Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 47 (2006), No. 1, 1-14.

Wild Kernels for Higher K-theory of Division and Semi-simple Algebras

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Abstract. Let Σ be a semi-simple algebra over a number field F. In this paper, we prove that for all $n \geq 0$, the wild kernel $WK_n(\Sigma) := Ker(K_n(\Sigma) \longrightarrow \prod_{\text{finite } v} K_n(\Sigma_v))$ is contained in the torsion part of the image of the natural homomorphism $K_n(\Lambda) \longrightarrow K_n(\Sigma)$, where Λ is a maximal order in Σ. In particular, $WK_n(\Sigma)$ is finite. In the process, we prove that if Λ is a maximal order in a central division algebra D over F, then the kernel of the reduction map $K_{2n-1}(\Lambda) \xrightarrow{\pi_v} \prod_{\text{finite } v} K_{2n-1}(d_v)$ is finite. In Section 3 we investigate the connections between $WK_n(D)$ and $\mathrm{div}(K_n(D))$ and prove that $\mathrm{div} K_2(\Sigma) \subset WK_2(\Sigma)$; if the index of D is square free, then $\mathrm{div}(K_2(D)) \simeq \mathrm{div}(K_2(F))$, $WK_2(F) \simeq WK_2(D)$ and $|WK_2(D)/\mathrm{div}(K_2(D))| \leq 2$. Finally we prove that if D is a central division algebra over F with $[D : F] = m^2$, then (1) $\mathrm{div}(K_n(D))_l = WK_n(D)_l$ for all odd primes l and $n \leq 2$; (2) if l does not divide m, then $\mathrm{div}(K_3(D))_l = WK_3(D)_l = 0$; (3) if $F = \mathbb{Q}$ and l does not divide m, then $\mathrm{div}(K_n(D))_l \subset WK_n(D)_l$ for all n.

MSC 2000: 19C99, 19F27, 11S45
Keywords: wild kernel, K_n group, semi-simple algebra

1. Introduction

Let F be a number field and R the ring of integers of F. For all $n \geq 0$, the wild kernel $WK_n(F)$ is defined in [4] by

$$WK_n(F) := Ker(K_n(F) \longrightarrow \prod_{\text{finite } v} K_n(F_v)),$$

where v runs through all the finite places of F and F_v is the completion of F at v. In Proposition A of [4], it is proved that $WK_n(F)$ is contained in the torsion part of $K_n(R)$ and in particular that $WK_n(F)$ is finite.

In this paper, we at first generalize this result to the non-commutative case. Let D be a central division algebra over F and let Λ be a maximal R-order in D. We define the wild kernel $WK_n(D)$ of D to be the kernel of

$$K_n(D) \longrightarrow \prod_{\text{finite } v} K_n(D_v),$$

and prove that $WK_n(D)$ is a finite group for all $n \geq 0$. We shall denote by $W^{'}K_n(D)$ the kernel of

$$K_n(D) \longrightarrow \prod_{\text{non complex } v} K_n(D_v),$$

which is a generalization of the definition

$$W^{'}K_2(F) = Ker(K_2(F) \longrightarrow \prod_{\text{non complex } v} K_2(F_v))$$

given in [3], with the observation that $W^{'}K_n(D)$ is a subgroup of $WK_n(D)$ for any $n \geq 0$. We shall refer to $W^{'}K_n(D)$ as pseudo-wild-kernel of D.

The above definitions of $WK_n(D)$ and $W^{'}K_n(D)$ extend naturally to $WK_n(\Sigma)$ and $W^{'}K_n(\Sigma)$, where Σ is a semi-simple algebra over F.

Soulé had proved in [22] that the natural homomorphism

$$K_n(R) \longrightarrow K_n(F)$$

is always injective for all $n \geq 1$. However this is not true in the non-commutative case since if n is odd and Λ is a maximal order in a semi-simple F-algebra Σ, then

$$K_n(\Lambda) \longrightarrow K_n(\Sigma)$$

is not always injective (cf. Theorem 2 of [11]). So $K_n(\Lambda)$ can not be regarded as a subgroup of $K_n(\Sigma)$ if n is odd, even though it is known that

$$SK_n(\Lambda) := Ker(K_n(\Lambda) \longrightarrow K_n(\Sigma))$$

is finite for all $n \geq 0$ and $SK_{2n}(\Lambda) = 0$ (see [13], [14]). We apply these considerations to the proof of finiteness of $WK_{2n-1}(D)$ (cf. Proposition 2.3).

Let R be the ring of integers in a number field F. In [1], Arlettaz and Banaszak proved that the kernel of the reduction map

$$K_{2n-1}(R) \xrightarrow{\pi_v} \prod_{\text{finite } v} K_{2n-1}(k_v)$$

is finite, where k_v is the residue field of R at the finite place v. First we generalize this result to the non-commutative case. Let D be a central division algebra over a number field F and Λ a maximal R-order in D. Then for any finite place v of F, the residue ring of Λ_v is a matrix ring over d_v, where d_v is a finite field extension of k_v (see [20], IV, Theorem 5.9). We prove (Theorem 2.2) that the kernel of the reduction map

$$K_{2n-1}(\Lambda) \xrightarrow{\pi_v} \prod_{\text{finite } v} K_{2n-1}(d_v)$$

is finite, and then deduce that the kernel of

$$K_{2n-1}(\Lambda) \longrightarrow K_{2n-1}(\Lambda_v)$$

is finite.

By making use of Theorem 2.2, we prove that for all $n \geq 1$, $WK_{2n-1}(D)$ is finite if D is a central division algebra over F (Proposition 2.3) and that also $WK_{2n}(D)$ is finite (see Proposition 2.4). We then generalize these two results to the case of semi-simple algebras in Theorem 2.5.

In [4], Banaszak et al. conjectured that for all number fields F and all $n \geq 0$, we should have

$$WK_n(F)_l = \operatorname{div}(K_n(F))_l.$$

They proved that under certain hypotheses, the above conjecture is equivalent to the Quillen-Lichtenbaum Conjecture (see Theorem C of [4]). They also proved that the above conjecture holds for all number fields F for $0 \leq n \leq 3$ and that when $F = \mathbb{Q}$, the conjecture is true for $n = 4$. Motivated by above results and considerations, we investigate the connection between $WK_n(D)$, $W'K_n(D)$ and $\operatorname{div}(K_n(D))$ in Section 3 of this paper.

At the beginning of Section 3, we prove (Theorem 3.1) that if F is a number field and Σ is a semi-simple algebra over F, then $WK_n(\Sigma)/W'K_n(\Sigma)$ is a finite 2-group with 8 rank 0 if $n \equiv 0,\ 4,\ 6 \pmod 8$, and with 16 rank 0 if $n \equiv 2 \pmod 8$. We also prove that if D is a central division algebra over a number field F, then $\operatorname{div}(K_2(D)) \subset WK_2(D)$. If the index of D is square free, then $\operatorname{div}(K_2(D)) \simeq \operatorname{div}(K_2(F))$, $WK_2(F) \simeq WK_2(D)$ and $|WK_2(D)/\operatorname{div}(K_2(D))| \leq 2$. This result is then extended to semi-simple algebras Σ (Theorem 3.3).

Finally we prove that if D is a central division algebra over number field F with $[D : F] = m^2$, then

(1) $\operatorname{div}(K_n(D))_l = WK_n(D)_l$ for all odd primes l and $n \leq 2$;

(2) if l does not divide m, then $\operatorname{div}(K_3(D))_l = WK_3(D)_l = 0$;

(3) if $F = \mathbb{Q}$ and l does not divide m, then $\operatorname{div}(K_n(D))_l \subset WK_n(D)_l$ for all n.

We conjecture that $\operatorname{div}(K_n(\Sigma)) \subset WK_n(\Sigma)$ for all n and all semi-simple algebras Σ.

Notes on Notations

If F is a number field, we shall write D for a central division algebra over F, Σ for a semi-simple algebra over F, Λ for a maximal order in D or Σ and $\Lambda_v,\ D_v,\ \Sigma_v$ for the completions of $\Lambda,\ D,\ \Sigma$ respectively at a place v of F.

For any ring S, define $K_n(S) = \pi_{n+1}(BQ(\mathbb{P}(S)))$ for all $n \geq 0$ (cf. [19]), where $\mathbb{P}(S)$ is the category of finitely generated projective S-modules, or $K_n(S) = \pi_n(BGL(S)^+)$ for $n \geq 1$ (cf. [15]). For an abelian group G, we shall write $\mathrm{div}(G)$ for $\bigcap\limits_{n\geq 1} G^n$ and G_l for $\bigcup\limits_{k\geq 1} G[l^k]$, the l-torsion subgroup of G, where $G[l^k] = \{g \in G | g^{l^k} = 1\}$. Call $\mathrm{div}(G)$ the subgroup of divisible elements of G. The group $SK_n(\Lambda)$ is defined for all $n \geq 1$ by $SK_n(\Lambda) := Ker(K_n(\Lambda) \longrightarrow K_n(\Sigma))$, where Λ is a maximal order in Σ.
The wild kernel $WK_n(\Sigma) := Ker(K_n(\Sigma) \longrightarrow \prod\limits_{\text{finite } v} K_n(\Sigma_v))$, and the pseudo-wild-kernel is $W'K_n(\Sigma) := Ker(K_n(\Sigma) \longrightarrow \prod\limits_{\text{non complex } v} K_n(\Sigma_v))$. For any group G, we shall write $|G|$ for the number of elements in G.

2. The wild kernel $WK_n(D)$ for central division algebras D

The aim of this section is to prove 2.2–2.5 below. However we start with proof of Lemma 2.1 which is used to prove the other results. We observe that Lemma 2.1 is proved in the K_2 case in [9].

Lemma 2.1. *Let D be a division algebra of dimension m^2 over its center F. For $n \geq 0$, let*

$$i_n : \ K_n(F) \longrightarrow K_n(D)$$

be the homomorphism induced by the inclusion map $i : \ F \hookrightarrow D$; and

$$tr_n : \ K_n(D) \longrightarrow K_n(F)$$

the transfer map. Then for all $n \geq 0$, each of $i_n \circ tr_n$ and $tr_n \circ i_n$ is multiplication by m^2.

Proof. Every element d of D acts on the vector space D of dimension m^2 over F via left multiplication, i.e., there is a natural inclusion

$$t : \ D \longrightarrow M_{m^2}(F).$$

This inclusion induces the transfer homomorphism of K-groups

$$t_n : \ K_n(D) \longrightarrow K_n(M_{m^2}(F)) \simeq K_n(F).$$

The composition of t with $i : \ F \hookrightarrow D$, namely,

$$F \xrightarrow{i} D \xrightarrow{t} M_{m^2}(F)$$

is diagonal, i.e.,

$$t \circ i(x) = diag(x, x, \dots, x).$$

So by Lemma 1 of [7], $tr_n \circ i_n$ is multiplication by m^2.

The composition

$$D \xrightarrow{t} M_{m^2}(F) \xrightarrow{i} M_{m^2}(D)$$

is not diagonal. But we will prove that it is equivalent to the diagonal map. By Noether-Skolem Theorem, there is an inner automorphism φ such that the following diagram commutes,

$$\begin{tikzcd} D \arrow[r, "t"] \arrow[d, "diag"'] & M_{m^2}(F) \arrow[d, "i"] \\ M_{m^2}(D) \arrow[r, "\varphi"'] & M_{m^2}(D) \end{tikzcd}$$

where *diag* is the diagonal map. By the Lemma 2 of [7], the induced homomorphism $K_n(\varphi)$ is an identity. So $i_n \circ tr_n$ is multiplication by m^2, also by Lemma 1 of [7]. □

Theorem 2.2. *Let F be a number field and D a central division algebra of dimension m^2 over F. Let R be the ring of integers of F and Λ a maximal R-order in D. For any place v of F, let k_v be the residue ring of R at v. Then the residue ring of Λ_v is a matrix ring over d_v, where d_v is a finite field extension of k_v and the kernel of the reduction map*

$$K_{2n-1}(\Lambda) \xrightarrow{(\pi_v)} \prod_{\text{finite } v} K_{2n-1}(d_v)$$

is finite. Hence the kernel of

$$K_{2n-1}(\Lambda) \xrightarrow{(\varphi)} \prod_{\text{finite } v} K_{2n-1}(\Lambda_v)$$

is finite.

Proof. It is well known that the residue ring of Λ_v is a matrix ring over d_v, where d_v is a finite field extension of k_v (see [11], [20]).

By [14] or [13], $K_{2n-1}(\Lambda)$ is finitely generated. So it suffices to prove that the kernel of the reduction map is a torsion group, in order to show that it is finite.

Let

$$i: \ K_{2n-1}(R) \longrightarrow K_{2n-1}(\Lambda)$$

be the homomorphism induced by inclusion and let

$$tr: \ K_{2n-1}(\Lambda) \longrightarrow K_{2n-1}(R)$$

be the transfer homomorphism. Then

$$i \circ tr(x) = x^{m^2}$$

for any $x \in K_{2n-1}(\Lambda)$ by a suitable modification of the proof of Lemma 2.1 above.

So if there is a torsion free element $x \in Ker(\pi_v)$, then $tr(x)$ is a torsion free element in $K_{2n-1}(R)$. Consider the following commutative diagram

$$\begin{tikzcd} K_{2n-1}(R) \arrow[r, "(\pi_v')"] \arrow[d, "\iota"'] & \prod_{\text{finite } v} K_{2n-1}(k_v) \arrow[d, "(\iota_v)"] \\ K_{2n-1}(\Lambda) \arrow[r, "(\pi_v)"'] & \prod_{\text{finite } v} K_{2n-1}(d_v)\,, \end{tikzcd}$$

By Theorem 1 of [1], the kernel of $(\pi_v^{'})$ is finite. So $(\iota_v) \circ (\pi_v^{'}) \circ tr(x)$ is torsion free. But $x \in ker(\pi_v)$ and so $(\iota_v) \circ (\pi_v^{'}) \circ tr(x)$ must be 0 since from the above diagram

$$(\iota_v) \circ (\pi_v^{'}) \circ tr(x) = (\pi_v)(x^{m^2}) = 0.$$

This is a contradiction. Hence $Ker(\pi_v)$ is finite.

The last statement follows from the following commutative diagram

$$\begin{tikzcd} K_{2n-1}(\Lambda) \arrow[rr] \arrow[dr, "\pi_v"'] & & \prod_{\text{finite } v} K_{2n-1}(\Lambda_v) \arrow[dl, "\varphi_v"] \\ & \prod_{\text{finite } v} K_{2n-1}(d_v) & \end{tikzcd}$$

and the fact that $Ker(\pi_v)$ is finite (as proved above). □

Proposition 2.3. *Let F be a number field, D a central division algebra over F. Then the wild kernel $WK_{2n-1}(D)$ is finite.*

Proof. By the Theorem 2 of [11], the following sequence is exact

$$0 \longrightarrow \bigoplus_{\text{finite } v} K_{2n-1}(d_v)/K_{2n-1}(k_v) \longrightarrow K_{2n-1}(\Lambda) \longrightarrow K_{2n-1}(D) \longrightarrow 0. \qquad \text{(I)}$$

Since $K_{2n-1}(d_v)/K_{2n-1}(k_v)$ is trivial for almost all v, it follows that

$$\bigoplus_{\text{finite } v} K_{2n-1}(d_v)/K_{2n-1}(k_v)$$

is a finite group. Kuku had proved in [13] and [14] that $K_{2n-1}(\Lambda)$ is finitely generated. So $K_{2n-1}(D)$ is finitely generated which implies that $WK_{2n-1}(D)$ is finitely generated. So it suffices to prove that $WK_{2n-1}(D)$ is a torsion group.

If $x \in WK_{2n-1}(D) \subset K_{2n-1}(D)$ is torsion free, then from (I) above, we can find an element $x_1 \in K_{2n-1}(\Lambda)$ such that the image of x_1 under the homomorphism

$$i: \ K_{2n-1}(\Lambda) \longrightarrow K_{2n-1}(D)$$

is x, and x_1 is also torsion free. By Theorem 2.2, the kernel of the composite of the following maps

$$K_{2n-1}(\Lambda) \longrightarrow \prod_{\text{finite } v} K_{2n-1}(\Lambda_v) \longrightarrow \prod_{\text{finite } v} K_{2n-1}(d_v)$$

is finite. If x_2 is the image of x_1 in $\prod\limits_{\text{finite } v} K_{2n-1}(\Lambda_v)$, then x_2 is torsion free. Consider the following commutative diagram (II) with the maps of elements illustrated in diagram (III)

$$\begin{array}{ccc} K_{2n-1}(\Lambda) & \longrightarrow & \prod\limits_{\text{finite } v} K_{2n-1}(\Lambda_v) \\ \big\downarrow & & \big\downarrow \\ K_{2n-1}(D) & \longrightarrow & \prod\limits_{\text{finite } v} K_{2n-1}(D_v)\,, \end{array} \tag{II}$$

$$\begin{array}{ccc} x_1 & \longrightarrow & x_2 \\ \big\downarrow & & \big\downarrow \\ x & \longrightarrow & x_3\,, \end{array} \tag{III}$$

where x_3 is the image of x_2 in $\prod\limits_{\text{finite } v} K_{2n-1}(D_v)$. Since D is ramified at finitely many places of F, $k_v = d_v$ for almost all v. So $K_{2n-1}(\Lambda_v) \simeq K_{2n-1}(D_v)$ for almost all v by Theorem 1 of [11]. Hence the kernel of the right vertical arrow in diagram (II) is finite. So x_3 is torsion free. However $x \in WK_{2n-1}(D)$ and so $x_3 = 0$. This is a contradiction. Hence $WK_{2n-1}(D)$ is finite. $\square$

Proposition 2.4. *Let F be a number field and D a central division algebra over F. Then for all $n \geq 0$ the wild kernel $WK_{2n}(D)$ is contained in the image of $K_{2n}(\Lambda) \longrightarrow K_{2n}(D)$. In particular, $WK_{2n}(D)$ is finite.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & WK_{2n}(D) & \longrightarrow & K_{2n}(D) & \xrightarrow{f} & \prod\limits_{v} K_{2n}(D) \\ & & \big\downarrow & & \big\| & & \big\downarrow \tau \\ 0 & \longrightarrow & K_{2n}(\Lambda) & \longrightarrow & K_{2n}(D) & \xrightarrow[g]{} & \prod\limits_{v} K_{2n-1}(d_v)\,, \end{array}$$

where the middle vertical arrow is an identity. By this commutative diagram,

$$g = \tau \circ f$$

which implies $\ker f \subset \ker g$. So $WK_{2n}(D) \subset K_{2n}(\Lambda)$.
Let

$$tr: \ K_{2n}(\Lambda) \longrightarrow K_{2n}(R)$$

be the transfer homomorphism and let

$$i: \ K_{2n}(R) \longrightarrow K_{2n}(\Lambda)$$

be the homomorphism induced by the inclusion. Then for any $x \in K_{2n}(\Lambda)$,

$$i \circ tr(x) = x^{m^2},$$

where m^2 is the dimension of D over F. Since $K_{2n}(R)$ is a torsion group, $K_{2n}(\Lambda)$ is also a torsion group. So it must be finite which implies $WK_{2n}(D)$ is finite. □

Theorem 2.5. *Let Σ be a semi-simple algebra over a number field F. Then the wild kernel $WK_n(\Sigma)$ is contained in the torsion part of the image of the homomorphism*

$$K_n(\Lambda) \longrightarrow K_n(\Sigma),$$

where Λ is a maximal order of Σ. In particular, $WK_n(\Sigma)$ is finite.

Proof. Assume $\Sigma = \prod_{i=1}^{k} M_{n_i}(D_i)$, where D_i is a finite dimensional F-division algebra with center E_i. Let Λ be a maximal order of Σ. We know that $\Lambda = \prod_{i=1}^{k} M_{n_i}(\Lambda_i)$, where Λ_i is maximal order of D_i. So $WK_n(\Sigma) = \prod_{i=1}^{k}(WK_n(D_i))$ and $K_n(\Lambda) = \prod_{i=1}^{k} K_n(\Lambda_i)$. This theorem follows from Proposition 2.3 and 2.4. □

3. Connections between $WK_n(D)$, $W'K_n(D)$ and $\mathrm{div}(K_n(D))$

Theorem 3.1. *Let F be a number field and Σ a semi-simple algebra over F. Then $WK_n(\Sigma)/W'K_n(\Sigma)$ is a finite 2-group with 8-rank 0 if $n \equiv 0,\ 4,\ 6 \pmod 8$, and with 16-rank 0 if $n \equiv 2 \pmod 8$.*

Proof. (1) If $n \equiv 0,\ 4,\ 6 \pmod 8$, then $K_n(\mathbb{R})$ is a uniquely divisible group by Corollary 2.9.2 of [23]. Let D be a central division algebra over F. If D is not ramified at a real place v, then $D_v = D \otimes F_v = D \otimes \mathbb{R}$ is a matrix ring over $\mathbb{R}$. So $K_n(D_v) \simeq K_n(\mathbb{R})$ is a uniquely divisible group. Since $K_n(F)$ is a torsion group for even n (see [5] or [6]) , then by Lemma 2.1 $K_n(D)$ is also a torsion group for even n. The image of a torsion element in a uniquely divisible group must be 0. So, if D is not ramified at a real place v, then the map

$$K_n(D) \longrightarrow K_n(D_v) \simeq K_n(\mathbb{R})$$

is 0. If D is ramified at a real place v, then D_v is a matrix ring over the Hamilton quaternion algebra $\mathbb{H}$. For any torsion element $x \in K_n(\mathbb{H})$, we have $x^4 = 0$ by Lemma 2.1. So if D is ramified at a real place v, then every element of the image of

$$K_n(D) \longrightarrow K_n(D_v) \simeq K_n(\mathbb{H})$$

is a 4-torsion element. Using the same arguments as in the proof of Theorem 2.5, we know the image of

$$K_n(\Sigma) \longrightarrow \prod_{\text{real places } v} K_n(\Sigma_v)$$

is a finite 2-group with 8-rank 0. By the definitions of wild kernel and pseudo-wild-kernel, $WK_n(\Sigma)/W'K_n(\Sigma)$ is isomorphic to a subgroup of the image of

$$K_n(\Sigma) \longrightarrow \prod_{\text{real places } v} K_n(\Sigma \otimes \mathbb{R}).$$

So $WK_n(\Sigma)/W'K_n(\Sigma)$ is a finite 2-group with 8-rank 0 if $n \equiv 0,\ 4,\ 6 \pmod 8$.

If $n \equiv\ 2 \pmod 8$, then the torsion part of $K_n(\mathbb{R})$ is $\mathbb{Z}/2\mathbb{Z}$. Using the same arguments as above, we have $WK_n(\Sigma)/W'K_n(\Sigma)$ is a finite 2-group with 16-rank 0. □

Theorem 3.2. *Let F be a number field and D a central division algebra over F. Then*

(1) $\operatorname{div}(K_2(D)) \subset WK_2(D)$.

If the index of D is square free, then

(2) $\operatorname{div}(K_2(D)) \simeq \operatorname{div}(K_2(F))$,

(3) $WK_2(D) \simeq WK_2(F)$,

(4) $|WK_2(D)/\operatorname{div}(K_2(D))| \leq 2$.

Proof. (1) Let v be a non-complex place of F. It is known that $K_2(F_v)$ is a direct sum of cyclic group r and an infinitely divisible torsion free group $(K_2(F_v))^s$, where s is the number of roots of unity of F (cf. Theorem A.14 of [18]). Let E be the maximal divisible subgroup of $K_2(D_v)$. By Theorem 3 in §5 of [10], E is a direct summand of $K_2(D_v)$, i.e., there is a subgroup T such that $K_2(D_v) = E \oplus T$. By Lemma 2.1, the reduced norm Nrd_2 induces an isomorphism $E \simeq (K_2(F_v))^s$. So T must be a torsion group. If $[D : F] = m^2$, then $T^{sm^2} = 1$ by Lemma 2.1.

Consider the following commutative diagram

$$\begin{tikzcd}
K_2(D) \arrow[r, "(i_D)"] \arrow[d, "\mathrm{Nrd}_2"'] & \prod_{\text{non complex } v} K_2(D_v) \arrow[d, "(\mathrm{Nrd}_2^v)"] \\
K_2(F) \arrow[r, "(i_F)"'] & \prod_{\text{non complex } v} K_2(F_v),
\end{tikzcd}$$

where the map Nrd_2 and (Nrd_2^v) are the reduced norms, (i_D) and (i_F) are the homomorphisms induced by the inclusion. For any $x \in \mathrm{div}(K_2(D))$, $(i_D)(x)$ is divisible and torsion in $\prod_{\text{non complex } v} K_2(D_v)$ since $K_2(D)$ is a torsion group. So $(i_D)(x) = 0$ which implies $\mathrm{div}(K_2(D)) \subset WK_2(D)$.

(2) If the index of D is square free, then by [17], Proposition 26.6 and Theorem 26.7 of [24], Nrd_2 is injective. So we need only to prove that the restriction map

$$\mathrm{div}(K_2(D)) \longrightarrow \mathrm{div}(K_2(F))$$

is surjective.

Since $K_2(F)$ is a torsion group and $K_2(\mathbb{R})$ is the direct sum of a divisible group and $\mathbb{Z}/2\mathbb{Z}$, we have

$$\mathrm{div}(K_2(F)) \subset Ker(K_2(F) \longrightarrow \bigoplus_{\text{real ramified } v} \mathbb{Z}/2\mathbb{Z}).$$

Let K_2^+F be the subgroup of K_2F generated by the Steinberg symbols $\{a,\ b\}$ with $a \in F^*$ and $b \in F^+ = \{b \in F | v(b) > 0$ for all real places v such that D is ramified at $v\}$. Since every element of F^+ is a norm of some element of D^*, K_2^+F is generated by the Steinberg symbols $\{a,\ n(d)\}$ with $a \in F^*$ and $d \in D^*$, where n is the reduced norm of D. By Theorem 1 of [2] and Theorem 2.2 of [8], the image of the reduced norm Nrd_2 is

$$K_2^+F = Ker(K_2(F) \longrightarrow \bigoplus_{\text{real ramified } v} \mathbb{Z}/2\mathbb{Z}).$$

So

$$\mathrm{div}(K_2(F)) \subset \mathrm{Nrd}_2(K_2(D)).$$

Since

$$K_2(F) \longrightarrow \bigoplus_{\text{real ramified } v} \mathbb{Z}/2\mathbb{Z}$$

is split (cf. 2.1 of [12]), we can write $K_2(F)$ as

$$K_2^+F \bigoplus (\bigoplus_{\text{real ramified } v} \mathbb{Z}/2\mathbb{Z}).$$

So

$$\begin{aligned} x \in \mathrm{div}(K_2(F)) &= \mathrm{div}(K_2^+F \bigoplus (\bigoplus_{\text{real ramified } v} \mathbb{Z}/2\mathbb{Z})) \\ &= \mathrm{div}(K_2^+F) \bigoplus \mathrm{div}(\bigoplus_{\text{real ramified } v} \mathbb{Z}/2\mathbb{Z}) \\ &= \mathrm{div}(K_2^+F) \subset \mathrm{Nrd}_2(K_2(D)). \end{aligned}$$

So for $x \in \mathrm{div}(K_2(F))$, we can find $y \in \mathrm{div}(K_2(D))$ such that

$$x = \mathrm{Nrd}_2(y).$$

So

$$\mathrm{Nrd}_2: \ \mathrm{div}(K_2(D)) \longrightarrow \mathrm{div}(K_2(F))$$

is an isomorphism.

(3) Consider the following commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & WK_2(D) & \longrightarrow & K_2(D) & \longrightarrow & \prod_{\text{non complex } v} K_2(D_v) \\
 & & \downarrow{\scriptstyle \mathrm{Nrd}_2^w} & & \downarrow{\scriptstyle \mathrm{Nrd}_2} & & \downarrow{\scriptstyle (\mathrm{Nrd}_2^v)} \\
1 & \longrightarrow & WK_2(F) & \longrightarrow & K_2(F) & \longrightarrow & \prod_{\text{non complex } v} K_2(F_v)
\end{array}$$

By [17] and Proposition 26.6, Theorem 26.7 of [24] Nrd_2 is injective. So Nrd_2^w is injective. Next we will prove that $\mathrm{Nrd}_2^w: \ WK_2(D) \longrightarrow WK_2(F)$ is surjective.

By Theorem 1 of [2],

$$\begin{aligned}
WK_2(F) &= Ker(K_2F \longrightarrow \prod_{\text{non complex } v} K_2(F_v)) \\
&\subset Ker(K_2F \longrightarrow \prod_{\text{real ramified } v} K_2(F_v)) \\
&= \mathrm{Image}(\mathrm{Nrd}_2: \ K_2D \longrightarrow K_2F).
\end{aligned}$$

By [17] and [25], (Nrd_2^v) is injective. So

$$\mathrm{Nrd}_2{}^{-1}(WK_2(F)) = WK_2(D).$$

So $\mathrm{Nrd}_2: \ WK_2(D) \longrightarrow WK_2(F)$ is surjective which implies it is bijective.

(4) Tate had proved that

$$|WK_2(F)/\mathrm{div}(K_2(F))| \leq 2$$

(cf. page 250 of [3]). So (4) follows from (2) and (3). □

By Theorem 3.2 and the arguments in the proof of Theorem 2.5, we have the following theorem.

Theorem 3.3. *Let F be a number field, $\Sigma = \prod_{i=1}^{k} M_{n_i}(D_i)$ a semi-simple algebra over F, where each D_i is a finite dimensional division algebra over F with square free index. Then $div(K_2(\Sigma)) \subset WK_2(\Sigma)$ and $WK_2(\Sigma)/div(K_2(\Sigma))$ is an elementary abelian 2-group, with 2-rank less than or equal to k.*

Theorem 3.4. *Let F be a number field and D a central division algebra over F with $[D : F] = m^2$. Then*

(1) *$div(K_n(D))_l = WK_n(D)_l$ for all odd primes l and $n \leq 2$;*

(2) *if l does not divide m, then $div(K_3(D))_l = WK_3(D)_l = 0$;*

(3) *if $F=\mathbb{Q}$ and l does not divide m, then $div(K_n(D))_l \subset WK_n(D)_l$ for all n.*

Proof. (1) If $n \leq 1$, then $\mathrm{div}(K_n(D)) = WK_n(D) = 0$. If $n=2$, this result follows from Theorem 3.3.

(2) By Theorem 5.5 of [4], $\mathrm{div}(K_3(F))_l = WK_3(F)_l$ for any odd prime l. However $K_3(F)$ is finitely generated, so $\mathrm{div}(K_3(F)) = 0$ which implies $WK_3(F)_l = 0$ for any odd prime l. Consider the composite of following maps.

$$WK_3(D) \xrightarrow{tr_3} WK_3(F) \xrightarrow{i_3} WK_3(D),$$

where tr_3 is the transfer map and i_3 is the map induced by the inclusion. For any $x \in WK_3(D)$,

$$i_3 \circ tr_3(x) = x^{m^2}.$$

Since $WK_3(F)_l = 0$, $tr_3(x) \in WK_3(F)_2$. So

$$x^{m^2} = i_3 \circ tr_3(x) \in WK_3(D)_2.$$

However l does not divide m, and so, $x \in WK_3(D)_2$, i.e., $WK_3(D)_l = 0$ for any odd prime l. So if l does not divide m, then

$$WK_3(D)_l = \mathrm{div}(K_3(D))_l = 0.$$

(3) If Λ is a maximal order of the semi-simple algebra Σ, then $K_{2n-1}(\Sigma)$ is a quotient of $K_{2n-1}(\Lambda)$ and $K_{2n-1}(\Lambda)$ is finite by Quillen's localization sequence (cf. [13]). Kuku proved in [13] and [14] that $K_n(\Lambda)$ is finitely generated if $n > 1$. So $K_{2n-1}(\Sigma)$ must be finitely generated which implies $\mathrm{div}(K_{2n-1}(\Sigma)) = 0$ if $n \geq 1$. So we need only to prove this assertion for K_{2n}. By (4.2) of [4], $K_{2n}(\mathbb{Q}_v)_l$ is finite.

By Lemma 2.1, the composite of following maps

$$K_{2n}(D_v)_l \xrightarrow{tr_{2n}^v} K_{2n}(\mathbb{Q}_v)_l \xrightarrow{i_{2n}^v} K_{2n}(D_v)_l$$

is multiplication by m^2, where tr_{2n}^v is the transfer and i_{2n}^v is the map induced by inclusion. Since $(l,\ m) = 1$, this composite is injective. So $K_{2n}(D_v)_l$ is a finite group also. Assume $|K_{2n}(D_v)_l| = a_v$. Consider the following sequence

$$0 \longrightarrow \mathrm{div}(K_{2n}(D)) \xrightarrow{i} K_{2n}(D) \xrightarrow{i_v} K_{2n}(D_v).$$

If $x \in \mathrm{div}(K_{2n}(D))_l$, then we can find $y \in K_{2n}(D)$ such that $y^{a_v} = x$. Since

$$i_v \circ i(x) \in (K_{2n}(D_v))_l,$$

we have

$$i_v \circ i(x) = i_v(y)^{a_v} = 0.$$

So

$$\mathrm{div}(K_{2n}(D))_l \subset Ker(K_{2n}(D) \longrightarrow K_{2n}(D_v))$$

for any finite v. So

$$\mathrm{div}(K_{2n}(D))_l \subset Ker(K_{2n}(D) \longrightarrow \prod_{\text{finite } v} K_{2n}(D_v))$$

which implies

$$\mathrm{div}(K_{2n}(D))_l \subset WK_{2n}(D)_l.$$

So if $F = \mathbb{Q}$, then $\mathrm{div}(K_n(D))_l \subset WK_n(D)_l$ for all n and all odd primes l such that l does not divide m. □

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Received April 20, 2005