A Classical Complex Analyst Encounters a Post-modern Mathematical Object*

Phillip Griffiths
Institute for Advanced Study,
Princeton, U.S.A.

Abstract

We show how an elementary problem in classical function theory and elementary projective geometry leads into a decidedly non-classical object studied in recent times by K-theorists and algebraic number theorists. No knowledge of algebraic K-theory will be assumed; rather the presentation will be appropriate for a reader with a basic background in the theory of functions of one complex variable.

This presentation has two historical sources:

- Geometry-specifically algebraic geometry
- Complex function theory-especially its use in algebraic geometry

Algebraic geometry studies the solutions to polynomial equations:

$$\begin{gathered} f_1(z_1, \dots, z_n) = 0 \\ \cdots\cdots \\ \cdots\cdots \\ f_m(z_1, \dots, z_n) = 0 \end{gathered}$$

It begins with the study of algebraic plane curves $f(x, y) = 0$; next come surfaces $f(x, y, z) = 0$ in 3-space. (See Figs 1 and 2).

The relation of algebraic geometry to complex analysis comes through the fundamental theorem of algebra:

*This article is a transcription made by Tom Berry of a talk given by the author at the Mathematics Department of the Universidad Simón Bolívar, Venezuela, on February 19th, 2001. It is reproduced here with permission from the author.

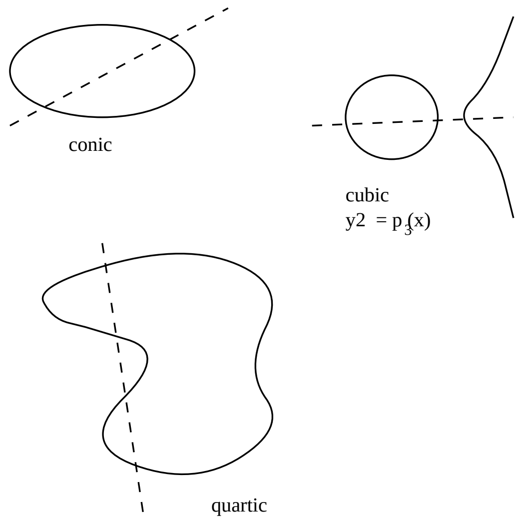

Figure 1: Some algebraic curves

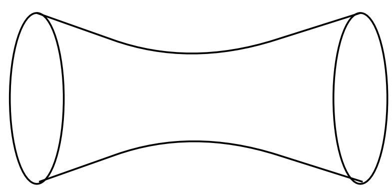

Figure 2: A quadric surface

The equation

$$f(z) = \sum_{i=1}^{n} a_i z^i = 0, \;\; a_n \neq 0$$

has n roots, (counting multiplicities) in the complex *z-plane. Moreover,* any *set of n points $\{z_1, \ldots, z_n\}$ are the solutions to such an equation*

$$f(z) = \prod_{i=1}^{n} (z - z_i) = 0.$$

We add the point $z = \infty$ to get the Riemann sphere, or *complex projective line* $\mathbf{P}^1 = \mathbf{C} \cup \infty$ (c.f. Fig. 3). Recall that the behaviour of $f(z)$ at ∞ is defined

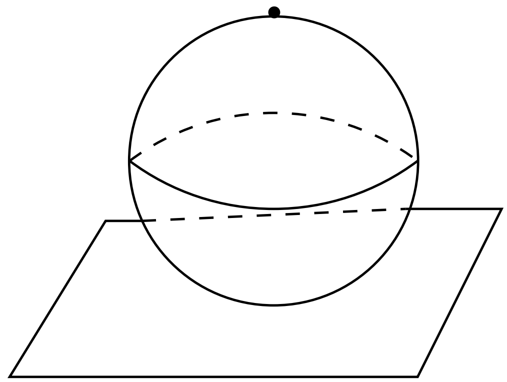

Figure 3: The Riemann Sphere

to be the behaviour of $f(1/z)$ at 0. Then the meromorphic functions on $\mathbf{P}^1$ have the same number of zeroes and poles, since

$$\begin{aligned}\#\text{zeroes}-\#\text{poles} &= \sum_{z\in\mathbf{P}^1} Res_z\left(\frac{df}{f}\right)\\ &= 0 \quad \text{by the Residue Theorem.}\end{aligned}$$

The meromorphic functions on $\mathbf{P}^1$ are just the rational functions $p(z)/q(z)$, p, q polynomials.

A configuration of points $z_i \in \mathbf{P}^1$ and multiplicities $n_i \in \mathbf{Z}$ is encapsulated as the *divisor* $\sum n_i[z_i]$ (i.e. a divisor is an element of the free abelian group on the points of $\mathbf{P}^1$). To any rational function f is associated its divisor (f) of zeroes and poles:

$$(f) = \sum ord_z f[z]$$

(where $ord_z f = n$ if f has a zero of order n at z and $-n$ if f has a pole of order n at z.) With this terminology, any configuration

$$\sum n_i[z_i], \quad \sum n_i = 0$$

is the divisor of the rational function

$$f(z) = \prod (z - z_i)^{n_i}$$

(Here, if $z_i = \infty$ the corresponding factor is suppressed.)

So much for polynomial equations in one variable. We turn to two variables and look at the algebraic curve

$$f(x, y) = 0$$

We draw the real solutions but consider all complex solutions. This is essentially

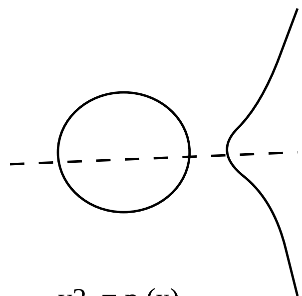

Figure 4:

because of the 18th century result known as *Bézout's theorem*:
The number of solutions to

$$f(x, y) = 0$$
$$g(x, y) = 0$$

is $\deg f \cdot \deg g$, *provided that*

1. *There are only finitely many solutions.*
2. *We use complex solutions.*
3. *We count multiplicities properly (c.f. Fig.5).*
4. *we add points at infinity corresponding to "asymptotic intersections" (c.f. Fig. 6).*

These conditions are better understood in the complex projective plane $\mathbf{P}^2(\mathbf{C})$. This is $\mathbf{C}^2$ completed by a line at infinity, whose points correspond to directions through the origin in $\mathbf{C}^2$. A (necessarily) schematic picture is given in Fig.7. Fig. 8 shows the hyperbola in the projective plane. We should note that in $\mathbf{P}^2$, all lines, and in particular the coordinate axes, are $\mathbf{P}^1$'s.

Not every set of mn points in $\mathbf{P}^2$ is $C \cap D$ where $C = \{f_n(x, y) = 0\}$ and $D = \{g_m(x, y) = 0\}$ are curves of degrees n and m respectively. We may

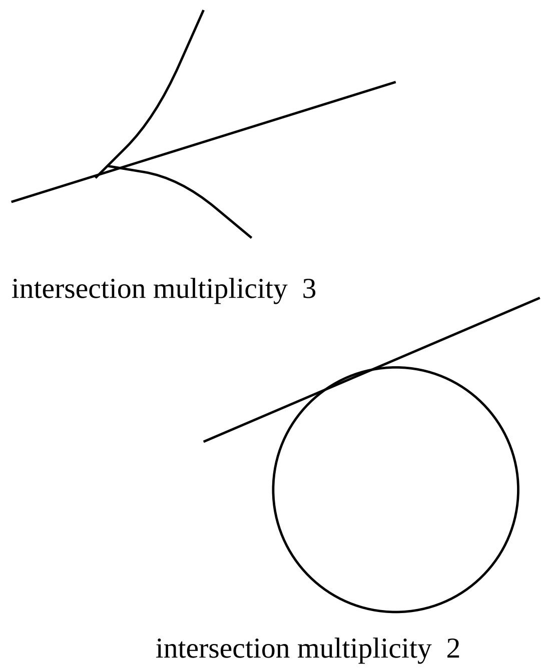

Figure 5: Intersection multiplicities

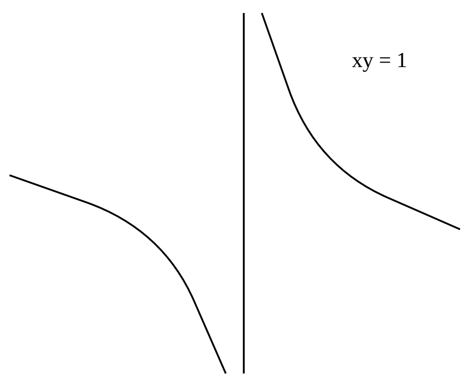

Figure 6: Asymptotic Intersections of $xy = 1$ with the coordinate axes

contrast the one and two variable cases by dimension counts.

$$\dim\{f(z) : f(z) = z^n + a_{n-1}z^{n-1} + \dots a_0\} = n,$$
$$\dim\{\text{sets of } n \text{ points in } \mathbf{C}\} = n$$

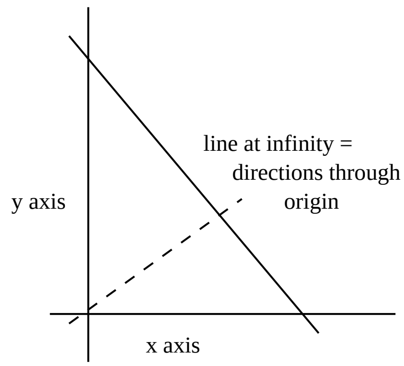

Figure 7: The Projective Plane $\mathbf{P}^2$

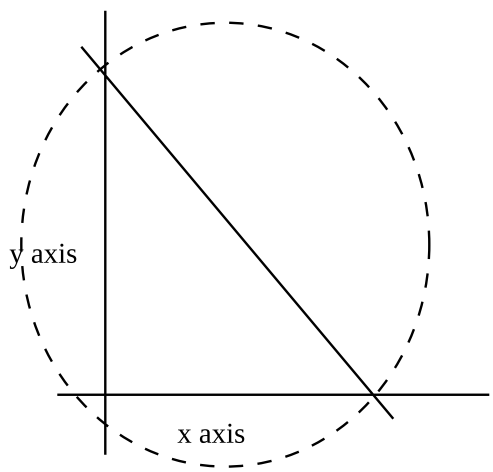

Figure 8: Hyperbola in $\mathbf{P}^2$

so that any n points of $\mathbf{C}$ will be the zeros of a monic polynomial of degree n. By contrast (taking for simplicity $m = n$)

$$\dim\{f_n(x,y), g_n(x,y)\} \approx \ n^2$$
$$\dim\{\text{sets of } n^2 \text{ points in } \mathbf{C}^2\} \approx 2n^2$$

The "≈" indicates that the error term has degree ≤ 1 in n. The first assertion follows observing that a polynomial of degree k in two variables has $(k+1)(k+$

2)/2 coefficients.

Thus not every set of 3 points is the intersection of a cubic and a line (the points have to be collinear!), and not every set of 9 points is the intersection of two cubics. We have the following

Problem. *Study the geometric constraints on a configuration of points in* $\mathbf{P}^2$ *to be* $C \cap D$ *for algebraic curves* C, D.

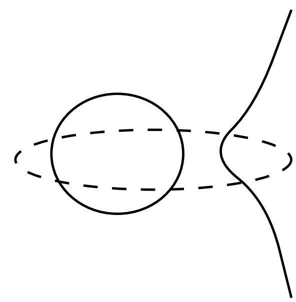

Figure 9: A complete intersection.

This turns out to be closely related to the

Problem. *Given* $D = \sum n_i p_i$, $p_i \in \mathbf{P}^2$, *when is there an algebraic curve* C *passing through* D *and a rational function*

$$f = \frac{p(x,y)}{q(x,y)}\Big|_C$$

on C, *with* $(f) = D$?

Here, (f) needs to be defined. Grosso modo, C defines a Riemann surface, almost all of whose points can be identified with points of C, and f is a meromorphic function on the surface. Then (f) means the zeros and poles of f taken with their orders, analogous to what we described for $\mathbf{P}^1$. At non-singular points P of C (i.e. points where there is a tangent line) $ord_P(p(x,y)/q(x,y)) = \{$ intersection multiplicity of $p = 0$ and C at $P\} - \{$ intersection multiplicity of $q = 0$ and C at $P\}$.

In this problem, as in the case of $\mathbf{P}^1$, a necessary condition on D is $\sum n_i = 0$, but this is no longer enough.

Although this 18th century question has a 19th century answer, a closely related question has a "new millennium" answer and leads to some of the deepest questions in modern algebraic geometry.

To explain this we go back to the 1-variable case, and ask the question:
When is $D = \sum_{i=1}^{k} n_i[z_i]$, $z_i \neq 0, \infty$ *the divisor of a rational function* $f(z)$ *with* $f(0) = f(\infty)$?

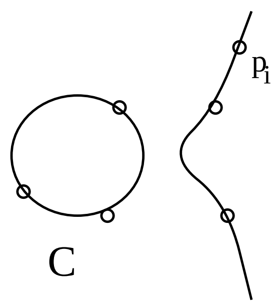

Figure 10:

As before $\sum Res \frac{df}{f} = 0$ gives $\sum n_i = 0$, but there is an additional constraint. Suppose $Im z_i \neq 0$. Then, integrating $\log z \frac{df}{f}$ around the contour shown in Fig. 11 and letting $R \to \infty$

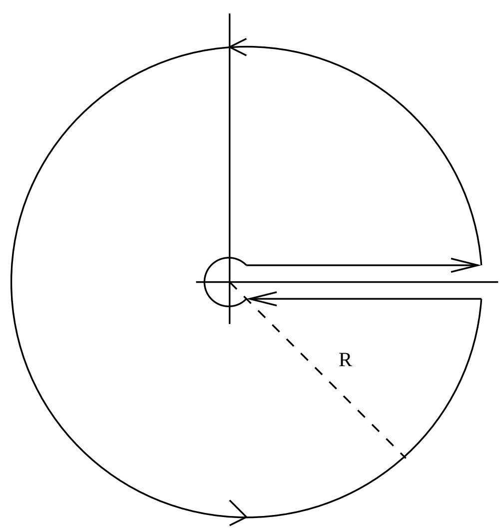

Figure 11:

$$\sum_{i=1}^{k} Res\left(\log z \frac{df}{f}\right) = \int_0^\infty \frac{df}{f}$$

But the LHS is $\sum n_i \log z_i$, while, by taking a primitive of f'/f in a simply connected set containing the positive real axis, we find the RHS is $2\pi\sqrt{-1}m$, where m is an integer. Thus

$$\sum n_i \log z_i = 2\pi\sqrt{-1}m$$

and, exponentiating, we obtain the further constraint

$$\prod z_i^{n_i} = 1$$

This, together with $\sum n_i = 0$, is necessary and sufficient for the existence of a solution to the problem, as is confirmed by a dimension count.

The simplest non-trivial function $f(z)$ with $f(0) = f(\infty)$ is

$$f(z) = \frac{(z-a)(z-b)}{(z-1)(z-ab)}$$

with divisor

$$\begin{aligned}(f) &= [a] + [b] - [1] - [ab] \\ &= ([a] - [1]) + ([b] - [1]) - ([ab] - [1])\end{aligned}$$

This has the following meaning: let

$$Div(\mathbf{P}^1; 0, \infty) = \left\{\sum n_i[z_i], \sum n_i = 0,\ z_i \neq 0, \infty\right\}$$

Then the divisors $D_a = [a] - [1]$ generate $Div(\mathbf{P}^1; 0, \infty)$, and with the function f defined above

$$(f) = D_a + D_b - D_{ab}$$

Define an equivalence relation "$\sim$" on the group $Div(\mathbf{P}^1; 0, \infty)$ by:

$$D \sim D' \text{ if } \exists \text{ a rational function } g, g(0) = g(\infty) \text{ and } (g) = D - D'$$

Then, using the function f

$$D_a + D_b \sim D_{ab}$$

Define $Div(\mathbf{P}^1; 0, \infty) \to \mathbf{C}^*$ by $\sum_i n_i[z_i] \mapsto \prod_i z_i^{n_i}$. This is surjective, since $D_a \mapsto a$, and the kernel is $\sum_i n_i z_i : \prod_i z_i^{n_i} = 1$.

Conclusion: *The map*

$$Div(\mathbf{P}^1; 0, \infty)/\sim\ \to \mathbf{C}^*$$
$$\textit{induced by } \sum n_i z_i \mapsto \prod z_i^{n_i}$$

is well-defined and is an isomorphism.

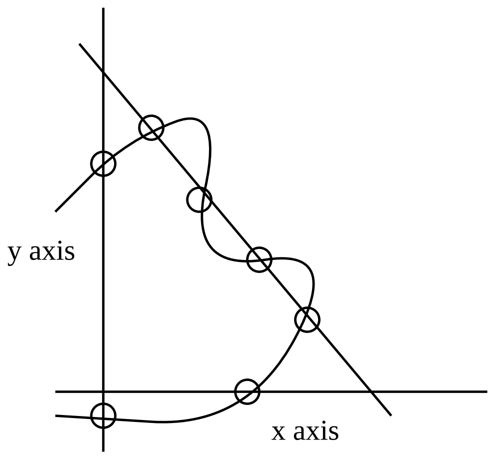

Figure 12: situation in $\mathbf{P}^2$

Now we go to configurations of points in $(\mathbf{P}^2, T)$ where T denotes the triangle formed by the coordinate axes. $\mathbf{P}^2 - T \cong \mathbf{C}^* \times \mathbf{C}^*$, since $\mathbf{P}^2 - T$ is just $\mathbf{C}^2 - \{$ coordinate axes$\}$. We set

$$Div(\mathbf{P}^2, T) = \left\{ \sum n_i p_i, \sum n_i = 0,\, p_i \notin T \right\}$$

and ask the question:
For $D \in Div(\mathbf{P}^2, T)$ when is there a curve $C = \{f(x,y) = 0\}$ and

$$g = \frac{p(x,y)}{q(x,y)}\Big|_C$$

with

$$\begin{aligned} g &= constant \ on \ T \cap C \\ (g) &= D \end{aligned}$$

As before, the residue theorem for $\log x \; dg/g$ and $\log y \; dg/g$ gives the conditions

$$\prod x_i^{n_i} = 1$$
$$\prod y_i^{n_i} = 1$$

Let $Div^0(\mathbf{P}^2, T) \subseteq Div(\mathbf{P}^2, T)$ be the subgroup satisfying these conditions. Define an equivalence relation "$\sim$" on $Div^0(\mathbf{P}^2, T)$ to be generated by

$D_1 \sim D_2$ if there exists a curve C passing through D_1 and D_2 and a rational funcion $g = \frac{p(x,y)}{q(x,y)}\Big|_C$ such that g is constant on $C \cap T$ and $(g) = D_1 - D_2$

Define

$$C(\mathbf{P}^2, T) = Div^0(\mathbf{P}^2, T)/\sim$$

Now $Div(\mathbf{P}^2, T)$ is generated by

$$D_{a,b} = (a,b) - (a,1) - (1,b) + (1,1)$$

Considering

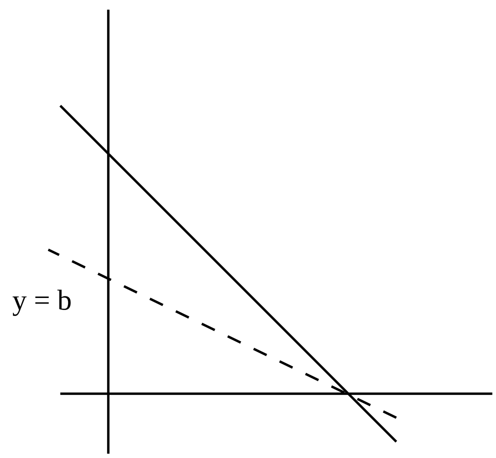

Figure 13:

$$\frac{(x-a_1)(x-a_2)}{(x-1)(x-a_1a_2)}$$

on $C = \{y = b\}$ gives (noting that, for any line C passing through a vertex of T, $(C, C \cap T) \cong (\mathbf{P}^1; 0, \infty)$)

$$D_{a_1,b} + D_{a_2,b} \sim D_{a_1a_2,b}$$
$$D_{a^2,b} \sim D_{a,b} + D_{a,b} \sim D_{a,b^2}$$

Conclusion:
The map

$$Div^0(\mathbf{P}^2, T) \to \mathbf{C}^* \otimes_{\mathbf{Z}} \mathbf{C}^*$$
$$D_{a,b} \mapsto a \otimes b$$

is well-defined.

It would have been much simpler if the story had ended here. But the above map is *not* an isomorphism, in contrast to its analogue in the 1-variable case. Geometrically, we also need to look at lines which do not pass through a vertex of T.

For $g = \prod(x - a_i)^{n_i}|_{x+y=1}$ where $\sum n_i = 0, \prod a_i^{n_i} = \prod(1 - a_i)^{n_i}$, we get

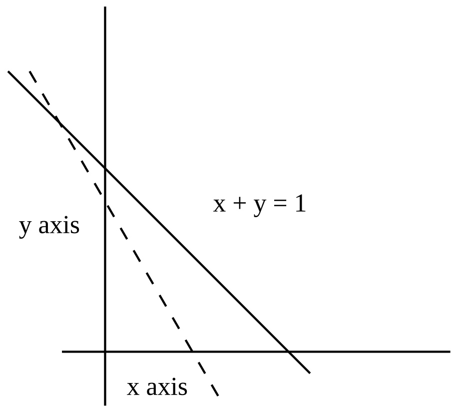

Figure 14: $x + y = 1$

$$\sum D_{a_i, 1-a_i} \sim 0$$

This intertwines x, y in a subtle way.

Definition.

$$K_2(\mathbf{C}) = \mathbf{C}^* \otimes_{\mathbf{Z}} \mathbf{C}^* / \{a \otimes (1 - a) : a \in \mathbf{C}^*, a \neq 1\}$$

The relations $a \otimes (1 - a) = 1$ are the *Steinberg relations.* Then one can prove:

Theorem. *The map* $D_{a,b} \mapsto a \otimes b$ *induces an isomorphism*

$$C(\mathbf{P}^2, T) \cong K_2(\mathbf{C})$$

Now $K_2(\mathbf{C})$ is a subtle *arithmetic* object. We set $\{a, b\}$ = image of $a \otimes b$ in $K_2(\mathbf{C})$. Then

$$\{a, 1\} = 1 = \{1, b\}$$
$$\{a, b\} = 1 \text{ if } a, b \in \bar{\mathbf{Q}}$$

For example, on $x = y$

$$(ab, ab) - (a, a) - (b, b) + (1, 1) \sim 0$$

hence

$$D_{a,b} + D_{b,a} \sim 0$$

which implies

$$\begin{aligned}\{a,b\} &= \{b,a\}^{-1}\\ &= \{\tfrac{1}{b}, a\}\end{aligned}$$

Now

$$\begin{aligned}\{a,1\} &= \{a,1-a\}\{a,\tfrac{1}{1-a}\}\\ &= \{a,1-a\}^{-1} = 1\end{aligned}$$

For λ a complex nth root of unity, $\lambda^n = 1$

$$1 = \{a,1\} = \{a,\lambda\}^n$$

so that $\{a,\lambda\}$ is torsion.

Corollary *Given $x_i, y_i \in \bar{\mathbf{Q}}$; $n_i \in \mathbf{Z}$, $i = 1 \ldots k$ so that $\sum_{i=1}^k n_i = 0$ and $\prod_{i=1}^k x_i^{n_i} = \prod_{i=1}^k y_i^{n_i} = 1$, then there exists a curve $C = \{f(x,y) = 0\}$ and g a rational function on C such that, with $p_i = (x_i, y_i)$*

$$(g) = \sum_{i=1}^k n_i p_i$$

and g is constant on $C \cap T$. Moreover C, g are defined over $\bar{\mathbf{Q}}$.

$K_2(\mathbf{C})$, which has provided the above corollary, is a "new millenium" object. For example,

$$\begin{aligned}\dim K_2(\mathbf{C}) &= \infty\\ TK_2(\mathbf{C}) &= \Omega^1_{\mathbf{C}/\mathbf{Q}}\end{aligned}$$

where $TK_2(\mathbf{C})$ indicates the tangent space (which of course needs a rigorous definition), and $\Omega^1_{\mathbf{C}/\mathbf{Q}}$ is the module of Kähler differentials of $\mathbf{C}/\mathbf{Q}$; that is, the complex vector space with generators $dz, z \in \mathbf{C}$ and relations:

$$\begin{aligned}d(z_1 + z_2) &= dz_1 + dz_2 && \forall z_1, z_2 \in \mathbf{C}\\ dqz &= qdz && \forall q \in \mathbf{Q}, z \in \mathbf{C}\\ d(z_1 z_2) &= z_1 dz_2 + z_2 dz_1 && \forall z_1, z_2 \in \mathbf{C}\end{aligned}$$

From these relations follow successively: $dq = 0\ \forall q \in \mathbf{Q}$; $d\alpha = 0$ for any $\alpha \in \bar{\mathbf{Q}}$ (if $f(\alpha) = 0$ where $f(x) \in \mathbf{Q}[x]$ is the minimal polynomial of α over $\mathbf{Q}$, then $df(\alpha) = f'(\alpha)d\alpha = 0$, hence $d\alpha = 0$). So $TK_2(\mathbf{C}) = \Omega^1_{\mathbf{C}/\mathbf{Q}}$ looks like $\mathbf{C}/\bar{\mathbf{Q}}$. $K_2(\mathbf{C})$ is a subtle mixture of arithmetic and geometry.