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$\mathbb{F}_1$-schemes and Toric Varieties

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Abstract. In this paper it is shown that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties. A description of the $\mathbb{F}_1$-zeta function in terms of toric geometry is given. Etale morphisms and universal coverings are introduced.

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Introduction

There are by now several attempts to make the theory of the field of one element $\mathbb{F}_1$ rigorous. In [10] the authors formalize the transition from rings to schemes on a categorial level and apply this machinery to the category of sets to obtain the category of $\mathbb{F}_1$-schemes as in [1]. In [3] and [5] the authors extend the definition of rings in order to capture a structure that deserves to be called $\mathbb{F}_1$. In [1] the author tried instead to fix the minimum properties any of these theories must share. The current paper extends this line of thought. We use terminology of [1] and [2].

In this paper, a ring will always be commutative with unit and a monoid will always be commutative. An *ideal* $\mathfrak{a}$ of a monoid A is a subset with $A\mathfrak{a} \subset \mathfrak{a}$. A *prime ideal* is an ideal $\mathfrak{p}$ such that $S_\mathfrak{p} = A \smallsetminus \mathfrak{p}$ is a submonoid of A. For a prime ideal $\mathfrak{p}$ let $A_\mathfrak{p} = S_\mathfrak{p}^{-1}A$ be the *localization* at $\mathfrak{p}$. The *spectrum* of a monoid A is the set of all prime ideals with the obvious Zariski-topology (see [1]). Similar to the theory of rings, one defines a structure sheaf $\mathcal{O}_X$ on $X = \mathrm{spec}\,(A)$, and one defines a *scheme over* $\mathbb{F}_1$ to be a topological space together with a sheaf of monoids, locally isomorphic to spectra of monoids.

A $\mathbb{F}_1$-scheme X is *of finite type*, if it has a finite covering by affine schemes $U_i = \operatorname{spec}(A_i)$ such that each A_i is a finitely generated monoid. For a ring R, we write X_R for the R-base-change of X, so $X_R = X_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$.

For a monoid A we let $A \otimes \mathbb{Z}$ be the monoidal ring $\mathbb{Z}[A]$. This defines a functor from monoids to rings which is left adjoint to the forgetful functor that sends a ring R to the multiplicative monoid $(R, \times)$. This construction is compatible with gluing, so one gets a functor $X \mapsto X_{\mathbb{Z}}$ from $\mathbb{F}_1$-schemes to $\mathbb{Z}$-schemes. In [2] we have shown that X is of finite type if and only if $X_{\mathbb{Z}}$ is a $\mathbb{Z}$-scheme of finite type.

We say that the monoid A is *integral*, if it has the cancellation property, i.e., if $ab = ac$ implies $b = c$ in A. This is equivalent to saying that A injects into its quotient group or A is a submonoid of a group.

By a *module* of a monoid A we mean a set M together with a map $A \times M \to M$; $(a, m) \mapsto am$ with $1m = m$ and $(ab)m = a(bm)$. A *stationary point* of a module is a point $m \in M$ with $am = m$ for every $a \in A$. A *pointed module* is a pair (M, m_0) consisting of an A-module M and a stationary point $m_0 \in M$.

1. Flatness

Recall the tensor product of two modules M, N of A:

$$M \otimes N = M \otimes_A N = M \times N / \sim,$$

where $\sim$ is the equivalence relation generated by $(am, n) \sim (m, an)$ for every $a \in A, m \in M, n \in N$. The class of (m, n) is written as $m \otimes n$. The tensor product $M \otimes N$ becomes a module via $a(m \otimes n) = (am) \otimes n$. For example, the module $A \otimes M$ is isomorphic to M.

Let now (M, m_0) and (N, n_0) be two pointed modules of A, then $(M \otimes N, m_0 \otimes n_0)$ is a pointed module, called the pointed tensor product.

The category $\operatorname{Mod}_0(A)$ of pointed modules and pointed morphisms has a terminal and initial object 0, so it makes sense to speak of kernels and cokernels. It is easy to see that every morphism f in $\operatorname{Mod}_0(A)$ possesses both. One defines the *image* of f as $\operatorname{im}(f) = \ker(\operatorname{coker}(f))$ and the *coimage* as $\operatorname{coim}(f) = \operatorname{coker}(\ker(f))$.

A morphism is called *strong*, if the natural map from $\operatorname{coim}(f)$ to $\operatorname{im}(f)$ is an isomorphism. Kernels and cokernels are strong. If $A \xrightarrow{f} B \xrightarrow{g} C$ is given with g being strong and $gf = 0$, then the induced map $\operatorname{coker}(f) \to C$ is strong. Likewise, if f is strong and $gf = 0$, then the induced map $A \to \ker g$ is strong. A map is strong if and only if it can be written as a cokernel followed by a kernel.

The usual notion of exact sequences applies, and we say that a sequence of morphisms is *strong exact* if it is exact and all morphisms in the sequence are strong.

A module $F \in \operatorname{Mod}_0(A)$ is called *flat*, if the functor $X \mapsto F \otimes X$ is strong-exact, i.e., if for every strong exact sequence

$$0 \to M \to N \to P \to 0$$

the induced sequence

$$0 \to F \otimes M \to F \otimes N \to F \otimes P \to 0$$

is strong exact as well.

It is easy to see that a pointed module F is flat if and only if for every injection $M \hookrightarrow N$ of pointed modules the map $F \otimes M \to F \otimes N$ is an injection.

Examples. If A is a group, then every module is flat. Let S be a submonoid of A. Then the localization $S^{-1}A$ is a flat A-module. The direct sum $G \oplus F$ of two flat modules is flat. Finally, consider the free monoid in one generator $C_+ = \{1, \tau, \tau^2, \dots\}$, then an A-module M is flat if and only if $\tau m = \tau m'$ implies $m = m'$ for all $m, m' \in M$. This is equivalent to saying that M is a C_+-submodule of a module of the quotient group $C_\infty = \tau^{\mathbb{Z}}$ of C_+. The same characterization holds for every integral monoid.

A morphism $\varphi : A \to B$ of monoids is called flat if B is flat as an A-module. A morphism of $\mathbb{F}_1$-schemes $f : X \to Y$ is called flat if for every $x \in X$ the morphism of monoids $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

The following is straightforward.

- A morphism of monoids $\varphi : A \to B$ is flat if and only if the induced morphism of $\mathbb{F}_1$-schemes $\operatorname{spec} B \to \operatorname{spec} A$ is flat.
- The composition of flat morphisms is flat.
- The base change of a flat morphism by an arbitrary morphism is flat.

Remark. It is easy to see that if $\mathbb{Z}[F]$ is flat as $\mathbb{Z}[A]$-module, then F is flat as A-module. The converse is already false if A is a group. As an example let k be a field and let A be the group of all matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in k$. Let A act on k^2 in the usual way and trivially on k. Consider the exact sequence of $\mathbb{Z}[A]$-modules,

$$0 \longrightarrow k \xrightarrow{\alpha} k^2 \xrightarrow{\beta} k \longrightarrow 0,$$

where $\alpha(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $\beta\begin{pmatrix} x \\ y \end{pmatrix} = y$. Let $F = \{1\}$ the trivial A-module, then for every $\mathbb{Z}[A]$-module M one has $M \otimes_{\mathbb{Z}[A]} \mathbb{Z}[F] = H_0(A, M)$. Note that $H_0(A, k) = k$ and that $H_0(\alpha) = 0$, so it is not injective, hence $\mathbb{Z}[F]$ is not flat.

2. Algebraic extensions

Let A be a submonoid of B. An element $b \in B$ is called *algebraic over* A, if there exists $n \in \mathbb{N}$ with $b^n \in A$. The extension B/A is called *algebraic*, if every $b \in B$ is algebraic over A. An algebraic extension B/A is called *strictly algebraic*, if for every $a \in A$ the equation $x^n = a$ has at most n solutions in B.

If B/A is algebraic, then $\mathbb{Z}[B]/\mathbb{Z}[A]$ is an algebraic ring extension, but the converse is wrong in general, as the following example shows: Let $A = \mathbb{F}_1$ and B be the set of two elements, 1 and b with $b^2 = b$.

A monoid A is called *algebraically closed*, if every equation of the form $x^n = a$ with $a \in A$ has a solution in A. Every monoid A can be embedded into an algebraically closed one, and if A is a group, then there exists a smallest such embedding, called the *algebraic closure* of A. For example, the algebraic closure $\bar{\mathbb{F}}_1$ of $\mathbb{F}_1$ is the group μ_∞ of all roots of unity, which is isomorphic to $\mathbb{Q}/\mathbb{Z}$.

3. Etale morphisms

Recall that a homomorphism $\varphi\colon A \to B$ of monoids is called a *local* homomorphism, if $\varphi^{-1}(B^\times) = A^\times$ (every φ satisfies "$\supset$"). For a monoid A let $m_A = A \setminus A^\times$ be its maximal ideal. It is easy to see that a homomorphism $\varphi\colon A \to B$ is local if and only if $\varphi(m_A) \subset m_B$.

A local homomorphism $\varphi\colon A \to B$ is called *unramified* if

- $\varphi(m_A)B = m_B$ and
- φ injects $A^\times$ into $B^\times$ and $B/\varphi(A)$ is a finite strictly algebraic extension.

Note that if φ is unramified, then so are all localizations $\varphi_\mathfrak{p} : A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{spec} B$.

A morphism $f : X \to Y$ of $\mathbb{F}_1$-schemes is called unramified, if for every $x \in X$ the local morphism $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified.

A morphism $f : X \to Y$ of $\mathbb{F}_1$-schemes is called *locally of finite type*, if every point in Y has an open affine neighborhood $V = \operatorname{spec} A$ such that $f^{-1}(V)$ is a union of open affines $\operatorname{spec} B_i$ with B_i finitely generated as a monoid over A. The morphism is *of finite type* if for every point in Y the number of B_i can be chosen finite. The morphism is called *finite*, if every $y \in Y$ has an open affine neighborhood $V = \operatorname{spec} A$ such that $f^{-1}(V)$ is affine, equal to $\operatorname{spec} B$, where B is finitely generated as A-module.

A morphism $f : X \to Y$ of finite type is called *étale*, if f is flat and unramified. It is called an *étale covering*, if it is also finite.

Proposition 3.1. *The étale coverings of* $\operatorname{spec} \mathbb{F}_1$ *are the morphisms of the form* $\operatorname{spec} A \to \operatorname{spec} \mathbb{F}_1$*, where A is a finite cyclic group. The scheme* $\operatorname{spec} \bar{\mathbb{F}}_1$ *has no non-trivial étale coverings.*

Proof. Clear. □

A connected scheme over $\mathbb{F}_1$, which has only the trivial étale covering, is called *simply connected*.

Proposition 3.2. *The schemes* $\operatorname{spec} \bar{\mathbb{F}}_1$, $\operatorname{spec} C_+ \times_{\mathbb{F}_1} \bar{\mathbb{F}}_1$ *and* $\mathbb{P}^1_{\bar{\mathbb{F}}_1}$ *are simply connected.*

Proof. The first has been dealt with. For the second, let $A = \mu_\infty \times C_+$. Then $\operatorname{spec} A = \operatorname{spec} C_+ \times_{\mathbb{F}_1} \bar{\mathbb{F}}_1$. Let $f : X \to \operatorname{spec} A$ be an étale covering. As f is finite, X is affine, say $X = \operatorname{spec} B$. Let $\varphi : A \to B$ denote the corresponding morphism of monoids. The space $\operatorname{spec} A$ consists of two points, the generic point η_A and the closed point c_A. Likewise, let η_B, c_B denote the generic and closed points of $\operatorname{spec} B$. One has $f(\eta_B) = \eta_A$. We will show that $f(c_B) = c_A$. Assume the contrary. Then

$\varphi^{-1}(m_B)$ is empty, hence φ maps A to the unit group $B^\times$. The localization at the closed point c_B then maps $\mu_\infty \times C_\infty$ to $B^\times$ and is unramified, hence injective. But as $C_+ \to C_\infty$ is not finite, neither can φ be finite, a contradiction. So we conclude $f(c_B) = c_A$, and so the corresponding localization, which is φ itself, is unramified. Let $s = \varphi(\tau)$, where τ is the generator of C_+. Then $\varphi(m_A)B = m_B$ implies $m_B = sB$, and so $B = B^\times \cup sB$ (disjoint union). Also, $B^\times$ is an algebraic extension of $A^\times \cong \mu_\infty$, hence equals $\varphi(A^\times)$. As B is finitely generated and flat as A-module, there are $b_1, \dots, b_r \in B$ with

$$sB = B^\times s^{\mathbb{N}} \cup B^\times s^{\mathbb{N}} b_1 \cup \cdots \cup B^\times s^{\mathbb{N}} b_r.$$

If we assume $r > 0$, then b_1 is algebraic over $\varphi(A) = B^\times \cup B^\times s^{\mathbb{N}}$, so let N be the smallest number in $\mathbb{N}$ such that $b_1^N \in \varphi(A)$. Then $b_1^N \notin B^\times \cong \mu_\infty$, because, as the extension is strictly algebraic, then b_1 would be in $B^\times$ already. So $b_1^N \in B^\times s^{\mathbb{N}}$. As the group $B^\times$ is divisible, we can replace b_1 with a $B^\times$ multiple to get $b_1^N = s^M$ for some $M \in \mathbb{N}$. Then $b_1 \notin B^\times s^{\mathbb{N}} b_1$, as $b_1 = b^* s^k b_1$ leads to $s^M = b_1^N = (b^*)^N s^{kN+M}$ which contradicts the injectivity of φ. But then b_1 must be in one of the other $B^\times s^{\mathbb{N}}$-orbits,which contradicts the disjointness of these orbits. We conclude $r = 0$, i.e. $B = B^\times \cup B^\times s^{\mathbb{N}} \cong A$ as claimed. The assertion for $\mathbb{P}^1_{\mathbb{F}_1}$ is an easy consequence. □

4. Toric varieties

Recall a *toric variety* is an irreducible variety V over $\mathbb{C}$ together with an algebraic action of the r-dimensional torus GL_1^r, such that V contains an open orbit.

As toric varieties can be constructed via lattices it follows that every toric variety is the lift $X_\mathbb{C}$ of an $\mathbb{F}_1$-scheme X. For integral schemes of finite type there is a converse direction given in the following theorem, which shows that integral $\mathbb{F}_1$-schemes of finite type are essentially the same as toric varieties.

Theorem 4.1. *Let X be a connected integral $\mathbb{F}_1$-scheme of finite type. Then every irreducible component of $X_\mathbb{C}$ is a toric variety. The components of $X_\mathbb{C}$ are mutually isomorphic as toric varieties.*

Proof. Let $U = \operatorname{spec} A$ be an open affine subset of X. Let η be the generic point of X, then the localization $G = A_\eta$ is the quotient group of A. At the same time, G is the stalk $\mathcal{O}_{X,\eta}$, so G does not depend on the choice of U up to canonical isomorphism. Let $\varphi : A \to G$ be the quotient map, which is injective as X is integral. The $\mathbb{C}$-algebra homomorphism,

$$\begin{array}{rcl} \mathbb{C}[A] & \to & \mathbb{C}[G] \otimes \mathbb{C}[A] \\ a & \mapsto & \varphi(a) \otimes a \end{array}$$

defines an action of the algebraic group $\mathcal{G} = \operatorname{spec} \mathbb{C}[G]$ on $\operatorname{spec} \mathbb{C}[A]$. Since this is compatible with the restriction maps of the structure sheaf, we get an algebraic action of the group scheme $\mathcal{G}$ on $X_{\mathbb{C}}$. As X is integral, $\mathcal{G} = \operatorname{spec} \mathbb{C}[G] = \operatorname{spec} \mathbb{C}[A_\eta]$ also is an open subset $V_{\mathbb{C}}$ of $X_{\mathbb{C}}$, and for $U_{\mathbb{C}} = \operatorname{spec} \mathbb{C}[A]$ the map

$$\mathcal{O}(U_{\mathbb{C}}) = \mathbb{C}[A] \xrightarrow{\varphi} \mathbb{C}[G] = \mathcal{O}(V_{\mathbb{C}})$$

is the restriction map of the structure sheaf $\mathcal{O}$ of $X_{\mathbb{C}}$. The map $\mathbb{C}[A] \to \mathbb{C}[G]$ is injective and $\mathbb{C}[G]$ has zero Jacobson radical, so it follows that $V_{\mathbb{C}}$ is dense in $X_{\mathbb{C}}$, so in particular it meets every irreducible component. The group G is a finitely generated abelian group, so $G \equiv \mathbb{Z}^r \times F$ for a finite abelian group F. Hence $\mathcal{G} \equiv \mathrm{GL}_1^r \times F$ as a group-scheme. As $\mathcal{G}$ meets every component of $X_{\mathbb{C}}$, the latter are permuted by F. Whence the claim. □

To formulate the next result, we will briefly recall the standard construction of toric varieties, see [4]. Let N be a *lattice*, i.e., a group isomorphic to $\mathbb{Z}^n$ for some n. A *fan* Δ in N is a finite collection of *proper convex rational polyhedral cones* σ in the real vector space $N_{\mathbb{R}} = N \otimes \mathbb{R}$, such that every face of a cone in Δ is in Δ and the intersection of two cones in Δ is a face of each. (Here zero is considered a face of every cone.) We explain the notation further: A *convex cone* is a convex subset σ of $N_{\mathbb{R}}$ with $\mathbb{R}_{\geq 0}\sigma = \sigma$, it is *polyhedral*, if it is finitely generated and *rational*, if the generators lie in the lattice N. Finally, a cone is called *proper* if it does not contain a non-zero sub vector space of $N_{\mathbb{R}}$.

Let a fan Δ be given. Let $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice. For a cone $\sigma \in \Delta$ the *dual cone* $\breve{\sigma}$ is the cone in the dual space $M_{\mathbb{R}}$ consisting of all $\alpha \in M_{\mathbb{R}}$ such that $\alpha(\sigma) \geq 0$. This defines a monoid $A_\sigma = \breve{\sigma} \cap M$. Set $U_\sigma = \operatorname{spec}(\mathbb{C}[A_\sigma])$. If τ is a face of σ, then $A_\tau \supset A_\sigma$, and this inclusion gives rise to an open embedding $U_\tau \hookrightarrow U_\sigma$. Along these embeddings we glue the affine varieties U_σ to obtain a variety X_Δ over $\mathbb{C}$, which has a given $\mathbb{F}_1$-structure. Then X_Δ is a toric variety, the torus being $U_0 \cong \mathrm{GL}_1^n$. Every toric variety is given in this way.

Lemma 4.2. *Let B be a submonoid of the monoid A of finite index. Then the map $\psi : \operatorname{spec} A \to \operatorname{spec} B$ defined by $\psi(\mathfrak{p}) = \mathfrak{p} \cap B$ is a bijection.*

Proof. Let $N \in \mathbb{N}$ be such that $a^N \in B$ for every $a \in A$. To see injectivity, let $\psi(\mathfrak{p}) = \psi(\mathfrak{q})$ and let $a \in \mathfrak{p}$. Then $a^N \in \mathfrak{q}$ and so $a \in \mathfrak{q}$ as $\mathfrak{q}$ is a prime ideal. This shows $\mathfrak{p} \subset \mathfrak{q}$ and by symmetry we get equality. For surjectivity, let $\mathfrak{p}_B \in \operatorname{spec} B$ and let $\mathfrak{p} = \{a \in A : a^N \in \mathfrak{p}_B\}$. Then $\psi(\mathfrak{p}) = \mathfrak{p}_B$. □

Proposition 4.3. *Suppose that Δ is a fan in a lattice of dimension n. For $j = 0, \dots, n$ let f_j be the number of cones in Δ of dimension j. Set*

$$c_j = \sum_{k=j}^{n} f_{n-k}(-1)^{k+j} \binom{k}{j}.$$

Let X be the corresponding toric variety, then the $\mathbb{F}_1$-zeta function of X equals

$$\zeta_X(s) = s^{c_0}(s-1)^{c_1} \cdots (s-n)^{c_n}.$$

Proof. Let $\sigma \in \Delta$ be a cone of dimension k. Let F be a face of $\breve{\sigma}$. Let $\mathfrak{p}_F = A_\sigma \smallsetminus F$. Then $\mathfrak{p}_F$ is a non-empty prime ideal in A_σ. The map $F \mapsto \mathfrak{p}_F$ is a bijection between the set of all faces of $\breve{\sigma}$ and the set of non-empty prime ideals of A_σ. The set $S_\mathfrak{p} = A \smallsetminus \mathfrak{p}$ equals $M \cap F$. The quotient group $\mathrm{Quot}(S_\mathfrak{p})$ is isomorphic to $\mathbb{Z}^f$, where f is the dimension of F. There is a bijection between the set of faces of σ and the set of faces of $\breve{\sigma}$ mapping a face τ to the face F of all $\alpha \in \breve{\sigma}$ with $\alpha(\tau) = 0$. The dimension of F then equals $n - \dim(\tau)$. So let f_j^σ denote the number of faces of σ of dimension j. Then the zeta polynomial of X_σ equals

$$N_\sigma(x) = \sum_{k=0}^{n} f_k^\sigma (x-1)^{n-k}.$$

Let N_Δ be the zeta polynomial of X_Δ. We get

$$\begin{aligned} N_\Delta(x) &= \sum_{k=0}^{n} f_k (x-1)^{n-k} \\ &= \sum_{k=0}^{n} f_k \sum_{j=0}^{n-k} \binom{n-k}{j} x^j (-1)^{n-k-j} \\ &= \sum_{k=0}^{n} f_{n-k} \sum_{j=0}^{k} \binom{k}{j} x^j (-1)^{k-j} \\ &= \sum_{j=0}^{n} x^j \sum_{k=j}^{n} f_{n-k} \binom{k}{j} (-1)^{k-j}. \end{aligned}$$

This implies the claim. □

5. Valuations

On the infinite cyclic monoid $C_+ = \{1, \tau, \tau^2, \ldots\}$ we have a natural linear order given by $\tau^k \le \tau^l \Leftrightarrow k \le l$. Let φ, ψ be two monoid morphisms from a monoid A to C_+. Then define $\varphi \le \psi \Leftrightarrow \varphi(a) \le \psi(a)\ \forall a \in A$. A *valuation* on A is a non-trivial homomorphism $v : A \to C_+$ which is minimal with respect to the order $\le$ among all non-trivial homomorphisms from A to C_+. Let $V(A)$ denote the set of valuations on A.

Lemma 5.1. *Let*

$$1 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} F \longrightarrow 1$$

be an exact sequence of monoids, where F is a finite abelian group. Then for every valuation $v \in V(A)$ there exists a unique valuation w on B and $k \in \mathbb{N}$ such that

$$w|_A = v^k.$$

Mapping v to w sets up a bijection from $V(A)$ to $V(B)$.

Proof. Let F' be a subgroup of F and let B' be the preimage of F' under φ. We get two exact sequences

$$1 \longrightarrow A \longrightarrow B' \longrightarrow F' \longrightarrow 1,$$

and

$$1 \longrightarrow B' \longrightarrow B \longrightarrow F/F' \longrightarrow 1.$$

Assume we have proven the lemma for each of these two sequences, then it follows for the original one. In this way we reduce the proof to the case when F is a finite cyclic group. We first show existence of w for given v. For this let f_0 be a generator of F and let l be its order. Choose a b_0 in the preimage $\varphi^{-1}(f_0)$. Then $b_0^l \in A$, and $v(b_0^l) = \tau^n$ for some $n \geq 0$. If $n = 0$, then set $k = 1$ and define $w : B \to C_+$ by $w(b_0^j a) = v(a)$ for $a \in A$ and $j \geq 0$. If $n > 0$, then set $k = l/\mathrm{gcd(l, n)}$ and let $w : B \to C_+$ be defined by $w(b_0^j a) = \tau^j v(a)^k$. This shows existence of the extension w. □

6. Cohomology

Cohomology is not defined over $\mathbb{F}_1$. I am grateful to Ofer Gabber for bringing the following example to my attention. Let X be the topological space consisting of three points η, X_+, x_-. The open sets besides the trivial ones are $U = \{\eta\}, U_+ = \{\eta, x_+\}, U_- = \{\eta, x_-\}$. Let A be a subgroup of the abelian group B and let $C = B/A$. Let $\mathcal{F}$ be the sheaf of abelian groups on X with $\mathcal{F}(U_\pm) = A$ and $\mathcal{F}(U) = B$ and the restriction being the inclusion. Let $\mathcal{G}$ be the constant sheaf B and let $\mathcal{H}$ be the quotient sheaf $\mathcal{G}/\mathcal{F}$. As $\mathcal{G}$ is flabby, the long cohomology sequence terminates and looks like this:

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to 0.$$

In concrete terms this is

$$0 \to A \to B \to C \times C \to (C \times C)/\Delta \to 0,$$

where Δ means the diagonal in $C \times C$. Let $f : X \to X$ be the homeomorphism with $f(x_+) = x_-$, $f(x_-) = x_+$, and $f(\eta) = \eta$. There is a natural isomorphism $f_*\mathcal{F} \cong \mathcal{F}$ and for the other sheaves as well. On the global sections of $\mathcal{F}$ and $\mathcal{G}$ this induces the trivial map, whereas on $H^0(\mathcal{H})$ it induces the flip $(a, b) \mapsto (b, a)$, which on $H^1(\mathcal{F})$ amounts to the same as the inversion $a \mapsto -a$. The naturality of these isomorphisms means that if the sheaves and the cohomology groups are defined over $\mathbb{F}_1$, then so must be the flip. This, however, is not the case, as for a set S the inversion on the abelian group $\mathbb{Z}[S]$ is not induced by a self-map of S.

Even more convincing is the fact that in this example there are different injective resolutions which produce different cohomology groups.

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