On new estimates for distances in analytic function spaces in the unit disk, the polydisk and the unit ball.

Romi Shamoyan, Olivera Mihic[*]

Abstract. We provide various new sharp estimates for distances of fixed analytic functions of a certain classical analytic class (analytic Besov space, Bloch type space) to its subspaces in the unit disk, the unit polydisk and the unit ball. We substantially enlarge the list of previously known assertions of this type.

Resumen. Ofrecemos varias nuevas estimaciones fuerte para las distancias de funciones analiticas fijas de una cierta clase de funciones analíticas clasicas (espacios analiticos de Besov, espacios de tipo Bloch) a sus subespacios en el disco unidad, el polydisco unidad y la bola unidad. Ampliamos sustancialmente la lista de afirmaciones previamente conocidas de este tipo.

1 Introduction and main notations

Let $\mathbf{D}$ be, as usual, the unit disk on the complex plane, $dA(z)$ be the normalized Lebesgue measure on $\mathbf{D}$ so that $A(\mathbf{D}) = 1$ and $d\xi$ be the Lebesgue measure on the circle $\mathbf{T} = \{\xi : |\xi| = 1\}$. Let further $H(\mathbf{D})$ be the space of all analytic functions on the unit disk $\mathbf{D}$.

For $f \in H(\mathbf{D})$ and $f(z) = \sum_k a_k z^k$, define the fractional derivative of the function f as usual in the following manner

$$D^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha a_k z^k, \ \alpha \in \mathbb{R}.$$

We will write $Df(z)$ if $\alpha = 1$. Obviously, for all $\alpha \in \mathbb{R}$, $D^\alpha f \in H(\mathbf{D})$ if $f \in H(\mathbf{D})$.

[*]Supported by MNZŽS Serbia, Project 144010

For $a \in \mathbf{D}$, let $g(z,a) = \log(\frac{1}{|\varphi_a(z)|})$ be the Green's function for $\mathbf{D}$ with pole at a, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. For $0 < p < \infty,\ -2 < q < \infty,\ 0 < s < \infty,\ -1 < q+s < \infty$, we say that $f \in F(p,q,s)$, if $f \in H(\mathbf{D})$ and

$$\|f\|^p_{F(p,q,s)} = \sup_{a \in \mathbf{D}} \int_{\mathbf{D}} |Df(z)|^p (1-|z|^2)^q g(z,a)^s dA(z) < \infty.$$

As we know [15], if $0 < p < \infty,\ -2 < q < \infty,\ 0 < s < \infty,\ -1 < q+s < \infty$, $f \in F(p,q,s)$ if and only if

$$\sup_{a \in \mathbf{D}} \int_{\mathbf{D}} |Df(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|)^s dA(z) < \infty.$$

It is known (see [15]) that $F(2,0,1) = BMOA$.

We recall that the weighted Bloch class $\mathcal{B}^\alpha(\mathbf{D}),\ \alpha > 0$, is the collection of the analytic functions on the unit disk satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbf{D}} |Df(z)|(1-|z|^2)^\alpha < \infty.$$

Space $\mathcal{B}^\alpha(\mathbf{D})$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha}$.Note $\mathcal{B}^1(\mathbf{D}) = \mathcal{B}(\mathbf{D})$ is a classical Bloch class (see [2], [8] and the references there).

For $k > s,\ 0 < p,q \le \infty$, the weighted analytic Besov space $\mathcal{B}^{q,p}_s(\mathbf{D})$ is the class of analytic functions satisfying (see [8])

$$\|f\|^q_{\mathcal{B}^{q,p}_s} = \int_0^1 \left(\int_{\mathbf{T}} |D^k f(r\xi)|^p |d\xi| \right)^{\frac{q}{p}} (1-r)^{(k-s)q-1} dr < \infty.$$

Quasinorm $\|f\|_{\mathcal{B}^{q,p}_s}$ does not depend on k. If $\min(p,q) \ge 1$, the class $\mathcal{B}^{q,p}_s(\mathbf{D})$ is a Banach space under the norm $\|f\|_{\mathcal{B}^{q,p}_s}$. If $\min(p,q) < 1$, then we have a quasinormed class.

The well-known so called "duality" approach to extremal problems in theory of analytic functions leads to the following general formula

$$dist_Y(g,X) = \sup_{l \in X^\perp, \|l\| \le 1} |l(g)| = \inf_{\varphi \in X} \|g-\varphi\|_Y,$$

where $g \in Y$, X is subspace of a normed space Y, $Y \in H(\mathbf{D})$ and $X^\perp$ is the ortogonal complement of X in Y^*, the dual space of Y and l is a linear functional on Y (see [7]).

Various extremal problems in H^p Hardy classes in $\mathbf{D}$ based on duality approach we mentioned were discussed in [3, Chapter 8]. In particular for a function $K \in L^q(\mathbf{T})$ the following equality holds (see [3]), $1 \le p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$,

$$dist_{L^q}(K, H^q) = \inf_{g \in H^q, K \in L^q} \|K-g\|_{H^q} = \sup_{f \in H^p, \|f\|_{H^p} \le 1} \frac{1}{2\pi} \left| \int_{|\xi|=1} f(\xi) K(\xi) d\xi \right|.$$

It is well known that if $p>1$ then the inf-dual extremal problem in the analytic H^p Hardy classes has a solution, it is unique if an extremal function exists (see [3]).

Note also that extremal problems for H^p spaces in multiply connected domains were studied before in [1], [9].

Various new results on extremal problems in A^p Bergman class and in its subspaces were obtained recently by many authors (see [6] and the references there).

In this paper we will provide direct proofs for estimation of $dist_Y(f,X) = \inf_{g\in X}\|f-g\|_Y,\ X\subset Y,\ X,Y\subset H(\mathbf{D}),\ f\in Y$, not only in unit disk, but also in higher dimension.

Let further $\Omega^k_{\alpha,\varepsilon} = \{z\in\mathbf{D} : |D^k f(z)|(1-|z|^2)^\alpha \geq \varepsilon\},\ \alpha\geq 0,\ \varepsilon>0,\ \Omega^0_{\alpha,\varepsilon} = \Omega_{\alpha,\varepsilon}$.

Applying famous Fefferman duality theorem, P. Jones proved the following

Theorem A. *([4], [15]) Let $f\in\mathcal{B}$. Then the following are equivalent:*
(a) $d_1 = dist_{\mathcal{B}}(f, BMOA)$;
(b) $d_2 = \inf\{\varepsilon>0 : \chi_{\Omega^1_{1,\varepsilon}(f)}(z)\frac{dA(z)}{1-|z|^2}$ is a Carleson measure$\}$,
where χ denotes the characteristic function of the mentioned set.

Recently, R. Zhao (see [15]) and W. Xu (see [14]), repeating arguments of R. Zhao in the unit ball, obtained results on distances from Bloch functions to some Möbius invariant function spaces in one and higher dimensions in a relatively direct way. The goal of this paper is to develop further their ideas and present new sharp theorems in the unit disk and higher dimension.

In next sections various sharp assertions for distance function will be given. We will indicate proofs of some assertions in details, short sketches of proofs in some cases will be also provided.

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

Given two non negative real numbers A, B we will write $A\lesssim B$ if there is a positive constant C such that $A<CB$.

2 New sharp assertions on $dist_X(f,Y)$ function in the unit disk

For the proof of one of the main results of this paper we will need the following estimate which can be found in [8].

Lemma 1. *(see [8]) Let $s>-1,\ r>0,\ t>0$ and $r+t-s>2$. If $t<s+2<r$ then we have* $\int_{\mathbf{D}}\frac{(1-|z|^2)^s dA(z)}{|1-\overline{w}z|^r|1-\overline{a}z|^t} \leq \frac{C}{(1-|w|^2)^{r-s-2}|1-\overline{a}w|^t},\ a,w\in\mathbf{D}$.

Note that $F(p,q,s) \subset \mathcal{B}^{\frac{q+2}{p}}$, $s \in (0,1]$, (see [15]). Hence for $\alpha \geq \frac{q+2}{p}$, the problem of finding $dist_{\mathcal{B}^\alpha}(f, F(p,q,s))$ appears naturally.

In the following theorem we show that in Zhao's theorems (see[15]) Möebius invariant Bloch classes can be replaced by Bloch classes with general weights.

Theorem 1. *Let* $1 \leq p < \infty,\ \alpha > 0,\ 0 < s \leq 1,\ \alpha \geq \frac{q+2}{p},\ q > \alpha(p-1) - s - 1,\ q > s - 2 + \alpha(p-1)$ *and* $f \in \mathcal{B}^\alpha$. *Then the following are equivalent:*

(a) $d_1 = dist_{\mathcal{B}^\alpha}(f, F(p,q,s))$;

(b) $d_2 = \inf\{\varepsilon > 0 : \chi_{\Omega^1_{\alpha,\varepsilon}}(z)\frac{dA(z)}{(1-|z|^2)^{\alpha p-q-s}}$ *is an* $s-Carleson\ measure\}$.

Proof. First we show $d_1 \leq Cd_2$. According to the Bergman representation formula (see [2]), we have $f(z) = C(\alpha)\int_{\mathbf{D}} Df(w)(1-|w|^2)^\alpha D^{-1}\frac{1}{(1-\overline{w}z)^{\alpha+2}}dA(w)$

$$= C(\alpha)\int_{\Omega^1_{\alpha,\varepsilon}} Df(w)(1-|w|^2)^\alpha D^{-1}\frac{1}{(1-\overline{w}z)^{\alpha+2}}dA(w)+$$

$$+C(\alpha)\int_{\mathbf{D}\setminus\Omega^1_{\alpha,\varepsilon}} Df(w)(1-|w|^2)^\alpha D^{-1}\frac{1}{(1-\overline{w}z)^{\alpha+2}}dA(w)) = f_1(z)+f_2(z),$$

where $C(\alpha)$ is the constant of the Bergman representation formula (see [2]).

By $Df_1(z) = C(\alpha)\int_{\Omega^1_{\alpha,\varepsilon}}\frac{Df(w)(1-|w|^2)^\alpha}{(1-\overline{w}z)^{2+\alpha}}dA(w)$,

$|Df_1(z)| \leq C\int_{\Omega^1_{\alpha,\varepsilon}}\frac{|Df(w)|(1-|w|^2)^\alpha}{|1-\overline{w}z|^{2+\alpha}}dA(w) \leq C\|f\|_{\mathcal{B}^\alpha}\frac{1}{(1-|w|)^\alpha}$.

Then $f_1 \in \mathcal{B}^\alpha$. By Lemma 1,

$$\int_{\mathbf{D}}|Df_1(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)$$

$$\leq C\|f_1\|^{p-1}_{\mathcal{B}^\alpha}\int_{\mathbf{D}}|Df_1(z)|(1-|z|^2)^{q-(p-1)\alpha}(1-|\varphi_a(z)|^2)^s dA(z)$$

$$\leq C\|f_1\|^{p-1}_{\mathcal{B}^\alpha}\int_{\mathbf{D}}\int_{\Omega^1_{\alpha,\varepsilon}}\frac{|Df(w)|(1-|w|^2)^\alpha}{|1-\overline{w}z|^{2+\alpha}}dA(w)(1-|z|^2)^{q-(p-1)\alpha}(1-|\varphi_a(z)|^2)^s dA(z)$$

$$\leq C\|f_1\|^{p-1}_{\mathcal{B}^\alpha}\|f\|_{\mathcal{B}^\alpha}\int_{\Omega^1_{\alpha,\varepsilon}}(1-|a|^2)^s\int_{\mathbf{D}}(1-|z|^2)^{q-(p-1)\alpha+s}dA(z)dA(w)$$

$$\leq C\int_{\Omega^1_{\alpha,\varepsilon}}\frac{(1-|a|^2)^s}{|1-|w|^2|^{p\alpha-q-s}|1-\overline{a}w|^{2s}}dA(w).$$

By $\chi_{\Omega^1_{\alpha,\varepsilon}}\frac{dA(z)}{(1-|z|^2)^{\alpha p-q-s}}$ is an s-Carleson measure, $f_1 \in F(p,q,s)$. Also we have

$$|Df_2(z)| \leq C\int_{\mathbf{D}\setminus\Omega^1_{\alpha,\varepsilon}}\frac{|Df(w)|(1-|w|^2)^\alpha}{|1-\overline{w}z|^{2+\alpha}}dA(w) \leq C\varepsilon\int_{\mathbf{D}}\frac{dA(w)}{|1-\overline{w}z|^{2+\alpha}} \leq \frac{C\varepsilon}{(1-|z|)^\alpha}.$$

So, $dist_{\mathcal{B}^\alpha}(f, F(p,q,s)) \leq \|f-f_1\|_{\mathcal{B}^\alpha} = \|f_2\|_{\mathcal{B}^\alpha} < \varepsilon$.

It remains to show that $d_1 \geq d_2$. If $d_1 < d_2$ then we can find two numbers $\varepsilon,\ \varepsilon_1$ such that $\varepsilon > \varepsilon_1 > 0$, and a function $f_{\varepsilon_1} \in F(p,q,s)$, $\|f - f_{\varepsilon_1}\|_{\mathcal{B}^\alpha} \leq \varepsilon_1$, and $\frac{\chi_{\Omega^1_{\alpha,\varepsilon}}(z)}{(1-|z|^2)^{\alpha p-q-s}}$ is not a s-Carleson measure.

Since $(|Df(z)| - |Df_{\varepsilon_1}(z)|)(1-|z|^2)^\alpha \leq \varepsilon_1$, we can easily obtain

$$(\varepsilon - \varepsilon_1)\chi_{\Omega^1_{\alpha,\varepsilon}}(z)dA(z) \leq C|Df_{\varepsilon_1}(z)|(1-|z|^2)^\alpha, \tag{1}$$

where $\chi_{\Omega^1_{\alpha,\varepsilon}}$ is defined above. Hence from (1) and the fact that $f_{\varepsilon_1} \in F(p,q,s)$ we arrive at a contradiction. The theorem is proved. □

Remark 1. Theorem 1 can be expanded similarly to more general analytic classes with quasinorms $\sup_{|z|<1} |D^\gamma f(z)|(1-|z|)^\alpha,\ \alpha > 0$, with some restrictions on $\alpha,\ \gamma$.

Let $\quad \widetilde{\mathcal{B}}^{-t} = D^{-1}\mathcal{B}^{-t} = \left\{f \in H(\mathbf{D}) : D^{-1}f \in \mathcal{B}^{-t}\right\},\ t < 0.$
It is well-known that $\mathcal{B}^{q,q}_s(\mathbf{D}) \subset \widetilde{\mathcal{B}}^{-t}(\mathbf{D}),\ t = s - \frac{1}{q},\ t < 0,\ s < 0$ (see [8]).

In the following theorem we calculate distances from a weighted Bloch class to Bergman spaces for $q \leq 1$.

Theorem 2. *Let* $0 < q \leq 1,\ s < 0,\ t \leq s - \frac{1}{q},\ \beta > \frac{1-sq}{q} - 2$ *and* $\beta > -1-t$. *Let* $f \in \widetilde{\mathcal{B}}^{-t}$. *Then the following are equivalent:*

(*a*) $l_1 = dist_{\widetilde{\mathcal{B}}^{-t}}(f, \mathcal{B}^{q,q}_s)$;

(*b*) $l_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{D}} \left(\int_{\Omega_{\varepsilon,-t}(f)} \frac{(1-|w|)^{\beta+t}}{|1-\overline{z}w|^{2+\beta}} dA(w)\right)^q (1-|z|)^{-sq-1}dA(z) < \infty\}$.

Proof. First we show that $l_1 \leq Cl_2$. For $\beta > -1-t$, we have
$f(z) = C(\beta)\left(\int_{\mathbf{D}\setminus\Omega_{\varepsilon,-t}} \frac{f(w)(1-|w|)^\beta}{(1-\overline{w}z)^{\beta+2}} dA(w) + \int_{\Omega_{\varepsilon,-t}} \frac{f(w)(1-|w|)^\beta}{(1-\overline{w}z)^{\beta+2}} dA(w)\right) = f_1(z) + f_2(z)$,
where $C(\beta)$ is a well-known Bergman representation constant (see [2], [8]).

For $t < 0$,

$$|f_1(z)| \leq C\int_{\mathbf{D}\setminus\Omega_{\varepsilon,-t}} \frac{|f(w)|(1-|w|)^\beta}{|1-\overline{w}z|^{\beta+2}} dA(w) \leq C\varepsilon \int_{\mathbf{D}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}} dA(w) \leq C\varepsilon \frac{1}{(1-|z|)^{-t}}.$$

So $\sup_{z\in\mathbf{D}} |f_1(z)|(1-|z|)^{-t} < C\varepsilon$.

For $s < 0,\ t < 0$, we have

$$\int_{\mathbf{D}} |f_2(z)|^q(1-|z|)^{-sq-1}dA(z) \leq C\int_{\mathbf{D}} \left(\int_{\Omega_{\varepsilon,-t}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}} dA(w)\right)^q (1-|z|)^{-sq-1}dA(z) \leq C.$$

So we finally have

$$dist_{\widetilde{\mathcal{B}}^{-t}}(f, \mathcal{B}^{q,q}_s) \leq C\|f - f_2\|_{\widetilde{\mathcal{B}}^{-t}} = C\|f_1\|_{\widetilde{\mathcal{B}}^{-t}} \leq C\varepsilon.$$

It remains to prove that $l_2 \le l_1$. Let us assume that $l_1 < l_2$. Then we can find two numbers $\varepsilon,\ \varepsilon_1$ such that $\varepsilon > \varepsilon_1 > 0$, and a function $f_{\varepsilon_1} \in \mathcal{B}^{q,q}_s$, $\|f - f_{\varepsilon_1}\|_{\widetilde{\mathcal{B}}^{-t}} \le \varepsilon_1$, and $\int_{\mathbf{D}} \left(\int_{\Omega_{\varepsilon,-t}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{z}w|^{\beta+2}} dA(w) \right)^q (1-|z|)^{-sq-1} dA(z) = \infty$. Hence as above we easily get from $\|f - f_{\varepsilon_1}\|_{\widetilde{\mathcal{B}}^{-t}} \le \varepsilon_1$ that $(\varepsilon - \varepsilon_1)\chi_{\Omega_{\varepsilon,-t}(f)}(z)(1-|z|)^t \le C|f_{\varepsilon_1}(z)|$, and hence

$$M = \int_{\mathbf{D}} \left(\int_{\mathbf{D}} \frac{\chi_{\Omega_{\varepsilon,-t}(f)}(z)(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}} dA(w) \right)^q (1-|z|)^{-sq-1} dA(z)$$
$$\le C \int_{\mathbf{D}} \left(\int_{\mathbf{D}} \frac{|f_{\varepsilon_1}(w)|(1-|w|)^{\beta}}{|1-\overline{w}z|^{\beta+2}} dA(w) \right)^q (1-|z|)^{-sq-1} dA(z).$$

Since for $q \le 1$, (see [2], [8])

$$\left(\int_{\mathbf{D}} \frac{|f_{\varepsilon_1}(z)|(1-|z|)^{\alpha} dA(z)}{|1-wz|^t} \right)^q \le C \int_{\mathbf{D}} \frac{|f_{\varepsilon_1}(z)|^q (1-|z|)^{\alpha q+q-2} dA(z)}{|1-wz|^{tq}}, \tag{2}$$

where $\alpha > \frac{1-q}{q}, t > 0, f_{\varepsilon_1} \in H(\mathbf{D}),\ w \in \mathbf{D}$, and

$$\int_{\mathbf{D}} \frac{(1-|z|)^{-sq-1}}{|1-\overline{w}z|^{q(\beta+2)}} dA(z) \le \frac{C}{(1-|w|)^{q(\beta+2)+sq-1}}, \tag{3}$$

where $s < 0,\ \beta > \frac{1-sq}{q} - 2,\ w \in \mathbf{D}$. We get

$$M \le C \int_{\mathbf{D}} |f_{\varepsilon_1}(z)|^q (1-|z|)^{-sq-1} dA(z).$$

So as in the proof of the previous theorem we arrive at a contradiction. $\square$

The following theorem is a version of Theorem 2 for the case $q > 1$.

Theorem 3. *Let $q > 1,\ s < 0,\ t \le s - \frac{1}{q},\ \beta > \frac{-1-sq}{q}$ and $\beta > -1 - t$. Let $f \in \widetilde{\mathcal{B}}^{-t}$. Then the following are equivalent:*

(a) $l_1 = dist_{\widetilde{\mathcal{B}}^{-t}}(f, \mathcal{B}^{q,q}_s)$;

(b) $\widehat{l}_2 = \inf\{\varepsilon > 0 : \int_{\mathbf{D}} \left(\int_{\Omega_{\varepsilon,-t}(f)} \frac{(1-|w|)^{\beta+t}}{|1-\overline{z}w|^{2+\beta}} dA(w) \right)^q (1-|z|)^{-sq-1} dA(z) < \infty\}$.

The proof of this theorem is similar to the proof of Theorem 2 but here we will use (4) (see below) instead of (2). For $\varepsilon > 0,\ q > 1,\ \beta > 0,\ \alpha > \frac{-1}{q}$, (see [8])

$$\left(\int_{\mathbf{D}} \frac{|f(z)|(1-|z|)^{\alpha}}{|1-\overline{w}z|^{\beta+2}} dA(z) \right)^q \le C \int_{\mathbf{D}} \frac{|f(z)|^q (1-|z|)^{\alpha q}}{|1-\overline{w}z|^{\beta q-\varepsilon q+2}} dA(z)(1-|w|)^{-\varepsilon q}, w \in \mathbf{D}, \tag{4}$$

which follows immediately from Hölder's inequality and (3) (see [8]).

Remark 2. In Theorems 2 and 3 we considered only the linear case ($p = q$). Estimates for distances of more general mixed norm $\mathcal{B}_s^{p,q}$ classes, $s < 0$, can be obtained similarly. We give an example in this direction.

It is known that $\mathcal{B}_s^{p,q}(\mathbf{D}) \subset \widetilde{\mathcal{B}}^{-(s-\frac{1}{q})}(\mathbf{D}),\ s < 0$ (see [8]).

Theorem 4. *Let* $o < q \le 1,\ p \le q \le 1,\ s < 0,\ t \le s - \frac{1}{q},\ \beta > \frac{-sq}{p} + \frac{1}{q} - 2$. *Let* $f \in \widetilde{\mathcal{B}}^{-t}$. *Then the following are equivalent:*

(*a*) $\widetilde{l}_1 = dist_{\widetilde{\mathcal{B}}^{-t}}(f, \mathcal{B}_s^{p,q})$;

(*b*) $\widetilde{l}_2 = \inf\{\varepsilon > 0 : \int_0^1 \left(\int_{\mathbf{T}} \left(\int_{\Omega_{\varepsilon,-t}(f)} \frac{(1-|w|)^{\beta+t} dA(w)}{|1-\overline{z}w|^{2+\beta}}\right)^q d\xi\right)^{\frac{p}{q}} (1-|z|)^{-sp-1} d|z| < \infty\}$.

Let X be a quasinormed class in the unit disk, $X \subset H(\mathbf{D})$. Let also

$$\sup_{z\in\mathbf{D}} |f(z)|(1-|z|)^{\alpha+\tau} \le C_{\alpha,\tau}\|f\|_X, \tag{5}$$

where $C_{\alpha,\tau}$ ($\alpha > 0,\ \tau > 0$) is an absolute constant. It is well known that for many classes the above estimate (5) holds (see [2], [8] and the references there).

The following result follows directly from arguments we provided above during the proof of previous theorem.

Theorem 5. *Let* $X, Y \subset H(\mathbf{D}),\quad X \subset Y,\ \alpha > 0,\ \tau > 0$. *Let* $f \in Y(\mathbf{D})$ *and* $\sup_{z\in\mathbf{D}} |f(z)|(1-|z|)^{\tau} \le C_\tau\|f\|_Y,\quad \sup_{z\in\mathbf{D}} |f(z)|(1-|z|)^{\alpha+\tau} \le \widehat{C}_{\alpha,\tau}\|f\|_X$. *Then* $B(f, X) \le C dist_Y(f, X),\ C = C(\alpha, \tau)$, *where*

$$B(f, X) = \inf\left\{\varepsilon > 0 : \sup_{|z|<1} \chi_{\Omega_{\tau,\varepsilon}}(z)(1-|z|)^{\alpha} < \infty\right\}.$$

Remark 3. Theorem 5 provides various new results for *dist*- function for different analytic classes. The general theorem we have presented is true even if the unit disk is replaced by the polydisk or the unit ball, since uniform estimates like (5), which is the base of proof, are well known in unit disk, unit ball and polydisk for various concrete classes of analytic functions (Bergman, Hardy, Bloch, BMOA, Q_p, etc.), see [2].

For $0 < p < \infty$ and $\alpha > 0$, let as above

$$B_{-\alpha}^{\infty,1}(\mathbf{D}) = \left\{f \in H(\mathbf{D}) : \sup_{r<1}\left(\int_{\mathbf{T}} |f(r\xi)||d\xi|\right)(1-r)^{\alpha} < \infty\right\},$$

$$B_{-\alpha}^{p,1}(\mathbf{D}) = \left\{f \in H(\mathbf{D}) : \int_0^1 \left(\int_{\mathbf{T}} |f(r\xi)||d\xi|\right)^p (1-r)^{\alpha p-1} dr < \infty\right\}.$$

It is easy to see that $B_{-\alpha}^{p,1}(\mathbf{D}) \subset B_{-\alpha}^{\infty,1}(\mathbf{D}),\ 0 < p < \infty,\ \alpha > 0$.

We now define a new subset of the unit interval and then using its characteristic function we will give a new sharp assertion concerning distance function.

For $\varepsilon > 0,\ f \in H(\mathbf{D})$, let $L_{\varepsilon,\alpha}(f) = \{r \in (0,1) : (1-r)^{\alpha}\int_{\mathbf{T}} |f(r\xi)||d\xi| \ge \varepsilon\}$.

Theorem 6. *Let* $f \in B^{\infty,1}_{-\alpha}$, $\alpha > 0$, $1 \le p < \infty$. *Then the following are equivalent:*

(*a*) $s_1 = dist_{B^{\infty,1}_{-\alpha}}(f, B^{p,1}_{-\alpha})$;

(*b*) $s_2 = \inf\{\varepsilon > 0 : \int_0^1 (1-r)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r)dr < \infty\}$.

Proof. First we prove $s_1 \ge s_2$. Let as assume that $s_1 < s_2$. Then we can find two numbers ε, ε_1 such that $\varepsilon > \varepsilon_1 > 0$, and a function $f_{\varepsilon_1} \in B^{p,1}_{-\alpha}$, $\|f - f_{\varepsilon_1}\|_{B^{\infty,1}_{-\alpha}} \le \varepsilon_1$, and $\int_0^1 (1-r)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r) = \infty$. Hence we have

$$(1-r)^\alpha \int_{\mathbf{T}} |f_{\varepsilon_1}(r\xi)||d\xi| \ge (1-r)^\alpha \int_{\mathbf{T}} |f(r\xi)||d\xi| - \sup_{r<1}(1-r)^\alpha \int_{\mathbf{T}} |f(r\xi) - f_{\varepsilon_1}(r\xi)||d\xi|$$

$$\ge (1-r)^\alpha \int_{\mathbf{T}} |f(r\xi)||d\xi| - \varepsilon_1.$$

Hence for any $s \in [-1, \infty)$,

$$(\varepsilon - \varepsilon_1)^p \int_0^1 (1-r)^s \chi_{L_{\varepsilon,\alpha}(f)}(r)dr \le C \int_0^1 \left(\int_{\mathbf{T}} |f_{\varepsilon_1}(r\xi)||d\xi| \right)^p (1-r)^{\alpha p + s} dr.$$

Thus we have a contradiction.

It remains to show $s_1 \le Cs_2$. Let $I = [0,1)$. We argue as above and obtain from the classical Bergman representation formula (see [16]).

$$f(\rho\zeta) = f(z) = C(t) \int_{L_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}} \frac{f(r\xi)(1-r)^t}{(1 - r\overline{\xi}\rho\zeta)^{t+2}} d\xi dr + C(t) \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}} \frac{f(r\xi)(1-r)^t}{(1 - \overline{r\xi}\rho\zeta)^{t+2}} d\xi dr$$

$= f_1(z) + f_2(z)$, where t is large enough. Then we have

$$(1-\rho)^\alpha \int_{\mathbf{T}} |f_2(\rho\zeta)||d\zeta| \le C(1-\rho)^\alpha \int_{\mathbf{T}} \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}} \frac{|f(r\xi)|(1-r)^t}{|1 - r\overline{\xi}\rho\zeta|^{t+2}} |d\xi||dr||d\zeta|$$

$$\le C(1-\rho)^\alpha \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}} |f(r\xi)|(1-r)^t \left(\int_{\mathbf{T}} \frac{1}{|1 - r\overline{\xi}\rho\zeta|^{t+2}} |d\zeta| \right) |d\xi| dr$$

$$\le C(1-\rho)^\alpha \int_{I \setminus L_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}} |f(r\xi)||d\xi| \frac{(1-r)^t}{(1-r\rho)^{t+1}} dr \le C\varepsilon(1-\rho)^\alpha \int_0^1 \frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}} dr \le C\varepsilon.$$

For $\alpha > 0$, $\int_{\mathbf{D}} (1-\rho)^{\alpha-1} |f_1(\rho\zeta)| dA(\rho\zeta)$

$$\le C \int_{\mathbf{D}} (1-\rho)^{\alpha-1} \int_{L_{\varepsilon,\alpha}(f)} \int_{\mathbf{T}} \frac{|f(r\xi)|(1-r)^t}{|1 - r\overline{\xi}\rho\zeta|^{t+2}} |d\xi| dr dA(\rho\zeta)$$

$$\le C \sup_{r<1} \left((1-r)^\alpha \int_{\mathbf{T}} |f(r\xi)||d\xi| \right) \int_{L_{\varepsilon,\alpha}(f)} \frac{(1-r)^{t-\alpha}}{(1-r)^{t+1-\alpha}} dr$$

$$= C \sup_{r<1} \left((1-r)^{\alpha} \int_{\mathbf{T}} |f(r\xi)||d\xi| \right) \int_{L_{\varepsilon,\alpha}(f)} \frac{1}{(1-r)} dr.$$

Note that the implication$\|f_1\|_{B^{p,1}_{-\alpha}} < \infty$ for $p \geq 1$ follows directly from the known estimate (see [8])

$$\left(\int_0^1 (1-\rho)^{\alpha p-1} \left(\int_{\mathbf{T}} |f_1(\rho\xi)| d\xi \right)^p d\rho \right)^{\frac{1}{p}} \leq C \int_{\mathbf{D}} (1-\rho)^{\alpha-1} |f_1(\rho\xi)| dA(\rho\xi),$$

$\alpha > 0,\ p \geq 1,\ f_1 \in H(\mathbf{D})$.

Hence $\inf_{g \in B^{p,1}_{-\alpha}} \|f-g\|_{B^{\infty,1}_{-\alpha}} \leq C\|f-f_1\|_{B^{\infty,1}_{-\alpha}} = \|f_2\|_{B^{\infty,1}_{-\alpha}} \leq C\varepsilon$.

The theorem is proved. □

For $0 < p < \infty$ and $\alpha > 0$, let as above

$$B^{p,\infty}_{-\alpha}(\mathbf{D}) = \left\{ f \in H(\mathbf{D}) : \int_0^1 (M_\infty(f,r))^p (1-r)^{\alpha p-1} dr < \infty \right\},$$

where $M_\infty(f,r) = \max_{\xi \in \mathbf{T}} |f(r\xi)|,\ r \in (0,1),\ f \in H(\mathbf{D})$. It is easy to see that $B^{p,\infty}_{-\alpha}(\mathbf{D}) \subset \widetilde{\mathcal{B}}^{\alpha}(\mathbf{D}),\ 0 < p < \infty,\ \alpha > 0$.

For $\varepsilon > 0,\ f \in H(\mathbf{D})$, let $\widehat{L}_{\varepsilon,\alpha}(f) = \{r \in (0,1) : (1-r)^{\alpha} M_\infty(f,r) \geq \varepsilon\}$.

Theorem 7. *Let $f \in \widetilde{\mathcal{B}}^{\alpha},\ \alpha > 0,\ 1 \leq p < \infty$. Then the following are equivalent:*
(*a*) $s_1 = dist_{\widetilde{\mathcal{B}}^{\alpha}}(f, B^{p,\infty}_{-\alpha})$;
(*b*) $s_2 = \inf\{\varepsilon > 0 : \int_0^1 (1-r)^{-1} \chi_{\widehat{L}_{\varepsilon,\alpha}(f)}(r) dr < \infty\}$.

The proof of Theorem 7 is repetition of arguments provided in Theorem 6.
We now provide a sharp version of Theorem 6 for $p \leq 1$ case.

Theorem 8. *Let $f \in B^{\infty,1}_{-\alpha},\ p \leq 1,\ t > \alpha - 1,\ \alpha > 0$. Then the following are equivalent:*
(*a*) $s_1 = dist_{B^{\infty,1}_{-\alpha}}(f, B^{p,1}_{-\alpha})$;
(*b*) $\widehat{s}_2 = \inf\{\varepsilon > 0 : \int_0^1 \left(\int_0^1 \chi_{L_{\varepsilon,\alpha}(f)}(r) \frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}} dr \right)^p (1-\rho)^{p\alpha-1} d\rho < \infty\}$.

Proof. The proof of Theorem 8 is similar to the one provided in Theorem 6. One part of the theorem follows directly from the estimate

$$\chi_{L_{\varepsilon,\alpha}(f)}(r) \leq C(\varepsilon,\varepsilon_1) \left(\int_T |f_{\varepsilon_1}(r\xi)||d\xi| \right)^q (1-r)^{\alpha q},\ r \in (0,1),\ 0 < q < \infty,\ \alpha > 0, \tag{6}$$

which were given in the proof of the previous theorem. Indeed, from (6) for $q = 1$, we get

$$\int_0^1 \left(\int_0^1 \frac{(1-r)^{t-\alpha} \chi_{L_{\varepsilon,\alpha}(f)}(r) dr}{(1-r\rho)^{t+1}} \right)^p (1-\rho)^{\alpha p-1} d\rho$$

$$\leq C\int_0^1\int_0^1\left(\int_{\mathbf{T}}|f_{\varepsilon_1}(r\xi)|d\xi\right)^p\frac{(1-r)^{\alpha p+p-1+(t-\alpha)p}}{(1-r\rho)^{(t+1)p}}(1-\rho)^{\alpha p-1}drd\rho\leq C\|f_{\varepsilon_1}\|_{B^{p,1}_{-\alpha}}.$$

The rest is clear. It remains to argue as in the previous theorem to arrive to a contradiction.

To prove the second part we note that as in Theorem 6 we get

$$f(z)=f_1(z)+f_2(z)\ \text{ and }\ f_2(z)\leq C(1-r)^\alpha\int_{\mathbf{T}}|f_2(r\xi)||d\xi|\leq C\varepsilon.$$

And moreover, arguing similarly as for $p\geq 1$ in Theorem 6 we will have

$$\int_0^1\left(\int_{\mathbf{T}}|f_1(r\xi)||d\xi|\right)^p(1-r)^{\alpha p-1}dr$$

$$\leq C\sup_{r<1}\left((1-r)^\alpha\int_{\mathbf{T}}|f(r\xi)||d\xi|\right)^p\int_0^1\left(\int_0^1\chi_{L_{\varepsilon,\alpha}(f)}(r)\frac{(1-r)^{t-\alpha}}{(1-r\rho)^{t+1}}dr\right)^p(1-\rho)^{\alpha p-1}d\rho.$$

Hence $\inf_{g\in B^{p,1}_{-\alpha}}\|f-g\|_{B^{\infty,1}_{-\alpha}}\leq C\|f-f_1\|_{B^{\infty,1}_{-\alpha}}=\|f_2\|_{B^{\infty,1}_{-\alpha}}\leq C\varepsilon.$

The theorem is proved. □

Remark 4. Proofs of Theorem 6, 7 and 8 can be easily extended to $B^{q,p}$ spaces with more general $w(1-r)$ weights under some natural restrictions on the function $w(r)$.

For $\alpha>-1,\ \beta>0$ and $0<p<\infty$, let

$$M^\alpha_\beta(\mathbf{D})=\{f\in H(\mathbf{D}):\sup_{r<1}(1-r)^\beta\int_{|w|\leq r}|f(w)|(1-|w|)^\alpha dA(w)<\infty\}$$

and

$$M^\alpha_{p,\beta}(\mathbf{D})=\{f\in H(\mathbf{D}):\int_0^1(1-r)^{\beta p-1}\left(\int_{|w|\leq r}|f(w)|(1-|w|)^\alpha dA(w)\right)^p dr<\infty\}.$$

$M^\alpha_\beta(\mathbf{D})$ and $M^\alpha_{p,\beta}(\mathbf{D})$ for $p\geq 1$ are Banach spaces and they were studied by various authors (see for example [5]). It is easy to show that $M^\alpha_{p,\beta}(\mathbf{D})\subset M^\alpha_\beta(\mathbf{D})$, where $p\in(0,\infty),\ \beta>0,\ \alpha>-1$.

In the following result we provide another sharp result on the dist function using the characteristic function of a new set. For $f\in H(\mathbf{D})$ and $\varepsilon>0$, let

$\mathcal{G}^\alpha_{\varepsilon,\beta}(f)=\{r\in(0,1):(1-r)^\beta\int_{|w|\leq r}|f(w)|(1-|w|)^\alpha dA(w)\geq\varepsilon\},\ \beta>0,\ \alpha>-1.$

Theorem 9. *Let $p\geq 1,\alpha>-1,\beta>0,f\in M^\alpha_\beta$. Then the following are equivalent:*

(*a*) $t_1=dist_{M^\alpha_\beta}(f,M^\alpha_{p,\beta})$;

(*b*) $t_2=\inf\{\varepsilon>0:\int_0^1(1-r)^{-1}\chi_{\mathcal{G}^\alpha_{\varepsilon,\beta}(f)}(r)dr<\infty\}$.

The proof of Theorem 9 will be omitted. It can be obtained by a small modification of the proof of the previous theorem.

3 Sharp assertions for the Dist function in the unit ball and in the polydisk

The goal of this section is to provide straightforward generalizations of some of the results of the previous section to the case of the unit ball and the polydisk in $\mathbb{C}^n$: Practically all results of the previous section can be generalized to the case of the polydisk and to the unit ball. The proofs of these assertions are mostly based on the same ideas as in case of one variable, though some technical difficulties arise on that way. For the proofs of the theorems we will formulate below in higher dimensions we simply replace the well- known Bergman integral representation in the unit disk that were used in the previous section by the corresponding known integral representation version in the unit ball or in the polydisk (see [2], [16]) and then we define appropriate sets which will allow us to estimate such integral representation.

To formulate our results we will need some standard definitions (see [2], [16]).

We denote the open unit ball in $\mathbb{C}^n$ by $\mathbf{B} = \{z \in \mathbb{C}^n : |z| < 1\}$. The boundary of $\mathbf{B}$ will be denoted by $\mathbf{S}$, $\mathbf{S} = \{z \in \mathbb{C}^n : |z| = 1\}$. By dv we denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B}) = 1$, and by $d\sigma$ we denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S}) = 1$.

As usual, we denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$.

We denote the unit polydisk by $\mathbf{D}^n = \{z \in \mathbb{C}^n : |z_k| < 1,\ 1 \le k \le n\}$ and the distinguished boundary of $\mathbf{D}^n$ by $\mathbf{T}^n = \{z \in \mathbb{C}^n : |z_k| = 1,\ 1 \le k \le n\}$. By dA_{2n} we denote the volume measure on $\mathbf{D}^n$ and by dm_n we denote the normalized Lebesgue measure on $\mathbf{T}^n$. Let $H(\mathbf{D}^n)$ be the space of all holomorphic functions on $\mathbf{D}^n$. We refer to [2] and [10] for further details.

For every function $f \in H(\mathbf{D^n})$ and $f(z_1, \dots, z_n) = \sum_{k_1,\dots k_n} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}$, we define the operator of fractional differentiation by

$$\mathcal{D}^\alpha f(z_1, \dots, z_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \prod_{j=1}^{n} (k_j + 1)^\alpha a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n},\ \alpha \in \mathbb{R}.$$

We will write $Df(z)$ if $\alpha = 1$. For any α, $\mathcal{D}^\alpha$ is an operator acting from $H(\mathbf{D^n})$ to $H(\mathbf{D^n})$ (see [2]).

We formulate now direct generalization of Theorem 6 in the unit ball, its proof is a simple repetition of arguments we provide above for the unit disk and will be omitted.

For $0 < p < \infty$ and $\alpha > 0$, let

$$B^{\infty,1}_{-\alpha}(\mathbf{B}) = \left\{ f \in H(\mathbf{B}) : \sup_{r<1} \left(\int_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)| \right) (1-r)^\alpha < \infty \right\},$$

$$B^{p,1}_{-\alpha}(\mathbf{B}) = \left\{ f \in H(\mathbf{B}) : \int_0^1 \left(\int_{\mathbf{S}} |f(r\xi)||d\sigma(\xi)| \right)^p (1-r)^{\alpha p - 1} dr < \infty \right\}.$$

It is easy to see that $B^{p,1}_{-\alpha}(\mathbf{B}) \subset B^{\infty,1}_{-\alpha}(\mathbf{B}),\ 0<p<\infty,\ \alpha>0.$

We now define a new set on the unit interval and then using its characteristic function we will give a new sharp assertion concerning the distance function.

For $\varepsilon>0,\ f\in H(\mathbf{B})$, let $L_{\varepsilon,\alpha}(f)=\{r\in(0,1):(1-r)^\alpha\int_{\mathbf{S}}|f(r\xi)||d\sigma(\xi)|\ge \varepsilon\}$.

Theorem 10. *Let $f\in B^{\infty,1}_{-\alpha}(\mathbf{B}),\ \alpha>0,\ 1\le p<\infty$. Then the following are equivalent:*

(a) $\widehat{s}_1 = dist_{B^{\infty,1}_{-\alpha}(\mathbf{B})}(f, B^{p,1}_{-\alpha}(\mathbf{B}))$;

(b) $\widehat{s}_2 = \inf\{\varepsilon>0:\int_0^1(1-r)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r)dr<\infty\}$.

Let $I=(0,1)$. We denote by $r\xi=(r\xi_1,\dots,r\xi_n)$ where $r\in I,\ \xi_j\in\mathbf{T}$, $j=1,\dots,n,\ \xi=(\xi_1,\dots,\xi_n)$ and also $\overrightarrow{r}\xi=(r_1\xi_1,\dots,r_n\xi_n)$, where $\overrightarrow{r}\in I^n$, $\overrightarrow{r}=(r_1,\dots,r_n),\ r_j\in I,\ \xi_j\in\mathbf{T},\ j=1,\dots,n.$

We formulate now a polydisk version of Theorem 6.

For $0<p<\infty$ and $\alpha>0$, let as above

$$B^{\infty,1}_{-\alpha}(\mathbf{D^n})=\left\{f\in H(\mathbf{D^n}):\sup_{r_1<1,\dots,r_n<1}\left(\int_{\mathbf{T^n}}|f(\overrightarrow{r}\xi)||dm_n(\xi)|\right)\prod_{k=1}^n(1-r_k)^\alpha<\infty\right\},$$

$$B^{p,1}_{-\alpha}(\mathbf{D^n})=\left\{f\in H(\mathbf{D^n}):\int_0^1\cdots\int_0^1\left(\int_{\mathbf{T^n}}|f(\overrightarrow{r}\xi)||dm_n(\xi)|\right)^p\prod_{k=1}^n(1-r_k)^{\alpha p-1}dr_1\cdots dr_n<\infty\right\}.$$

It is easy to see that $B^{p,1}_{-\alpha}(\mathbf{D^n})\subset B^{\infty,1}_{-\alpha}(\mathbf{D^n}),\ 0<p<\infty,\ \alpha>0.$

We now define a new set on I^n and then using its characteristic function we will give a new sharp assertion concerning the distance function.

For $\varepsilon>0,\ f\in H(\mathbf{D^n})$, let

$$L_{\varepsilon,\alpha}(f)=\{\overrightarrow{r}=(r_1,\dots r_n)\in I^n:\prod_{k=1}^n(1-r_k)^\alpha\int_{\mathbf{T^n}}|f(\overrightarrow{r}\xi)||dm_n(\xi)|\ge\varepsilon\}.$$

Theorem 11. *Let $f\in B^{\infty,1}_{-\alpha},\ \alpha>0,\ 1\le p<\infty$. Then the following are equivalent:*

(a) $\widehat{\widehat{s}}_1 = dist_{B^{\infty,1}_{-\alpha}(\mathbf{D^n})}(f, B^{p,1}_{-\alpha}(\mathbf{D^n}))$;

(b) $\widehat{\widehat{s}}_2 = \inf\{\varepsilon>0:\int_0^1\cdots\int_0^1\prod_{k=1}^n(1-r_k)^{-1}\chi_{L_{\varepsilon,\alpha}(f)}(r_1,\dots r_n)dr_1\cdots dr_n<\infty\}$.

Proof. The proof is a repetition of arguments of the one dimensional case and we omit details. □

Now we formulate the polydisk version of Theorem 2 the proof is quite similar to one dimensional case and will be also omitted.

Let $\widetilde{\mathcal{B}}^{\alpha}(\mathbf{D^n}),\ \alpha>0$, be the collection of the analytic functions on the polydisk satisfying

$$\|f\|_{\widetilde{\mathcal{B}}^{\alpha}(\mathbf{D^n})}=\sup_{z_1\in\mathbf{D},\dots,z_n\in\mathbf{D}}|f(z_1,\dots,z_n)|\prod_{k=1}^{n}(1-|z_k|^2)^{\alpha}<\infty.$$

$\widetilde{\mathcal{B}}^{\alpha}(\mathbf{D^n})$ is a Banach space with the norm $\|f\|_{\widetilde{\mathcal{B}}^{\alpha}(\mathbf{D^n})}$.

For $k>s,\ 0<p,q\le\infty$, $\mathcal{B}^{q,p}_{s}(\mathbf{D^n})$ let be the class of analytic functions on the polydisk satisfying (see [8])

$$\|f\|^q_{\mathcal{B}^{q,p}_{s}(\mathbf{D^n})}=\int_0^1\cdots\int_0^1\left(\int_{\mathbf{T^n}}|D^kf(\overrightarrow{r}\xi)|^p|dm_n(\xi)|\right)^{\frac{q}{p}}\prod_{k=1}^{n}(1-r_k)^{(k-s)q-1}dr_1\cdots dr_n<\infty.$$

It is known that $\mathcal{B}^{q,q}_{s}(\mathbf{D^n})\subset\widetilde{\mathcal{B}}^{-(s-\frac{1}{q})}(\mathbf{D^n}),\ s<0$ (see [2]).

Theorem 12. *Let* $0<q\le1,\ s<0,\ t\le s-\frac{1}{q},\ \beta>\frac{1-sq}{q}$ *and* $\beta>-1-t$. *Let* $f\in\widetilde{\mathcal{B}}^{-t}(\mathbf{D^n})$. *Then the following are equivalent:*

(*a*) $l_1=dist_{\widetilde{\mathcal{B}}^{-t}(\mathbf{D^n})}(f,\mathcal{B}^{q,q}_{s}(\mathbf{D^n}))$;

(*b*) $l_2=\inf\{\varepsilon>0:\int_{\mathbf{D^n}}\left(\int_{\Omega_{\varepsilon,-t}(f)}\frac{\prod_{k=1}^{n}(1-|w_k|)^{\beta+t}dA_{2n}(w_1,\dots,w_n)}{\prod_{k=1}^{n}|1-\overline{z_k}w_k|^{2+\beta}}\right)^q\times$
$\times\prod_{k=1}^{n}(1-|z_k|)^{-sq-1}dA_{2n}(z_1,\dots,z_n)<\infty\}$.

We will formulate now a sharp theorem for analytic classes on the subframe. Let $\mathcal{R}^sf(z)=\sum_{k_1,\dots k_n\ge0}(k_1+\cdots+k_n+1)^sa_{k_1,\dots,k_n}z_1^{k_1}\cdots z_n^{k_n}$ and $\widetilde{\mathbf{D}}^n=(0,1]\times\mathbf{T}^n$. It is obvious $\mathcal{R}^sf\in H(\mathbf{D}^n)$ if $f\in H(\mathbf{D}^n)$.

It is easy to note that $\|f\|_{B^{\infty,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n)}=\sup_{r<1}(1-r)^{\alpha}\int_{\mathbf{T^n}}|\mathcal{R}^sf(r\xi)|dm_n(\xi)$

$$\le C\left(\int_0^1\left(\int_{\mathbf{T^n}}|\mathcal{R}^sf(r\xi)|dm_n(\xi)\right)^p(1-r)^{\alpha p-1}dr\right)^{\frac{1}{p}}=\|f\|_{B^{p,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n)},$$

where $s\in\mathbb{R},\ \alpha>0,\ 0<p<\infty$.

The analytic classes on the subframe $\widetilde{\mathbf{D}}^n$ were studied in [11], [12], [13]. For $\varepsilon>0$ and $f\in H(\mathbf{D}^n)$, let $K_{\varepsilon,\alpha,s}=\{r\in I:(1-r)^{\alpha}\int_{\mathbf{T}^n}|\mathcal{R}^sf(r\xi)|dm_n(\xi)\ge\varepsilon\}$.

Theorem 13. *Let* $f\in B^{\infty,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n),\ \alpha>0,\ 1\le p<\infty,\ s\in\mathbb{R}$. *Then the following are equivalent:*

(*a*) $\nu_1=dist_{B^{\infty,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n)}(f,B^{p,1}_{-\alpha,s}(\widetilde{\mathbf{D}}^n))$;

(*b*) $\nu_2=\inf\{\varepsilon>0:\int_0^1(1-r)^{-1}\chi_{K_{\varepsilon,\alpha,s}(f)}(r)dr<\infty\}$.

This paper only concerns with the situation when some Bergman (or mixed norm) space is acting as a subspace of a larger analytic class where sup can be seen somehow in quasinorm. It is also easy to notice that we systematically use the classical Bergman integral representation formula in all our proofs.

We note that based on similar arguments we obtained corresponding sharp results in cases when mentioned above Bergman classes are replaced by analytic (weighted) Hardy type spaces in the unit disk. The only visible difference is that in such cases proofs are based on the classical Causchy integral representation formula. We give an example of such a result in the unit disk.

Theorem 14. *Let $\alpha > \frac{1}{p}$, $p \geq 1$, $f \in B^{\infty,p}_{\frac{1}{p}-\alpha}(\mathbf{D})$. Then the following are equivalent:*

(*a*) $\upsilon_1 = dist_{\widetilde{\mathcal{B}}^{\alpha}(\mathbf{D})}(f, B^{\infty,p}_{\frac{1}{p}-\alpha}(\mathbf{D}))$;

(*b*) $\upsilon_2 = \inf\{\varepsilon > 0 : \sup_{r<1}\left(\frac{1}{1-r}\int_{\mathbf{T}} \chi_{\Omega_{r,\varepsilon,\alpha}}(\xi) d\xi\right) < \infty\}$,

where $\Omega_{r,\varepsilon,\alpha} = \{\xi \in \mathbf{T} : |f(r\xi)|(1-r)^{\alpha} \geq \varepsilon\}$.

In [15] the author provided several obvious corollaries of his results. Similar corollaries can be obtained immediate from our theorems 6 - 9. As example we note that from Theorem 6 we have the following

Proposition 1. *Let $\alpha > 0$, $1 \leq p_1 < p_2 < \infty$. Then*

$$dist_{B^{\infty,1}_{-\alpha}}(f, B^{p_1,1}_{-\alpha}) = dist_{B^{\infty,1}_{-\alpha}}(f, B^{p_2,1}_{-\alpha}).$$

Proposition 2. *Let $\alpha > 0$, $1 \leq p_1 < p_2 < \infty$. Then the closures of $B^{p_1,1}_{-\alpha}$ and $B^{p_2,1}_{-\alpha}$ in $B^{\infty,1}_{-\alpha}$ are the same and f is in closure of $B^{p_1,1}_{-\alpha}$ in $B^{\infty,1}_{-\alpha}$ if and only if $\int_0^1 (1-r)^{-1}\chi_{L_{\varepsilon,-\alpha}(f)}(r) < \infty$, for every $\varepsilon > 0$.*

Similar results obviously are true also in higher dimension. We omit details.

References

[1] L. V. Ahlfors:Bounded analytic functions, Duke Math Journal, 14 (1947), 1-14.

[2] M. M. Djrbashian, F. A. Shamoian:Topics in the theory of A^p_α classes, Teubner Texte zur Mathematics, 1988, v 105.

[3] P. L. Duren:Theory of H^p spaces, Academic Press, New York, 1970.

[4] P. G. Ghatage, D. Zheng:Analytic functions of bounded mean oscillation and the Bloch space, Integr. Equat. Oper. Theory, 17 (1993), 501-515.

[5] M. Jevtić, M. Pavlović, R. F. Shamoyan:A note on diagonal mapping theorem in spaces of analytic functions in the unit polydisk, Publ. Math. Debrecen, 74/1-2, 2009, 1-14.

[6] D. Khavinson, M. Stessin:Certain linear extremal problems in Bergman spaces of analityc function , Indiana Univ. Math. J., 3(46) (1997).

[7] S. Ya Khavinson:On an extremal problem in the theory of analityc function , Russian Math. Surv., 4(32) (1949), 158-159.

[8] J. Ortega, J. Fabrega:Hardy's inequality and embeddings in holomorphic Triebel-Lizorkin spaces, Illinois J. Math. 43 (1999), 733-751.

[9] W. Rudin:Analytic functions of class H^p, Trans AMS, 78 (1955), 46-66.

[10] W. Rudin:Function theory in polydisks, Benjamin, New York, 1969.

[11] R. F. Shamoyan, O. Mihić:On some inequalities in holomorphic function theory in polydisk related to diagonal mapping, Czechoslovak Mathematical Journal 60(2) (2010), 351-370.

[12] R. F. Shamoyan, O. Mihić:Analytic classes on subframe and expanded disk and the $\mathcal{R}^s$ differential operator in polydisk, Journal of Inequalities and Applications,Volume 2009, Article ID 353801, 22 pages.

[13] Songxiao Li, R. F. Shamoyan:On some properties of a differential operator on the polydisk, Banach Journal of Math. Analysis, v.3, n.1, (2009), 68-84.

[14] W. Xu:Distances from Bloch functions to some Möbius invariant function spaces in the unit ball of $\mathbb{C}^n$, J. Funct. Spaces Appl., v.7, n.1, (2009), 91-104.

[15] R. Zhao:Distances from Bloch functions to some Möbius invariant spaces, Ann. Acad. Sci. Fenn. Math. 33 (2008), 303-313.

[16] K. Zhu:Spaces of Holomorphic Functions in the Unit Ball, Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.

Romi Shamoyan
Bryansk University, Bryansk, Russia.

Olivera Mihic
Fakultet organizacionih nauka, Jove Ilića 154, Belgrade, Serbia.
e-mail: rshamoyan@yahoo.com, oliveradj@fon.rs