cos\varphi, 1-x)$ for some $x < \sin\alpha$ since S_2 overlaps S', and thus $N = (-\sin\varphi, 1-x-\cos\varphi-\sin\varphi)$. The line going through N and having direction vector $(1,1)$ is a support line of S_2. The intersection point of this line with the line AB has coordinates $(0, x-1+\cos\varphi)$ and this point is on the left of J since

$$x-1+\cos\varphi < \sin\alpha - 1 + \cos\varphi \leq \sin\alpha \leq \sin\alpha(\sin\alpha+\cos\alpha).$$

Case 2. Without loss of generality we may assume that A lies on $\overline{KL}$. For convenience, introduce the angle $\varphi = \measuredangle DAL$. We will distinguish three cases, (2.1) $\varphi \leq \alpha$, (2.2) $\alpha < \varphi \leq \frac{\pi}{4}$, (2.3) $\frac{\pi}{4} < \varphi$.

Case 2.1. The line going through K and having direction vector $(1,1)$ is a support line of S_2. The intersection point of this line with the line AB has coordinates $(\overline{AK}(\sin\varphi+\cos\varphi), 0)$ and this point is on the left of J since $\sin\varphi+\cos\varphi \leq \sin\alpha+\cos\alpha$ and $\overline{AK} < \sin\alpha$. The first estimate follows from the fact that the function $\sin\varphi+\cos\varphi$ is monotone increasing on the interval $0 \leq \varphi \leq \frac{\pi}{4}$ while the second estimate follows from the fact that the vertex L must lie above the lowest vertex of S'.

Case 2.2. The line going through K and having direction vector $(1,1)$ is a support line of S_2. The intersection point of this line with the line AB has coordinates $(\overline{AK}(\sin\varphi+\cos\varphi), 0)$.

Now $\overline{AL}$ is greater than the distance of A from S' which is $\cos\alpha$ and thus $\overline{AK} < 1 - \cos\alpha$. On the other hand $\sin\varphi + \cos\varphi \leq \sqrt{2}$ holds trivially. To prove that the above intersection point is on the left of J it is enough to show that

$$\sqrt{2}(1 - \cos\alpha) < \sin\alpha(\sin\alpha + \cos\alpha)$$

or, equivalently,

$$\sqrt{2} < \sin^2\alpha + \sin\alpha\cos\alpha + \sqrt{2}\cos\alpha \qquad (2)$$

for $0 < \alpha < \frac{\pi}{4}$. The derivative of the right hand side of (2) is

$$\sin 2\alpha + \cos 2\alpha - \sqrt{2}\sin\alpha = \sqrt{2}\sin\left(2\alpha + \frac{\pi}{4}\right) - \sqrt{2}\sin\alpha > 0$$

for $0 < \alpha < \frac{\pi}{4}$ hence the right hand side of (2) is a monotone increasing function on the interval $0 \leq \alpha \leq \frac{\pi}{4}$. Taking the fact that the right hand side of (2) is $\sqrt{2}$ for $\alpha = 0$ into account the assertion follows.

Case 2.3. The line going through N and having direction vector $(1,1)$ is a support line of S_2. The distance of N from S' is greater than one since the second coordinate of the lowest vertex of S' is $1-\sin\alpha > 1-\frac{\sqrt{2}}{2}$ while the second coordinate of N is less than $-\sin\varphi \leq -\frac{\sqrt{2}}{2}$. Rotate S_2 around N counterclockwise until S_2 and S' touch each other. During this rotation S_2 cannot overlap S. Observe that after the rotation L cannot be on the boundary of S' otherwise $\measuredangle DLN$ would be smaller than $\frac{\pi}{2}$. However, $\measuredangle DLN \geq \measuredangle DLA + \measuredangle KLN > \frac{\pi}{2}$ since $\measuredangle LDA < \frac{\pi}{2}$ and $\overline{LD} \leq \overline{DA}$ in the triangle ADL. Therefore after the rotation the lowest vertex of S' lies on $\overline{LK}$ and the line LK does not intersect the interior of S. Now translate S_2 along the line LK until L coincides with the lowest vertex of S' and then rotate S_2 around L counterclockwise until S_2 touches S (see Figure 3.2).

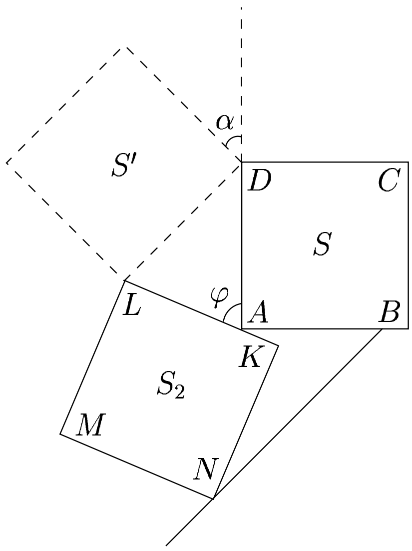

Figure 3.2

Note that during the above three transformations the line going through N and having direction vector $(1,1)$ remains a support line of S_2 and the first coordinate of the intersection point of this line with the line AB does not decrease. Moreover, after the transformations this intersection point has coordinates $(\sqrt{2}(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2})-1+\sin\alpha-\cos\alpha,0)$. To prove that this intersection point is on the left of J it is enough to show that

$$\sqrt{2}\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)-1+\sin\alpha-\cos\alpha<\sin\alpha(\sin\alpha+\cos\alpha)$$

or, equivalently,

$$\sqrt{2}\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)+(1-\cos\alpha)(\sin\alpha-\cos\alpha)<2$$

for $0<\alpha<\frac{\pi}{4}$. But this inequality holds trivially since $\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}<\sqrt{2}$ and $\sin\alpha-\cos\alpha\le 0$ for $0<\alpha<\frac{\pi}{4}$.

Thus the halfline emanating at that point of $\overline{AB}$ whose distance from A is $\sin\alpha(\sin\alpha+\cos\alpha)$ and having direction vector $(-1,-1)$ does not intersect S_2. Repeating the above argument with the square with vertices $(1,1)$, $(1+\cos\beta,1-\sin\beta)$, $(1+\sin\beta+\cos\beta,1+\cos\beta-\sin\beta)$, $(1+\sin\beta,1+\cos\beta)$ we conclude that the halfline emanating at that point of $\overline{AB}$ whose distance from B is $\sin\beta(\sin\beta+\cos\beta)$ and having direction vector $(1,-1)$ does not intersect S_3.

To complete the proof, i.e. to prove that $\mathcal{S}$ cannot be a protecting system for S, it is enough to show that

$$\sin\alpha(\sin\alpha+\cos\alpha)+\sin\beta(\sin\beta+\cos\beta)\le 1$$

if $\alpha+\beta\le\frac{\pi}{4}$. Indeed, in this case the square whose diameters are parallel to the coordinate axes and whose topmost vertex is J does not overlap the elements of $\mathcal{S}$. Since the function $\cos\beta+\sin\beta$ is monotone increasing on the interval $0\le\beta\le\frac{\pi}{4}$

$$\begin{aligned}\sin\beta(\sin\beta+\cos\beta)&\le\sin\left(\frac{\pi}{4}-\alpha\right)\left(\sin\left(\frac{\pi}{4}-\alpha\right)+\cos\left(\frac{\pi}{4}-\alpha\right)\right)\\&=\cos\alpha(\cos\alpha-\sin\alpha)\end{aligned}$$

from which the assertion follows. This finishes the proof of Theorem 2.

4. Proof of Theorem 3

We have already seen that the protecting number is at least three thus it is enough to show that three non-overlapping congruent copies K_1,K_2,K_3 of a regular $6n$-gon K can protect K. Let A_1,A_2,A_3 be the midpoints of the sides of K with indices $2n,4n,6n$, respectively, where the sides are counted counterclockwise. Set K_i such that the center O_i of K_i is collinear with A_i and the center O of K, and A_i is a vertex of K_i, $i=1,2,3$. Let K' be a further congruent copy of K which does not overlap K,K_1,K_2,K_3. Without loss of generality we may assume that the center O' of K' lies in the convex region bounded by the halflines emanating at O and going through O_1 and O_2, respectively. Let B_1 be the image of A_1 under the rotation around O_1 clockwise by an angle $\frac{\pi}{3}$. Similarly, let B_2 be the image of A_2 under the rotation around O_2 counterclockwise by an angle $\frac{\pi}{3}$. Then B_1 and B_2 are vertices of K_1

and K_2, respectively. Let r and R denote the radii of the incirle and the circumcircle of K, respectively. The incircle of K' does not overlap K_1 and K_2 hence the distance of O' from the triangles $A_1B_1O_1$ and $A_2B_2O_2$ is at least r. Therefore the distance of O' from K is at least R with equality if and only if O' coincides with the intersection point of the lines O_1B_1 and O_2B_2. In the case of equality B_1 and B_2 are midpoints of sides of K' which implies that $\overline{OO'}$ goes through the midpoint of a side of K'. Thus K' is disjoint from K, i.e., K_1, K_2, K_3 protect K.

5. Concluding remarks

In this paper we have proved that the protecting number of any regular polygon is three or four, and both of these values are indeed attained. We conjecture that this statement remains true for the class of all plane convex bodies as well.

Conjecture 1. *The protecting number of any plane convex body is three or four.*

Finally we note that the protecting number problem seems to be interesting for not necessarily convex discs as well (a disc is a subset of the plane homeomorphic to B^2). It is easy to see that the protecting number of any disc is at least two. On the other hand, certain discs can be protected by two non-overlapping congruent copies of themselves (see Figure 5.1).

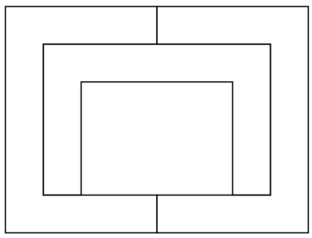

Figure 5.1 Non-convex disc with protecting number two

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