Geometry on the space of positive functions

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Abstract

This note is devoted to the study of geometric properties and the relationships between a projective space and an exponential class, both naturally associated with the positive elements in a commutative Banach algebra. Even though the motivating problem consists of understanding the geometry of the class of densities with respect to a given measure, the formulation can be carried out in general in a generic commutative Banach algebra set up.

Resumen

Este artículo esta destinado al estudio las propiedades geométricas y las relaciones entre el espacio proyectivo y la clase exponencial, ambas asociadas de manera natural a los elementos positivos en un álgebra conmutativa de Banach. Aunque la motivación del problema consiste en entender la geometría de la clase de densidades respecto de una medida, la formulación se puede realizar en general sobre una álgebra conmutativa de Banach.

1 Introduction and preliminaries

In two previous notes [GR1] and [GR2] we began exploring an intrinsic geometry in the commutative Banach algebra $\mathcal{A}$ consisting of all bounded, measurable, complex valued functions defined on a measure space $(S, \mathcal{S}, m)$. There we considered separately the finite and infinite dimensional cases. Even though the constructs are the same, in the finite dimensional case it is easy to visualize geometrically what goes on. The original aim was to provide a framework in which curves like

$$\rho(t) = \frac{\rho_0^{1-t}\rho_1^t}{E_m[\rho_0^{1-t}\rho_1^t]}$$

were related to geodesics in some geometry. Here ρ_0 and ρ_1 are densities (positive functions a such that the integral $\int a dm \equiv E_m[a] = 1$). Even though the model should be kept in mind, from now we assume that $\mathcal{A}$ is a commutative, complex Banach algebra, with a unit, denoted by 1 and a conjugation operation denoted by *.

After briefly describing the contents of this paper, we devote the remainder of this section to recalling some basics from [GR1] and [GR2]. In section 2 we study several aspects of the geometry of the projective space $\mathbb{P}^+$ obtained by identyfying the positive elements G^+ in the group G of invertible elements in $\mathcal{A}$. In particular we shall study the action of an affine group naturally associated to the projection of G^+ onto $\mathbb{P}^+$. In particular we relate some vector bundles over $\mathbb{P}^+$. In section 3 we take up the concluding comments in [GR1] and explore the geometry of an hyperbolic space which can be regarded as a class of representatives for $\mathbb{P}$ which inherits the geometry from G^+. In section 4 we conclude the study of the geometry on $\mathbb{P}^+$. We direct the reader tho the mentioned references for references to the necessary literature.

To describe the geometric structure, we considered in [GR1] and [GR2] the group G of invertible (with respect to the product operation) elements in $\mathcal{A}$. The group acts on the algebra according to (the right action)

$$L_g(a) = (g^*)^{-1} a g^{-1} = |g|^{-2} a$$

where the middle term stays as is in the non-commutative case. As usual, we shall say that an element a is real or self-adjoint whenever $a = a*$ and a positive when there is a b such that $a = bb^*$. We shall denote by G^+ the class of positive invertible elements in $\mathcal{A}$. It is clear the action of G on G^+ is transitive. To obtain G^+ as a homogeneous reductive space the idea was to fix an $a \in G^+$ and define $\pi_a : G \longrightarrow G^+$ by $\pi_a(g) = L_g(a)$. In the commutative case the conjugation operation on G is trivial, that is, if $g \in G$ and $C_g(g') = g g' g^{-1} = g$, but in general the setup is such that the following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{C_g} & G \\ \pi_a \downarrow & & \downarrow \pi_{L_g(a)} \\ G^+ & \xrightarrow{Lg} & G^+ \end{array}$$

One also defines the isotropy group of $a \in G^+$ by $I_a = \{g \in G \,|\, L_g(a) = a\}$, and the standard result here is that $G^+ = G/I_a$. This setup makes G^+ a homogeneous space, and (G, G^+, π_a) a fiber bundle with fibers isomorphic to I_a. there is a well established way of defining a connection on G^+ and render (G, G^+, π_a) a homogeneous reductive structure. let us recall the very basics and direct the reader to [KN] for the basics and to [CPR] for the specifics in the general non-commutative case. The basic constructs at this stage are: the tangent space at $1 \in G$ which happens to be $\mathcal{A}$ since G is open in $\mathcal{A}$, the tangent space to G^+ at a which happens to be $\mathcal{A}^s$, the symmetric elements in $\mathcal{A}$. To simplify notations, we shall denote the tangent map induced by π_a by $\tilde{\pi}_a$. The connection 1-form κ_b is defined on $G+$ in such a way that $\tilde{\pi}_b \circ \kappa_b = id|_{\mathcal{A}!^s}$. Here $\pi_b = \pi_{L_g(a)}$ for some

$g \in G$ which exists due to the transitivity of the action. The differential version of the commutative diagram helps us verify that the construction can be made equivariant starting from $\kappa_a : (TG^+)_a \simeq \mathcal{A}^s \longrightarrow (TG)_1 \simeq \mathcal{A}$, which is defined by $\kappa_a(X) = \frac{1}{2}a^{-1}X$. We leave for the reader to verify that $\tilde{\pi}_a \circ \kappa_a = id|_{\mathcal{A}^s}$. This construction is moved around by means of the group action and an equivariant setup is obtained.

With respect to this connection a geodesic through a_0 with initial speed X happens to be $a(t) = a_0 e^{tX}$. Also given any two points a_0 and a_1, the geodesic going from a_0 to a_1 in a unit of time is obtained starting with speed $X = \ln\left(\frac{a_1}{a_0}\right)$.

Comment 1.1 *Note that commutativity of $\mathcal{A}$ ensures that a_1/a_0 is well defined and being a positive element in $\mathcal{A}$, its logarithm is well defined*

Definition 1.1 *Given a differentiable curve $a(t)$ in G^+, the transport curve $g(t) \in G$ in associated to $a(t)$ is defined to be the solution to the* **transport equation**

$$\dot{g}(t) = \kappa_{a(t)}(\dot{a}(t))g(t); \quad g(0) = 1. \tag{1}$$

It is easy to see that $g(t) = (a_0/a(t))^{1/2}$ is the desired solution to (1) and that

Lemma 1.1 *With the notations just introduced, the following holds:*

$$(i)\pi_{a_0}(g(t)) = a(t) \quad \text{and} \quad (ii)\tilde{\pi}_{a(t)}\left(g^{-1}(t)\dot{g}(t)\right) = \dot{a}(t).$$

Proof Both assertions are easy to verify. To better understand the second, it is emphasizing that the tangent space to G at g is $g\mathcal{A}$ where $\mathcal{A}$ is the tangent space at 1. □

What is important at this stage is to realize that parallel transport along a curve $a(t) \in G^+$ is realized by means of the group action of the associated transport curve, and we have

Definition 1.2 *we say that the vector field $X(t)$ along the differentiable curve $a(t) \in G^+$ is parallel if $\tilde{L}_{g(t)}(X(0)) = X(t)$, where $L_{g(t)}(a_0) = a(t)$.*

Comment 1.2 *Note that if $L_g : G^+ \longrightarrow G^+$ then linearity implies that $\tilde{L}_g : (TG^+)_a \longrightarrow (TG^+)_a$ is given by $\tilde{L}_g(X) = L_g(X)$ as in the algebra.*

2 Geometry in $\mathbb{P}^+$

2.1 $\mathbb{P}^+$ as a homogeneous space

Let $\mathcal{B}$ be a sub algebra of $\mathcal{A}$ and let $\Phi_{\mathcal{B}} : \mathcal{A} \longrightarrow \mathcal{A}$ be a projection operator satisfying $\Phi_{\mathcal{B}}(ab) = b\Phi_{\mathcal{B}}(a)$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$. In our standard model $\mathcal{B}$ can be though of as a class of functions measurable with respect to a smaller σ-algebra and $\Phi_{\mathcal{B}}$ can be thought of as a conditional expectation, and when $\mathcal{B} = \mathbb{C}$, it can be thought of as an expectation. Let us begin with

Definition 2.1 *(a) We shall say that* $a \sim_{\mathcal{B}} \tilde{a}$ *whenever* $\tilde{a}a^{-1} \in \mathcal{B}^+$*, or equivalently, (b) when there exists an element* $g \in G_{\mathcal{B}}$ *such that* $\tilde{a} = L_g(a)$*. Let* (a, X) *and* $(\tilde{a}, \tilde{X})$ *be elements in* TG^+*. (c) We shall say that* $(a, X) \sim_{\mathcal{B}} (\tilde{a}, \tilde{X})$ *whenever* $\tilde{X}/\tilde{a} - X/a \in \mathcal{B}^s$

Comment 2.1 *The equivalence of (a) and (b) is left for the reader. Here* $G_{\mathcal{B}}$*,* $\mathcal{B}^+$ *and* $\mathcal{B}^s$ *denote, respectively, the invertible elements, the positive elements and the self-adjoint (real) elements in* $\mathcal{B}$*.*

Definition 2.2 *Set* $\mathbb{P}^+ = G^+/\sim_{\mathcal{B}}$ *and denote by* $\Psi : G^+ \longrightarrow G^+/\sim_{\mathcal{B}}$ *the canonical projection mapping.*

Notice to begin with that the action of G on G^+ induces an action on $\mathbb{P}^+$ in th obvious way. We shall denote this action by the same symbol. Let $\alpha = [a] \in \mathbb{P}^+$, an set

$$L_g([\alpha]) = L(\Psi(a)) \equiv \Psi(L_g(a)).$$

To see that this is independent of the representative $a \in [\alpha]$ is standard: Note that

$$L(\Psi(\tilde{(a)})) = \Psi(L_g(\tilde{a}))\Psi(L_g(L_h(a)) = \Psi(L_h(L_g(a))) = \Psi(L_g(a)).$$

That is, L_g maps "rays" in G^+ onto "rays" in G^+. To visualize $\mathbb{P}^+$ as a homogeneous space we need $\alpha_1 = \Psi \circ \pi_a(1)$ and set

$$I_{\alpha_1} = \{g \in G \,|\, L_g(\alpha_1) = \alpha_1\}.$$

Note that $g \in I_{\alpha_1}$ whenever $(g^*)^{-1}ag^{-1} \sim_{\mathcal{B}} a$ or $g^*g \sim_{\mathcal{B}} 1$ if you prefer. It should perhaps be more accurate to write $I_{\alpha_1} = S(\mathcal{A}, \mathcal{B})$, the $\mathcal{B}$-similarities of $\mathcal{A}$. An easy calculation shows that the Lie algebra of I_{α_1} is given by

$$\mathcal{I}_{\alpha_1} = \mathcal{S}(\mathcal{A}, \mathcal{B}) = \{X \in \mathcal{A} \,|\, X + X^* \in \mathcal{B}\}.$$

The next result renders $\mathbb{P}^+$ as a homogeneous space, with the obvious group action.

Proposition 2.1 *With the notations employed above,* $\mathbb{P}^+ \simeq G/S(\mathcal{A}, \mathcal{B})$*, where now the quotient denotes the class of cosets of* $g \sim_{S(\mathcal{A},\mathcal{B})} g' \Longleftrightarrow g = g'h$ *for some* $h \in S(\mathcal{A}, \mathcal{B})$*.*

Proof Let $[g] \in G/S(\mathcal{A}, \mathcal{B})$ be the class of $g \in G$. Define $p_{\alpha_1} : G \longrightarrow \mathbb{P}^+$ be defined by $p_{\alpha_1}(g) = L_g(\alpha_1)$. Note that if $g \sim_{S(\mathcal{A},\mathcal{B})} g'$, i.e., $g' = gh$ with $h \in S(\mathcal{A}, \mathcal{B})$. Then

$$p_{\alpha_1}(g') = L_{g'}(\alpha_1) = L_{gh}(\alpha_1) = L_g L_h(\alpha_1) = L_g(\alpha_1) = p_{\alpha_1}(g).$$

That is, the action of the group is constant on the classes of $\sim_{S(\mathcal{A},\mathcal{B})}$, and it can be naturally transported on to the quotient, that is, the mapping $p_{\alpha_1} : G/S(\mathcal{A},\mathcal{B}) \longrightarrow \mathbb{P}^+$ can be defined as above. □

To define the inverse to p_{α_1}, recall that given $\tilde{a} \in G^+$, there exists $g \in G$ such that $\tilde{a} = L_g(a)$. Actually $g = e^{-X/2}$ with $X = \ln(\tilde{a}/a)$. So, let $\tilde{\alpha} \in \mathbb{P}^+$ and set $\pi_{\alpha_1}^{-1}(\tilde{\alpha}) = g$. Again, it is easy to see that this mapping is well defined, for if $\Psi(c) = [\alpha]$ and $\pi_{\alpha_1}^{-1}(\tilde{\alpha}) = g_1$, then $g_1 = gh$

Now that we have obtained $\mathbb{P}^+$ as a homogeneous space. we can define a connection on it and verify that it admits a homogeneous reductive structure.

Proposition 2.2 *There exists a subspace $\mathcal{K}$ of $\mathcal{A}$ which is an invariant complement for $\mathcal{I}_{\alpha_1}$ which verifies: (i) $\mathcal{K} + \mathcal{I}_{\alpha_1}\mathcal{A}$, (ii) $\mathcal{K} = Ker(\Phi_{\mathcal{B}}) \cap \mathcal{A}^s$ and (iii) $h\mathcal{K}h^{-1} = \mathcal{K}$ for any $h \in \mathcal{I}_{\alpha_1}$.*

Proof We shall exhibit $\mathcal{I}_{\alpha_1}$ and $\mathcal{K}$ respectively as the kernel and the range of an idempotent mapping on $\mathcal{A}$. Note that $x + x^* \in \mathcal{B}^s$ is equivalent to $(Id - \Phi_{\mathcal{B}})(\Re(x)) = 0$, where $\Re(x) = (x_x^*)/2$ is a real idempotent on $\mathcal{B}_{\mathbb{R}}$ regarded as sub algebra of $\mathcal{K}$. Note as well that $Id - \Phi_{\mathcal{B}}$ is also an idempotent and that both of these idempotents commute. Therefore $(Id - \Phi_{\mathcal{B}}) \circ \Re$ is an idempotent and its range is a complement for $\mathcal{I}_{\alpha_1}$, that is $\mathcal{K} \equiv R((Id - \Phi_{\mathcal{B}}) \circ \Re)$ satisfies (i).

To verify that $\mathcal{K} = Ker(\Phi_{\mathcal{B}}) \cap \mathcal{A}^s$ is simple. Let $x \in \mathcal{K}$, then $x = x^*$ and $\Phi_{\mathcal{B}} \circ (Id - \Phi_{\mathcal{B}}) \circ \Re = 0$ trivially. The converse is equally simple.

To verify (iii) is simple in the commutative case and it is left for the reader. □

To define a linear connection on $\mathbb{P}^+$ we proceed as follows. As above let $\alpha_1 = \Psi \circ \pi_\alpha(1)$, therefore the tangent map $(dr_{\alpha_1})_1 : \mathcal{A} \longrightarrow (T\mathbb{P}^+)_{\alpha_1}$ is onto with kernel $\mathcal{I}_{\alpha_1}$. Therefore, the restriction

$$\delta_{\alpha_1} = (dr_{\alpha_1})|_{\mathcal{K}} : \mathcal{K} \longrightarrow (T\mathbb{P}^+)_{\alpha_1}$$

is an isomorphism. Define now the 1-form of the connection by

Definition 2.3 *Define*

$$\kappa_{\alpha_1} : (T\mathbb{P}^+)_{\alpha_1} \longrightarrow \mathcal{K} \text{ by } \kappa_{\alpha_1} = (\delta_{\alpha_1})^{-1}. \tag{2}$$

Lemma 2.1 *At any other point $\alpha = L_{\alpha_1} \in \mathbb{P}^+$, set $\delta_\alpha = (dr_\alpha|\mathcal{K})_1$. Then $\kappa_\alpha = Ad_g \circ \kappa_{\alpha_1} \circ L_{g^{-1}}$ is an inverse for δ_α*

To compute κ_α explicitly consider a differentiable curve $g(t) \in G$ such that $g(0) = 1$ and $\dot{g}(0) = X$. then

$$\frac{d}{dt} r_{\alpha_1}(g(t)) = \frac{d}{dt} \left(\Psi(g(t)^*)^{-1} a g(t)^{-1} \right) |_{t=0} = \tilde{\Psi}\left(a, -(X + X^*)a\right).$$

The restriction of this mapping to $\mathcal{K}$ provides us with δ_{α_1}. As element of $T\mathbb{P}^+$, $\tilde{\Psi}(a, -(X + X^*)a) = \{(b, w) \,|\, w/b + (X + X^*) \in \mathcal{B}^s\}$, therefore the obvious candidate for κ_{α_1} is

$$\kappa_{\alpha_1}(b, w) = -\frac{1}{2}\left(b^{-1}\left(w - \Phi_{\mathcal{B}}(b^{-1}w)\right)\right) \tag{3}$$

It is an exercise to verify that $\kappa_{\alpha_1}(b, w) \in \mathcal{K} = Ker(\Phi_{\mathcal{B}}) \cap \mathcal{A}^s$, that it has the desired properties and that the defining map is independent of the representative chosen.

Definition 2.4 *Let $a(t)$ be a differentiable curve in G^+ and $\alpha(t) = \Psi(a(t))$. Let $X(t)$ de a differentiable vector field along $a(t)$ and let us use the same symbol to define its equivalence class in $T\mathbb{P}^+$. The covariant derivative of $X(t)$ is defined to be*

$$\frac{DX}{dt} = \delta_{\alpha(t)}\left(\frac{d}{dt}\kappa_{\alpha(t)}(X(t))\right) \tag{4}$$

2.2 An affine group determined by $\mathcal{B}$

As at the beginning of this section, let $\mathcal{B}$ be a sub-algebra of $\mathcal{A}$. We can define an action of the group $G^+_{\mathcal{B}}$ on the real algebra $\mathcal{B}^s$ as follows

$$G^+_{\mathcal{B}} \times \mathcal{B}^s \longrightarrow \mathcal{B}^s \;\; (b, b') \to bb'. \tag{5}$$

Similarly, we can define an action of $\mathcal{B}^s$ on itself by means of

$$\mathcal{B}^s \times \mathcal{B}^s \longrightarrow \mathcal{B}^s \;\; (b, b') \to b + b'. \tag{6}$$

Definition 2.5 *Let us denote by $Af_{\mathcal{B}^s}$ de semi direct product of the multiplicative group $G^+_{\mathcal{B}}$ and the additive group $\mathcal{B}^s$. The group operation is $(\hat{b}, \hat{b}'), (b, b') = (\hat{b}b, \hat{b}b' + \hat{b}')$.*

Comment Notice that $\mathcal{B}^s$ can be thought of as the tangent space to $G^+_{\mathcal{B}}$ at the identity.

That that is a well defined group operation is standard exercise, and it is simple to verify the following

Lemma 2.2 *With the notations introduced above and in definition 2.1 we have*

(i)The mapping $Af_{\mathcal{B}^s} \times \mathcal{B}^s \to \mathcal{B}^s$ defined by $(b, b')(b'') = bb'' + b'$ is a well defined action of $Af_{\mathcal{B}^s}$ on $\mathcal{B}^s$.

(ii) The mapping $Af_{\mathcal{B}^s} \times TG^+ \to TG^+$, defined by $(b, b')(a, X) = (ba, bX + b'a)$ is a group action.

(iii)The affine group action is compatible with the equivalence relation $\sim_{\mathcal{B}}$.

(iv)$Af_{\mathcal{B}^s}$ acts on $T\mathbb{P}^+$ by means of $(b, b')[a, X] = [(b, b')(a, X)]$, where $[a, X]$ denotes the equivalence class of $(a, X) \in TG^+$ under $\sim_{\mathcal{B}}$.

Proof We shall just sketch the proof of the third assertion. Let $(a, X) \sim_{\mathcal{B}} (\tilde{a}, \tilde{X})$. It is just a computation to verify definition 2.1, namely that

$$(b, b')(a, X) \sim_{\mathcal{B}} (b, b')(\tilde{a}, \tilde{X}),$$

which we leave for the reader to complete. The fourth assertion is clear from this. □

2.3 Tangent bundles over $\mathbb{P}^+$

To better understand the apparition of $Af_{\mathcal{B}^s}$ and what comes below, let us go back to definition 2.1, and notice that the equivalence class of $(1, 0)$ with respect to $\sim_{\mathcal{B}}$ is $[1, 0] = \{(b, b') \mid b \in G^+_{\mathcal{B}}, b' \in \mathcal{B}^s\} = Af_{\mathcal{B}^s}$. Thus if we write the tangent space at $1 \in G^+$ as $\mathcal{A}^s = \mathcal{B}^s \oplus V$, then under (the lifting of) Ψ, $\mathcal{B}^s$ projects down to 0. Actually, we have the simple

Lemma 2.3 *With the notations introduced above*

$$[a, X] = \{(b, b')(a, X) \mid (b, b') \in Af_{\mathcal{B}^s}\}.$$

Another way in which $\mathcal{B}^s$ comes up as the part of the tangent bundle which is tangent to the rays is the following. Consider a smooth curve $b(t)$in $G^+_{\mathcal{B}}$ such that $b(0) = 1$ and derivative $\dot{b}(0) = X \in \mathcal{B}^s$. Then for $a \in G^+$, $b(t)a$ lies along the ray through a, and its tangent is aX. Therefore, we may call the vector bundle introduced below the radial bundle. We have the easy

Lemma 2.4 *Consider the vector bundle*

$$\mathcal{R} = \{(a, X) \in TG^+ \mid a^{-1}X \in \mathcal{B}^s\}$$

which is contained in TG^+. Then, $\mathcal{R}$ is stable under the action of $G^+_{\mathcal{B}}$.

Recall that the action of $G^+_{\mathcal{B}}$ on G^+ produces $\mathbb{P}^+$ as quotient space. Let us now examine the equivalence classes of action of $G^+_{\mathcal{B}}$ on TG^+.

Definition 2.6 *We shall say that $(a, X) \sim_{G^+_{\mathcal{B}}} (a', X')$ whenever there exists $b \in \mathcal{B}^+$ such that $a' = ba$ and $X' = bX$.*

Comments The classes on the action of $G^+_{\mathcal{B}}$ on TG^+ are bigger that those of the action on $\mathcal{R}$

Lemma 2.5 *The following sequence is exact:*

$$0 \to \mathcal{R} \xrightarrow{i} TG^+ \xrightarrow{\Psi_*} T\mathbb{P}^+ \to 0,$$

where i denotes the inclusion mapping.

Proof From the comments above, it is clear that if $(a, X) \in \mathcal{R}$ then $[a, X] = [a, 0]$, of $\mathcal{R} \subset \ker \Psi_*$. The rest is easy. □

For the next proposition we need the following

Lemma 2.6 *With the notations from above, Ψ_* preserves $G^+_{\mathcal{B}}$.*

Proof If $(a, X) \sim_{G^+_{\mathcal{B}}} (\tilde{a}, \tilde{X})$ or, equivalently¡ if $(\tilde{a}, \tilde{X}) = b_0(a, X)$ for some $b_0 \in G^+_{\mathcal{B}}$, then $[\tilde{a}, \tilde{X}] = b_0[a, x]$. To see why this is so, notice that according to 2.3

$$\begin{aligned}[\tilde{a}, \tilde{X}] &= \{(b, b')(\tilde{a}, \tilde{X}) \mid (b\tilde{a}, b\tilde{X} + \tilde{a}b') \text{ for (b, b')} \in \mathrm{Af}_{\mathcal{B}^s}\} \\ &= \{b_0(ba, bX + ab') \mid (b, b') \in Af_{\mathcal{B}^s}\} = b_0[a, X]\end{aligned}$$

from which the conclusion drops out. □

Proposition 2.3 *There exists a mapping $\hat{\Psi}_*$ such that the following diagram is commutative, and furthermore the lower row is exact.*

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{R} & \xrightarrow{i} & TG^+ & \xrightarrow{\Psi_*} & T\mathbb{P}^+ & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{R}/\sim_{G^+_{\mathcal{B}}} & \xrightarrow{i} & TG^+/\sim_{G^+_{\mathcal{B}}} & \xrightarrow{\hat{\Psi}_*} & T\mathbb{P}^+/\sim_{G^+_{\mathcal{B}}} & \rightarrow & 0, \end{array}$$

where the vertical mappings in all cases are the implied quotient mappings.

Proof According to the previous lemma, the last arrow is well defined. The existence of $\hat{\Psi}_*$ is a standard argument when dealing with quotient structures. See [D] or [P]. □

3 The class $\mathcal{E}_o = \exp \mathcal{K}$

We shall now explore the properties of the class $\mathcal{E}_o = \exp \mathcal{K} = \{\exp^z : |z \in \mathcal{K}\}$. The original ideas in the non-commutative case can be found in [PR]. This class happens to be isometric with $\mathbb{P}^+$ and its geometry is easier to deal with. Let us begin with

Proposition 3.1 *With the notations introduced above, any $a \in G^+$ can be uniquely factored as $a = be^z$, with $z \in \mathcal{K}$ and $b \in \mathcal{B}^+$. In other words, the mapping $G^+ = G^+_{\mathcal{B}} \times \mathcal{K}$, sending a onto (b, z) is a homeomorphism. Certainly $G^+_{\mathcal{B}}$ denotes the positive invertible elements in $\mathcal{B}$.*

Proof Commutativity readily implies that $a = e^{\ln a} = e^{\Phi_{\mathcal{B}}(\ln a)} e^{\ln a - \Phi_{\mathcal{B}}(\ln a)}$ for $a \in G^+$. □

Comment 3.1 *One way of thinking about the starting point of the proof is that a is the end point of the geodesic $\gamma(t) = e^{tX}$ that joins $a \in G^+$ to $1 \in G^+$, with initial speed $X = \ln a$. The rest is clear for the decomposition is multiplicative.*

The geometric properties of $\mathcal{E}_o$ are inherited from G^+. Let us begin with

Proposition 3.2 *(a) The connection on G^+ reduces to $\mathcal{E}_o$: If $a \in \mathcal{E}_o$ and $X \in (T\mathcal{E}_o)_a$ is tangent to a differentiable curve $c(t)$ in G^+ that passes through a, then $\nabla_X Y \in (T\mathcal{E}_o)_a$.*

(b) A geodesic of G^+, which starts tangent to $\mathcal{E}_o$, remains in $\mathcal{E}_o$, that is, if $\gamma(t)$ is a geodesic in G^+ such that $\gamma(0) = a \in \mathcal{E}_o$ and $\dot{\gamma}(0) \in (T\mathcal{E}_o)_a$, then $\gamma(t) \in \mathcal{E}_o$ for all t.

(c) $\mathcal{E}_o$ is geodesically convex. That is, if c_1 and c_2 are n $\mathcal{E}_o$, the geodesic in G^+ joining $c-1$ to c_2 is in $\mathcal{E}_o$.

Proof Let us begin with a useful remark: If $Z \in (T\mathcal{E}_o)_a$, then $\Phi_{\mathcal{B}}(a^{-1}Z) = 0$. To see why this is clear, let $c(t)$ be a differentiable curve in $\mathcal{E}_o$ such that $c(0) = a$ and $\dot{c}(0) = Z$, therefore $a^{-1}c(t) = e^{X(t)} \equiv \delta(t)$ for some differentiable curve $X(t) \in \mathcal{K}$. Thus $0 = \frac{d}{dt}\Phi_{\mathcal{B}}(\delta(t))_{t=0} = \Phi_{\mathcal{B}}(a^{-1}Z)$.

To proof (a) recall that if X is tangent to $a(t)$ and $Y(t)$ is tangent to $\mathcal{E}_o$ in a neighborhood of a, then $\nabla_X Y = \frac{d}{dt}Y - a^{-1}XY$. Multiply by a^{-1} both sides and keep in mind that $X = \dot{a}$, then

$$\Phi_{\mathcal{B}}(a^{-1}) = \Phi_{\mathcal{B}}\left(a\frac{dY}{dt} - a^{-1}\dot{a}a^{-1}Y\right) = \frac{d}{dt}\Phi_{\mathcal{B}}\left(a{-}1Y\right) = 0.$$

(b)Let now $\gamma(t) = a_0 e^{tX} = e^{\xi_o + tX}$ be a geodesic in G^+ such that $\gamma(0) = e^{\xi_o} \in \mathcal{E}_o$ and $\Phi_{\mathcal{B}}(\gamma(0)^{-1}\dot{\gamma}) = \Phi_{\mathcal{B}}(X) = 0$. Therefore $\gamma(t) \in \mathcal{E}_o$.

(c)Let $c_1 = e^{Z_1}$ and $c_2 = e^{Z_2}$ be such that $Z_1\ Z_2 \in \mathcal{K}$. We saw in section 1 that the geodesic in G^+ through these points is $c(t) = c_1 e^{t\ln(c_2/c_1)} = \exp\left(Z_1 + t(Z_2 - Z_1)\right) \in \mathcal{E}_o$. □ We also have

Proposition 3.3 *The restriction $\Psi|_{\mathcal{E}_o} : \mathcal{E}_o \longrightarrow \mathbb{P}^+$ is a diffeomorphism.*

Proof Note first that $\Psi|_{\mathcal{E}_o}$ is bijective. If $e^X \sim e^Y$, with $X, Y \in \mathcal{K}$, then there exists $b \in \mathcal{B}^+$ such that $e^X = be^Y$. By the uniqueness of the factorization, $b = 1$ and $X = Y$. Also, if $\Psi(a) \in \mathbb{P}^+$ for some $a \in G^+$, then $a = be^X$ and therefore $a \sim e^X$ and we have produced an $X \in \mathcal{K}$ such that $\Psi(e^X) = \Psi(a)$.

Clearly, the mapping is continuous and has inverse $\mathbb{P}^+ \longrightarrow \mathcal{E}_o$ given by $\Psi(a) \rightarrow e^X$. To verify the continuity of the inverse mapping, assume that a_n and $\mathrm{a} \in \mathrm{G}^+$, are such that $\Psi(a_n) \rightarrow \Psi(a)$. This means that there exists a sequence $b_n \in \mathcal{B}^+$ such that $b_n a_n \rightarrow a$. Now let $a_n = d_n e^{X_n}$ and $a = de^X$. Therefore $b_n d_n e^{X_n} \rightarrow de^X$ which implies that $X_n \rightarrow X$. □
and we finish with

Proposition 3.4 *The mapping $\Psi : G^+ \longrightarrow \mathbb{P}^+$ is a fiber bundle.*

Proof Suffices to exhibit a global section, namely

$$\mathbb{P}^+ \longrightarrow \mathcal{E}_o \subset G^+; \quad \text{given by } \ \Psi(\mathrm{a}) \rightarrow \mathrm{e}^{\mathrm{X}}$$

where $a = be^X$ with $b \in \mathcal{B}^+$ and $X \in \mathcal{K}$. □

4 The geometry on $\mathbb{P}^+$ concluded

In the previous section we saw how the geometry of G^+ restricts well to $\mathcal{E}_o$. We shall now see how to obtain the geometry of $\mathbb{P}^+$ from that of $\mathcal{E}_o$. The diffeomorphism $\Psi|_{\mathcal{E}_o} : \mathcal{E}_o \longrightarrow \mathbb{P}^+$ yield a linear isomorphism

$$\tilde{\Psi}|_{\mathcal{E}_o} : (T\mathcal{E}_o)_a \longrightarrow (T\mathbb{P}^+)_\alpha$$

where of course, $\alpha = \Psi(a)$. Also recall that

$$(T\mathbb{P}^+)_\alpha = \Big\{(a,X) \in G^+ \times \mathcal{A}^s \,|\, (a,X) \sim (\tilde{a},\tilde{X}) \\ \Longleftrightarrow \tilde{a}a^{-1} \in \mathcal{B}^+ \text{ and } \tilde{\mathrm{X}}/\tilde{\mathrm{a}} - \mathrm{X}/\mathrm{a} \in \mathcal{B}^{\mathrm{s}}\Big\}.$$

Let us denote by $\|a\|$ the norm in $\mathcal{A}$, and begin with

Definition 4.1 *For $(a,X) \in (T\mathbb{P}^+)_\alpha$ define the (projective) norm*

$$\|(a,X)\|_{\Phi_\mathcal{B}} = inf\left\{\|\tilde{X}\|_{a,\Phi_\mathcal{B}} \,|\, (\tilde{a},\tilde{X}) \sim (a,X)\right\} \tag{7}$$

where

$$\|\tilde{X}\|_{a,\Phi_\mathcal{B}} \equiv \|a^{-1/2}Xa^{-1/2}\|_{\Phi_\mathcal{B}} \equiv \|\Phi_\mathcal{B}(a^{-2}X^2)\|^{1/2}$$

Proposition 4.1 *With the same notation as above, the mapping*

$$\Psi|_{\mathcal{E}_o} : \mathcal{E}_o \longrightarrow \mathbb{P}^+$$

is isometric.

Proof Let (a,X) be a representative of the class of a tangent vector at $(T\mathbb{P}^+)_\alpha$, where $\alpha = \Psi(a)$. Since $\Psi|_{\mathcal{E}_o} : \mathcal{E}_o \longrightarrow \mathbb{P}^+$ is a diffeomorphism, there exists a pair (c,V) with $c \in \mathcal{E}_o$ and $V \in (T\mathcal{E}_o)_c$, such that $(a,X) \sim (c,V)$. Recall that $(T\mathcal{E}_o)_c = \{Y \in \mathcal{A} \,|\, Y = Y^*,$ and $\Phi_\mathcal{B}(\mathrm{c}^{-1}\mathrm{Y}) = 0\}$. Then $V/c - X/a \in \mathcal{B}^s$ or

$$V/c - X/a = \Phi_\mathcal{B}(V/c - X/a) = -\Phi_\mathcal{B}(X/a)$$

or $V/c = X/a - \Phi_\mathcal{B}(X/a)$ and therefore

$$\begin{aligned}\Phi_\mathcal{B}(c^{-2}V^2) &= \Phi_\mathcal{B}(a^{-2}X^2) - 2\Phi_\mathcal{B}\left(a^{-1}X\Phi_\mathcal{B}(a^{-1}X)\right) + \left(\Phi_\mathcal{B}(a^{-1}X)\right)^2 \\ &= \Phi_\mathcal{B}(a^2X^2) - \left(\Phi_\mathcal{B}(a^1X)\right)^2 \leq \Phi_\mathcal{B}(a^{-2}X^2).\end{aligned}$$

That is $\|V\|_{c,\Phi_\mathcal{B}} \leq \|\Phi_\mathcal{B}(a^2X^2)\|^{1/2}$ holds for any pair $(a,X) \sim (c,V)$, or in other words $\|V\|_{c,\Phi_\mathcal{B}} \leq \|\tilde{(V)}\|_{\Psi(c),\Phi_\mathcal{B}}$.
The converse inequality is proved similarly. □

Let us now verify that the connection on $\mathbb{P}^+$ transported from $\mathcal{E}_o$ by means of $\Psi|_{\mathcal{E}_o}$ coincides with the connection defined in section 2 by means of the reductive structure. Let us begin by explicitely computing the idempotent $\kappa_\alpha \circ \delta_\alpha$ for $\alpha = \Psi(a) \in \mathbb{P}^+$. For $X \in \mathcal{A}$

$$\begin{aligned}\kappa_\alpha \circ \delta_\alpha(X) = \kappa_\alpha(a, (-(X+X^*)a) &= \frac{1}{2}(Id - \Phi_{\mathcal{B}})\left(a^{-1}(X+X^*)a\right) \\ &= \frac{1}{2}\left(X + X^* - \Phi_{\mathcal{B}}(X+X^*)\right).\end{aligned}$$

Proposition 4.2 *The diffeomorphism $\Psi|_{\mathcal{E}_o}$ preserves linear connections.*

Proof Let $X(t)$ be a tangent field to $\mathbb{P}^+$ along a differentiable curve $\alpha(t)$ Let us denote by D^r/dt the covariant derivative determined by the reductive connection and denote by D^Ψ/dt the connection induced by $\Psi|_{\mathcal{E}_o}$ In order to compare them, we shall use κ_α to translate both to $\mathcal{A}$ (regarded as tangent space to G at 1). Let $V(t)$ be a vector field in $\mathcal{E}_o$ along the curve $(\Psi|_{\mathcal{E}_o})^{-1}(\alpha(t)) \equiv c(t)$, that is

$$X(t) = (\tilde{\Psi})_{c(t)}(V(t)).$$

Being tangent to $\mathcal{E}_o$, $V(t)$ verifies $\Phi_{\mathcal{B}}(c(t)^{-1}V(t)) = 0$. Therefore

$$\kappa_\alpha\left(\frac{D^r X}{dt}\right) = \kappa_\alpha \circ \delta_\alpha\left(\frac{d}{dt}\kappa_\alpha(X(t))\right),$$

and now note that $\kappa_\alpha(X(t)) = \kappa_\alpha(\tilde{\Psi})_{c(t)}(V(t)) = \frac{c(t)^{-1}}{2}V(t)$. Then

$$\frac{d}{dt}\kappa_\alpha(X(t)) = \frac{1}{2}c(t)^{-2}\dot{c}(t)V(t) - \frac{1}{2}c(t)^{-1}\dot{V}(t).$$

Using the computation carried out above for $\kappa_\alpha \circ \delta_\alpha$ with $a(t) = c(t)$ we obtain

$$\kappa_\alpha\left(\frac{D^r X}{dt}\right) = \frac{1}{2}\left(c(t)^{-2}\dot{c}(t)V(t) - c^{-1}\dot{V}(t)\right)$$

because $\Phi_{\mathcal{B}}(c(t)^{-2}\dot{c}(t)V(t) - c^{-1}\dot{V}(t)) = \frac{d}{dt}\Phi_{\mathcal{B}}(c^{-1}V) = 0$. On the other hand

$$\frac{D^\Psi X}{dt} = \tilde{\Psi}_{c(t)}\left(\frac{D^{\mathcal{E}_o}X}{dt}\right) = \tilde{\Psi}_{c(t)}\left(\dot{V}(t) - c^{-1}\dot{c}(t)V(t)\right).$$

Now apply κ_α to both sides to obtain

$$\begin{aligned}\kappa_\alpha\left(\frac{D^\Psi X}{dt}\right) &= -\frac{1}{2}c^{-1}\left(\dot{V}(t) - c^{-1}\dot{c}(t)V(t)\right) - \Phi_{\mathcal{B}}\left(c^{-1}\left(\dot{V}(t) - c^{-1}\dot{c}(t)V(t)\right)\right) \\ &= \frac{1}{2}\left(c(t)^{-2}\dot{c}(t)V(t) - c^{-1}\dot{V}(t)\right)\end{aligned}$$

for exactly the same reasons as in the previous computation. □

The following corollary, the proof of which is for the reader, asserts that $\mathbb{P}^+$ inherits geometric properties from G^+ via $\mathcal{E}_o$.

Corollary 4.1 *The Finsler metric defined in section 1, $\mathbb{P}^+$ inherits the following properties from $\mathcal{E}_o$:*
(i) Any two points in $\mathbb{P}^+$ are joined by a unique geodesic, which is the shortest possible curve in $\mathbb{P}^+$ with such end points.
(ii) If $\alpha_1(t)$ and $\alpha_2(t)$ are two geodesics in $\mathbb{P}^+$, and $d(a,b)$ denotes the distance in the Finsler metric, then the mapping $t \longrightarrow d(\alpha_1(t), \alpha_2(t))$ is a convex function.
(iii) If $\alpha = \Psi(a)$ and $\beta = \Psi(b)$, with $a, b \in \mathcal{E}_o$, then the unique geodesic joining them is given by

$$\gamma_{\alpha,\beta} = \Psi\left(a^{1-t} b^t\right).$$

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