Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 1, 227-232.

On Some Maximal S-Quasinormal Subgroups of Finite Groups

K. Al-Sharo

Department of Mathematics, Al al-Bayt University, Jordan

Abstract. A subgroup H of a group G is permutable subgroup of G if for all subgroups S of G the following condition holds $SH = HS = < S, H >$. A subgroup H is S-quasinormal in G if it permutes with every Sylow subgroup of G. In this article we study the influence of S-quasinormality of subgroups of some subgroups of G on the supersolvability of G.

1. Introduction

When H and K are two subgroups of a group G, then HK is also a subgroup of G if and only if $HK = KH$. In such a case we say that H and K permute. Furthermore, H is a permutable subgroup of G, or H permutable in G, if H permutes with every subgroup of G. Permutable subgroups where first studied by Ore [7] in 1939, who called them quasinormal. While it is clear that a normal subgroup is permutable, Ore proved that a permutable subgroup of a finite group is subnormal. We say, following Kegel [6], that a subgroup of G is S-quasinormal in G if it permutes with every Sylow subgroup of G. Several authors have investigated the structure of a finite group when some subgroups of prime power order of the group are well-situated in the group. Buckley [2] proved that if all minimal subgroups of an odd order group are normal, then the group is supersolvable. It turns out that the group which has many S-quasinormal subgroups have well-described structure.

In this article we study the influence of the S-quasinormal subgroups on the structure of finite group and prove the results generalizing mentioned above:

0138-4821/93 \$ 2.50

1. Let $\pi(G) = \{p_1, p_2, \dots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$ and P_i be a Sylow p_i-subgroup of G, where $i = 1, 2, \dots, n$. If all maximal subgroups of P_i are S-quasinormal in G for each $i = 1, 2, \dots n$. Then G is supersolvable.
2. Let $\pi(G) = \{p_1, p_2, \dots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$ and P_i be a Sylow p_i-subgroup of G, where $i = 1, 2, \dots, n$. If all maximal subgroups of $\Omega(P_i)$ are S-quasinormal in G for each $i = 1, 2, \dots, n$. Then G is supersolvable.

Throughout this article the term group always means a group of finite order.

2. Notation

Let π be a set of primes. A π-group is a group whose order is a π-number, i.e. a positive integer whose prime divisors lie in π. Set $\pi' = \{\text{primes } p \text{ with } p \notin \pi\}$. A Hall subgroup of a finite group G is a subgroup H of G such that $|H|$ and $|G : H|$ are coprime. A Hall π-subgroup of G is a subgroup H of G such that $|H|$ is a π-number and $|G : H|$ is a π'-number. We write $\mathrm{Hall}_\pi(G)$ to mean the set of all Hall π-subgroups of G. We say that the group G is *p-decomposable* if $G = P \times K$ for some Sylow p-subgroup P of G and a Hall p'-subgroup K of G.

The characteristic subgroup $O^\pi(G)$ is the smallest normal subgroup of G with the property that its quotient group is a π-group. For a finite p-group P, we write

$$\Omega(P) = \begin{cases} \Omega_1(P) & \text{if } p > 2 \\ \Omega_2(P) & \text{if } p = 2 \end{cases}$$

where

$$\Omega_i(P) = < x \in P | x^{p^i} = 1 > .$$

Observe that the subgroup $\Omega(P)$ is cyclic if and only if the p-group P is cyclic.

3. Basic results

The following results are applied in this article. Any of the results 3.1–3.3 can be found in [8] on page 202.

3.1. *If H_i is a permutable subgroup of G for all $i \in I$, then $< H_i : i \in I >$ is a permutable subgroup of G.*

3.2. *Let H and K be subgroups of G such that $K \leq H$ and $K \trianglelefteq G$. Then H is a permutable subgroup of G if and only if H/K is a permutable subgroup of G/K.*

3.3. *If H is a permutable subgroup of G and S is a subgroup of G, then $H \cap S$ is a permutable subgroup of S.*

3.4. *Let H be a p-subgroup of G for some prime p. Then $H \in Syl(G)^\perp$ if and only if $N_G(H) = O^p(G)$.*

Proof. See [9, Lemma A].

Theorem 3.5. *Let p be the smallest prime dividing $|G|$. If P is a Sylow p-subgroup of G such that every maximal subgroup of P is S-quasinormal in G, then G has a normal p-complement.*

Proof. Let H be a maximal subgroup of P. It follows from 3.4 that $N_G(H)$ contains $O^p(G)$. Since $P \leq N_G(H)$ we have that H is normal in G.

Suppose that P has at least two distinct maximal subgroups H_1 and H_2. Then $H_1H_2 = P$. Hence P is normal in G. Let r be a prime different from p and R be a Sylow r-subgroup of G. By the above and 3.4 R normalizes each maximal subgroup of P. Since p is a smallest prime dividing $|G|$, we have that R induces a trivial automorphism group on $P/\Phi(P)$ ($\Phi(P)$ is a Frattini subgroup of P). By Theorem 5.1.4 in [4] R centralizes P. This implies $G = P \times T$ by Schur Theorem.

Now we may assume that P has only one maximal subgroup H. Then P is cyclic and the assertion follows from Burnside's transfer theorem.

Remark. It follows from 3.4 that if a maximal subgroup of a Sylow p-subgroup of a group G is S-quasinormal, then it is also normal in G. Moreover G is even p-decomposable, if its Sylow p-subgroup for smallest prime p is non-cyclic and every maximal subgroup of its Sylow p-subgroup is S-quasinormal.

Corollary 3.6. *Put $\pi(G) = \{p_1, p_2, \ldots, p_n\}$. Let P_i be a Sylow p_i-subgroup of G, where $i = 1, 2, \ldots, n$. If every maximal subgroup of P_i is S-quasinormal in G for all $i \in \{1, 2, \ldots, n\}$, then G is supersolvable.*

Proof. Let $p_1 > p_2 > \cdots > p_n$. By Theorem 3.5 G has a normal p_n-complement K. If a Sylow p_n-subgroup P_n is non-cyclic, then by Remark we have $G = K \times P_n$. By induction, K is supersolvable. Therefore, G is supersolvable too. Suppose that P_n is cyclic. Then $G = K \rtimes P_n$, a semidirect product of a normal subgroup K and P_n. By induction K is supersolvable. Moreover all non-cyclic Sylow subgroups of K are normal in G.

Denote by H the direct product of all non-cyclic Sylow subgroups of G. Clearly, H is a nilpotent normal Hall subgroup of G. The Frattini subgroup $\Phi(H)$ is normal in G and the group $G/\Phi(H)$ by 3.2 satisfies the condition of the corollary. By induction we may assume that $G/\Phi(H)$ is a supersolvable group provided $\Phi(H) \neq 1$. Since the formation $\mathfrak{U}$ of all supersolvable groups is saturated, this implies that G is supersolvable. Hence we may assume that $\Phi(H) = 1$. By Theorem 5.1.4 in [4] we have that H is a direct product of elementary abelian p_i-subgroups for all $p_i \in \pi(H)$.

By Schur-Zassenhaus theorem on existence of complements (see [4], p. 221) we have $G = H \rtimes L$ where L is a Hall subgroup of G with cyclic Sylow p-subgroups for all $p \in \pi(L)$. Now it is enough to show that $P \rtimes L$ is a supersolvable group for each Sylow p-subgroup of H. But every maximal subgroup of P is normal in G (see Remark) and the result follows.

Theorem 3.7. *If a group G has a normal p-subgroup P such that G/P is supersolvable and every maximal subgroup of P is S-quasinormal in G, then G is supersolvable.*

Proof. We prove the theorem by induction on $|G|$. Let P_1 be a Sylow p-subgroup of G.

If $P = P_1$, then by Remark after Theorem 3.5 we have $G = P_1 \rtimes R$ where R is a Hall p'-subgroup of G, isomorphic to G/P. It is easy to see that the Frattini subgroup $\Phi(P)$ is in the Frattini subgroup of G. If $\Phi(G)$ is non-trivial, then $G/\Phi(G)$ is supersolvable by 3.2 and induction. Since the formation $\mathfrak{U}$ of all supersolvable groups is saturated this implies the supersolvability of G. Hence we may assume that $\Phi(P) = 1$. By Theorem 5.1.4 in [4] P is an elementary abelian group. Now the result follows from Remark after Theorem 3.5. If $P = P_1$ is cyclic, then G is clearly supersolvable.

Suppose that $P < P_1$. We may assume that P is non-cyclic. Since G is solvable, it has a Hall p'-subgroup H. By Remark after Theorem 3.5 it follows that the subgroup $K = HP = H \times P$. Clearly P is normal in P_1. Hence $Z(P_1) \cap P$ is non-trivial. Let Z be a cyclic subgroup of order p in $P \cap Z(P_1)$. Since $G = P_1H$, we have Z is normal in G. By induction and 3.2 we get G/Z is supersolvable. Now we obtain the required assertion from the definition of supersolvable group.

Corollary 3.8. *Let N be a normal subgroup of G such that G/N is supersolvable and $\pi(N) = \{p_1, p_2, \ldots, p_s\}$. Let P_i be a Sylow p_i-subgroup of N, where $i = 1, 2, \ldots, s$. Suppose that all maximal subgroups of each P_i are S-quasinormal in G. Then G is supersolvable.*

Proof. We prove the theorem by induction on $|G|$. From Corollary 3.6 we have N has an ordered Sylow tower. Hence if p_1 is the largest prime in $\pi(N)$, then P_1 is normal in N. Clearly, P_1 is normal in G. Observe that $(G/P_1)/(N/P_1) \cong G/N$ is supersolvable. Therefore we conclude that G/P_1 is supersolvable by induction on $|G|$. Now it follows from Theorem 3.7 that G is supersolvable.

4. A characterization of supersolvable groups

Theorem 4.1. *Let P be a Sylow p-subgroup of G where p is the smallest prime dividing $|G|$. Suppose that all maximal subgroups of $\Omega(P)$ are S-quasinormal in G. Then G has a normal p-complement.*

Proof. Let H be a maximal subgroup of $\Omega(P)$. Our hypothesis implies that H is S-quasinormal in G and so $O^p(G) \leq N_G(H) \leq G$ by 3.4. Clearly, $HO^p(G) \leq N_G(H) \leq G$. If $HO^p(G) \leq N_G(H) < G$, then $HO^p(G)$ has a normal p-complement K by induction. Thus K is a normal Hall p'-subgroup of G and so G has a normal p-complement.

Now we may assume that $N_G(H) = G$, i.e. H is normal in G. If G has no normal p-complement, then by Frobenius theorem, there exists a nontrivial p-subgroup L of G such that $N_G(L)/C_G(L)$ is not a p-group. Clearly we can assume that $L \leq P$. Let r be any prime dividing $|N_G(L)|$ with $r \neq p$ and let R be a Sylow r-subgroup of $N_G(L)$. Then R normalizes L and so $\Omega(L)R$ is a subgroup of $N_G(L)$. Since H is normal in G, we have $H\Omega(L)R$ is a subgroup of G. Now Theorem 3.5 implies that $(H\Omega(L))R$ has a normal p-complement and so also does $\Omega(L)R$.

Since $\Omega(L)R$ has a normal p-complement, R, and $\Omega(L)$ is normalized by R, then $\Omega(L)R = \Omega(L) \times R$ and so by [5, Satz 5.12, p. 437], R centralized L. Thus for each prime r dividing $|N_G(L)|$ with $r \neq p$, each Sylow r-subgroup R of $N_G(L)$ centralized L and hence $N_G(L)/C_G(L)$ is a p-group; a contradiction. Therefore G has a normal p-complement.

As an immediate consequence of Theorem 4.1 we have:

Corollary 4.2. *Put $\pi(G) = \{p_1, p_2, \ldots, p_n\}$ where $p_1 > p_2 > \cdots > p_n$. Let P_i be a Sylow p_i-subgroup of G where $i = 1,2,\ldots,n$. Suppose that all maximal subgroups of $\Omega(P_i)$ are S-quasinormal in G. Then G possesses an ordered Sylow tower.*

Lemma 4.3. *Suppose that P be a normal Sylow p-subgroup of G and that $\Omega(P)K$ is supersolvable, where K is a Hall p'-subgroup of G. Then G is supersolvable.*

Proof. See [3, Lemma 3.3.1].

Lemma 4.4. *Suppose that P is a normal p-subgroup of G such that G/P is supersolvable. Suppose that all maximal subgroups of $\Omega(P)$ are S-quasinormal in G. Then G is supersolvable.*

Proof. We prove the lemma by induction on $|G|$. Let P_1 be a Sylow p-subgroup of G. We treat the following two cases:

Case 1. $P = P_1$. Then by Schur-Zassenhous theorem, G possesses a Hall p'-subgroup K which is a complement to P in G. The $G/P \cong K$ is supersolvable. Since $\Omega(P)$ *char* P and P is normal in G, it follows that $\Omega(P)$ is normal in G. Then $\Omega(P)K$ is a subgroup of G. If $\Omega(P)K = G$, then $G/\Omega(P)$ is supersolvable. Therefore G is supersolvable by Theorem 3.7. Thus we may assume that $\Omega(P)K < G$. Since $\Omega(P)K/\Omega(P) \cong K$ is supersolvable, it follows by Theorem 3.7 that $\Omega(P)K$ is supersolvable. Applying Lemma 4.3, we conclude the supersolvability of G.

Case 2. $P < P_1$. Put $\pi(G) = \{p_1, p_2, \ldots, p_n\}$, where $p_1 > p_2 > \cdots > p_n$. Since G/P is supersolvable, it follows by [1] that G/P possesses supersolvable subgroups H/P and K/P such that $|G/P : H/P| = p_1$ and $|G/P : K/P| = p_n$. Since H/P and K/P are supersolvable, it follows that H and K are supersolvable by induction on $|G|$. Since $|G : H| = |G/P : H/P| = p_1$ and $|G : K| = |G/P : K/P| = p_n$, it follows again by [1] that G is supersolvable.

As an immediate consequence of Corollary 4.2 and Lemma 4.3, we have:

Theorem 4.5. *Put $\pi(G) = \{p_1, p_2, \ldots, p_n\}$ where $p_1 > p_2 > \cdots > p_n$. Let P_i be a Sylow p_i-subgroup of G where $i = 1, 2, \ldots, n$. Suppose that all maximal subgroups of $\Omega(P_i)$ are S-quasinormal in G. Then G is supersolvable.*

Proof. We prove the theorem by induction on $|G|$. By Theorem 4.1 and Lemma 4.3 we have that G possesses an ordered Sylow tower. Then P_1 is normal in G. By Schur-Zassenhaus' theorem, G possesses a Hall p'-subgroup K which is a complement to P_1 in G. Hence K is supersolvable by induction. Now it follows from Lemma 4.4 that G is supersolvable.

Corollary 4.6. *Let N be a normal subgroup of G such that G/N is supersolvable. Put $\pi(N) = \{p_1, p_2, \dots, p_s\}$, where $p_1 > p_2 > \cdots > p_s$. Let P_i be a Sylow p_i-subgroup of N. Suppose that all maximal subgroups of $\Omega(P_i)$ are S-quasinormal in N. Then G is supersolvable.*

Proof. We prove the corollary by induction $|G|$. Theorem 4.5 implies that N is supersolvable and so P_1 is normal in N, where P_1 is Sylow p_1-subgroup of N and p_1 is the largest prime dividing the order of N. Clearly, P_1 is normal in G. Since $(G/P_1)(N/P_1) \cong G/N$ is supersolvable, it follows that G/P_1 is supersolvable by induction on$|G|$. Therefore G is supersolvable by Lemma 4.4. The corollary is proved.

Acknowledgments. The author thanks the referee for his or her comments and suggestions.

References

[1] Asaad, M.: *On the supersolvability of finite groups I.* Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math. **18**(3–7) (1975). Zbl 0345.20021

[2] Buckley, J.: *Finite groups whose minimal subgroups are normal.* Math. Z. **116** (1970), 15–17. Zbl 0202.02303

[3] Ezzat, M.: *Finite groups in which some subgroups of prime power order are normal.* M.Sc. Thesis, Cairo University (1995).

[4] Gorenstein, D.: *Finite groups.* Harper and Row Publishers, New York-Evanston-London 1968. Zbl 0185.05701

[5] Huppert, B.: *Endliche Gruppen I.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen **134**. Springer-Verlag, Berlin-Heidelberg-New York 1967. Zbl 0217.07201

[6] Kegel, O. H.: *Sylow-Gruppen und Subnormalteiler endlicher Gruppen.* Math. Z. **78** (1962), 205–221. Zbl 0102.26802

[7] Ore, O.: *Contributions to the theory of groups.* Duke Math. J. **5** (1939), 431–460. Zbl 0021.21104 or JFM 65.0065.06

[8] Schmidt, R.: *Subgroup Lattices of Groups.* De Gruyter Expositions in Mathematics **14**, Walter de Gruyter Berlin 1994. Zbl 0843.20003

[9] Schmid, P.: *Subgroups Permutable with All Sylow Subgroups.* J. Algebra **207** (1998), 285–293. Zbl 0910.20015

Received March 14, 2007