On the controllability of a two-cell CNN

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Abstract. In this paper we study and characterize the controllability of a constant 2-cells CNN (Cellular Neural Network) with feedback resembling a symmetric or antisymmetric matrix and input with all entries set to zero except its first element . We characterize and give a precise description of the control in each case. This problem has been attacked already in order to study complete stability and in the seek of chaotic attractor; but this time the controllability is addressed.

Resumen. En este trabajo se estudia y caracteriza la capacidad de control de una red celular neural, (CNN por sus siglas en inglés) constante de 2 células con retroalimentación parecida a una matriz simétrica o antisimétrica y de entradas todas cero, excepto por el primer elemento. Caracterizamos y damos una descripción precisa del control en cada caso. Este problema se ha atacado con el fin de estudiar la estabilidad completa y en la busca de atractores caóticos; pero esta vez consideramos la capacidad de control.

0   Introduction

Since its introduction by Chua and Yang in 1988 ([6], [7]), the cellular Neural Network model (linear, nonlinear and delayed) referred as CNN, has been shown to have a host of desirable properties long sought after by the neural network community. CNNs are cellular, analog, programmable and multidimensional processing arrays with distributed logic and memory. The processing elements are locally connected. The extension of the CNN paradigm is the CNN universal machine in which distributed and global memories and logic functions support the execution of complex analogical algorithms. The key feature of the CNN architecture is its high operation speed ([18]). Several variations of he original CNN have been proposed and used for black and white image processing tasks, like edge detections, noise removal, horizontal and vertical line filtering and many many others ([5], [9], [10], [13], [16]).

Chaotic CNN has been used for image segmentation ([17]). Another application can be found in [18] where CNN is used in bubble debris classification

problem to distinguish debris particles from air bubbles. This is not an easy task due to the coarse resolution of the images and to requirements for an extremely low false alarm for misclassified bubbles at a very high processing speed.

In its simplest form, a CNN is either a linear or planar array of locally interconnected processors, in which each processor has a transfer of signal functions which is piece-wise linear function mapping the internal state of each given processor to the output of that processor. Letting x represent the internal state of a processor in such an array, the standard signal function (from [7] onward), takes the form

$$f(x) = \frac{1}{2}(|x+1| + |x-1|). \tag{1}$$

In the most general case f is sigmoid function, that is, differentiable, bounded and $f'(x) > 0$, for any $x \in \mathbf{R}$.

In the context of a simple CNN model, locally connected means the processor is connected to its nearest neighbors. For instance, in the linear array, processors are connected to a single line such that each processor which is not at either end of the line being connected to two other processors, which are its nearest neighbors.

Controllability of CNN for the linear case (f as in (1)) has been studied by imposing different boundary conditions; for instance, periodic boundary conditions in [12] (constant input) and [3] (periodic input); von Neumann boundary conditions in [2] (constant input) and [4] (periodic input).

In this paper we address the controllability of a modified version of the two-cell CNN given by

$$\begin{aligned} x_1' &= -x_1 + py_1 + sy_2 + bu \\ x_2' &= -x_2 + ry_1 + py_2 \end{aligned} \tag{2}$$

with the output function $y = g(x)$ and g being a sigmoid function. For this type of problem Zou and Nossek discovered a chaotic attractor ([19], [20], [21]) with $u = b\sin(\frac{2\pi t}{T})$ and $r = -s$. In [15] bifurcation and chaos are studied in a general problem (2) with output function (1) and

$$p > 1, \ p - 1 < r, \ p - 1 < -s. \tag{3}$$

In its more general form the equation for the CNN model (array of $M \times N$ Cells) is given as

$$\dot{x} = -x + AG(x) + Bu$$

where A and B are $n \times n$ matrices, $n = MN$, $x \in \mathbf{R}^n$, $G(x) = \text{col}(g(x_1), g(x_2) \cdots, g(x_n))$, $x = \text{col}(x_1, x_2, \cdots, x_n)$ and g a sigmoid function.

1   Controllability

In this section we set the problem in a more convenient way and prove the results on controllability. We begin with the following

Definition 1.1 *([11], [14]) A control system*

$$\dot{z} = f(z, u),$$

with $f(z, u) \in \mathcal{C}^1$ in $\mathbf{R}^n \times \mathbf{R}^m$ is said to be controllable in $[t_0, t_1]$ if for each pair of points z_0 and z_1 both in $\mathbf{R}^n$ there exists a bounded measurable controller $u(t)$ on $[t_0, t_1]$ (with values in $\mathbf{R}^m$) such that the corresponding response $z(t)$ steers $z(t_{0)} = z_0$ to $z(t_1) = z_1$.

We consider, as said before, g sigmoid function, $g(0) = 0$, and rewrite it as $g(x) = g'(0) + o(|x|)$ $(x \in \mathbf{R})$ as $x \to 0$, and (2) now looks like

$$x = Ax + Bu + o(||x||), \tag{4}$$

but now

$$x = \mathrm{col}(x_1, x_2), \ \ u = \mathrm{col}(u_1, u_2), \ \ B = (\,b\,)\,000$$

and

$$A = (\,-\,)\,1 + g'(0)p\,g'(0)s\,g'(0)r - 1 + g'(0)p,$$

The following result from [11] will be used in order to guarantee the controllability of (4) in the interval $[t_0, t_1]$

Theorem 1.2 *A system like (4) is controllable en $[t_0, t_1]$ if and only if matrix*

$$W(t_0, t_1) = \int_{t_0}^{t_1} [F(t_1, t)B(t)][F(t_1, t)B(t)]^T \, dt$$

is non singular. Where $F(t_1, t)$ is the transition matrix of the system.

We may notice that in our case (autonomous case) the above matrix is $F(t_1, t) = \exp(A(t_1 - t))$ for all t_1, t reals.

Our main result now runs as follows

Theorem 1.3 *The matrix $W(t_0, t_1)$, with A and B given as in (4), is invertible for all $t_0, t_1 \in \mathbf{R}$, $t_0 < t_1$.*

Proof. The eigenvalues of A are $\lambda = -1 + g'(0)p + g'(0)\sqrt{-sr}\,i$ and $\bar{\lambda} = -1 + g'(0)p - g'(0)\sqrt{-sr}\,i$ with corresponding eigenvector $\begin{pmatrix} \frac{\sqrt{-sr}}{r}\,i \\ 1 \end{pmatrix}$, indeed $\begin{pmatrix} \frac{\sqrt{-sr}}{r}\,i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{-sr}}{r} \\ 0 \end{pmatrix}\,i$. Now we define the matrix $P =$

$\left(\begin{array}{cc} \frac{\sqrt{-sr}}{r} & 0 \\ 0 & 1 \end{array} \right)$, therefore $P^{-1} = \left(\begin{array}{cc} \frac{r}{\sqrt{-sr}} & 0 \\ 0 & 1 \end{array} \right)$. But then

$$P^{-1}AP = \left(\begin{array}{cc} -1 + g'(0)p & -g'(0)\sqrt{-sr} \\ g'(0)\sqrt{-sr} & -1 + g'(0)p \end{array} \right).$$

Let us set $-1 + g'(0)p = H$ y $g'(0)\sqrt{-sr} = K$ and define a new matrix given by

$$M = \left(\begin{array}{cc} H & -K \\ K & H \end{array} \right).$$

In this context matrix A looks like $A = PMP^{-1}$, $e^{At} = Pe^{Mt}P^{-1}$, and

$$e^{At} = Pe^{Ht} \left(\begin{array}{cc} \cos(Kt) & -\sin(Kt) \\ \sin(Kt) & \cos(Kt) \end{array} \right) P^{-1};$$

that is,

$$e^{At} = \left(\begin{array}{cc} e^{Ht}\cos(Kt) & -\frac{\sqrt{-sr}}{r}e^{Ht}\sin(Kt) \\ -\frac{\sqrt{-sr}}{s}e^{Ht}\sin(Kt) & e^{Ht}\cos(Kt) \end{array} \right),$$

matrix $F(t_1, t)$ now becomes

$$\left(\begin{array}{cc} e^{H(t_1-t)}\cos(K(t_1-t)) & -\frac{\sqrt{-sr}}{r}e^{H(t_1-t)}\sin(K(t_1-t)) \\ -\frac{\sqrt{-sr}}{s}e^{H(t_1-t)}\sin(K(t_1-t)) & e^{H(t_1-t)}\cos(K(t_1-t)) \end{array} \right).$$

Consequently $[F(t_1, t)B(t)][F(t_1, t)B(t)]^T$ is

$$\left(\begin{array}{cc} e^{2H(t_1-t)}(\cos(K(t_1-t)))^2 b^2 & \frac{e^{2H(t_1-t)}\cos(K(t_1-t))b^2\sin(K(t_1-t))r}{\sqrt{-sr}} \\ \frac{e^{2H(t_1-t)}\cos(K(t_1-t))b^2\sin(K(t_1-t))r}{\sqrt{-sr}} & \frac{e^{2H(t_1-t)}rb^2\left(-1+(\cos(K(t_1-t)))^2\right)}{s} \end{array} \right).$$

We calculate $\int_{t_0}^{t_1} [F(t_1, t)B(t)][F(t_1, t)B(t)]^T\, dt$ and obtain

$$W(t_0, t_1) = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right),$$

with

$$a_{11} = \frac{b^2}{4H(H^2+K^2)} \left[e^\alpha (H^2\cos\beta + HK\sin\beta + H^2 + K^2) - 2H^2 - K^2 \right]$$

$$a_{12} = a_{21} = -\frac{b^2\sqrt{-sr}}{4s(H^2+K^2)} \left[e^\alpha (H\sin\beta - K\cos\beta) + K \right]$$

$$a_{22} = \frac{b^2 r}{4sH(H^2+K^2)} \left[e^\alpha (H^2\cos\beta + HK\sin\beta - H^2 - K^2) + K^2 \right],$$

here $\alpha = 2H(t_1 - t_0)$ and $\beta = 2K(t_1 - t_0)$.

On the other hand

$$\det(W(t_0, t_1)) = -\frac{b^4 r e^\alpha}{16 s H^2 (H^2 + K^2)} \left[-2H^2 + 2H^2 \cos\beta + K^2 e^{-\alpha} + K^2 e^\alpha - 2K^2 \right]$$

and $\det(W(t_0, t_1)) = 0$ if and only if $-2H^2 + 2H^2 \cos\beta + K^2 e^{-\alpha} + K^2 e^\alpha - 2K^2 = 0$ if and only if $H^2 \cos\beta + K^2 \cosh\alpha = H^2 + K^2$ and this last equality takes place if and only if $\alpha = \beta = 0$ which is impossible to happen; in conclusion, for our system $\det(W(t_0, t_1)) \neq 0$ and $W(t_0, t_1)$ is non singular matrix. ∎

Corollary 1.4 *System (4) is controllable in $[t_0, t_1]$, $t_0 < t_1$.*

2 The Control

Here we find the optimal control that we know already exists because of previous section, for this purpose the Pontryagin Maximum Principle and Filippov Theorem will be used ([1], [8], [14]). We rewrite system (4) in a more convenient way in order to use the mentioned results, first of all we set $b = 1$ (or $b = -1$) since b is a fix constant, and set the controls as the interval $[-1, 1]$ also

$$f(X, u) = AX + B(u) 0, \ A = (-)1 + g'(0)p g'(0)s g'(0)r - 1 + g'(0)p, \ B = (1)000, \ X = (x)_1 x_2.$$

Notice that f is defined on $\mathbf{R}^2 \times [-1, 1]$, linear, and the set of controls is compact. The control-dependent Hamiltonian function of Pontryagin Maximum Principle is now $(\xi = (\xi_1, \xi_2))$

$$h_u(\xi, X) = (\xi_1, \xi_2)(\ (\ ) - 1 + g'(0)p)x_1 + g'(0)s x_2 + u g'(0)r x_1 + (-1 + g'(0)p)x_2$$

or

$$h_u(\xi, X) = (-1 + g'(0)p)x_1\xi_1 + g'(0)s x_2\xi_1 + u\xi_1 + g'(0)r x_1\xi_2 + (-1 + g'(0)p)x_2\xi_2$$

and the corresponding Hamiltonian system has the form

$$\begin{cases} \dot{X} = \frac{\partial h_u}{\partial \xi} \\ \dot{\xi} = -\frac{\partial h_u}{\partial X}. \end{cases}$$

In coordinates this system splits into two independent subsystems

$$\begin{cases} \dot{x}_1 = (-1 + g'(0)p)x_1 + g'(0)s x_2 + u \\ \dot{x}_2 = g'(0)r x_1 + (-1 + g'(0)p)x_2, \end{cases} \quad \begin{cases} \dot{\xi}_1 = -(-1 + g'(0)p)\xi_1 - g'(0)r\xi_2 \\ \dot{\xi}_2 = -g'(0)s\xi_1 - (-1 + g'(0)p)\xi_2. \end{cases}$$

By Pontryagin Maximum Principle, if a control $\tilde{u}(\cdot)$ is time optimal, then the Hamiltonian system has a nontrivial solution $(\xi(t), X(t))$, $\xi(t) \neq 0$ such that

$$h_{\tilde{u}}((\xi(t), X(t))) = \max_{|u| \leq 1} h_u((\xi(t), X(t))),$$

but notice that

$$\max_{|u| \leq 1} h_u((\xi(t), X(t))) = (-1 + g'(0)p)x_1\xi_1 + g'(0)sx_2\xi_1 + g'(0)rx_1\xi_2 + (-1 + g'(0)p)x_2\xi_2 + |\xi_1|,$$

if $\xi_1(t) \neq 0$, then

$$\tilde{u} = \text{sgn}\xi_1.$$

In order to determine ξ_1 we look at the system

$$\begin{cases} \dot{\xi}_1 = -(-1 + g'(0)p)\xi_1 - g'(0)r\xi_2 \\ \dot{\xi}_2 = -g'(0)s\xi_1 - (-1 + g'(0)p)\xi_2 \end{cases}$$

which can be written as

$$\dot{\xi} = -A^T \xi$$

where A is given as before, and T means transposed. But then this system can be solved and ξ_1 is uniquely determined, so is $\tilde{u}$ and of course (4) as well for this $\tilde{u}$.

Example 2.1 *Let g be the sigmoid function given as $g(x) = \arctan(x)$, with $x \in [-1, 1]$. Then $g'(0) = 1$ and*

$$A = (-)1 + psr - 1 + p, \quad -A^T = (1) - p - r - s1 - p.$$

Therefore the foregoing equation can be solved for ξ_1 and ξ_2, actually

$$\xi_1 = \exp[(1 - p)t][C_1 \cos(\sqrt{-rs})t + C_2 \sin(\sqrt{-rs})t]$$

where C_1 and C_2 are constants, which means that $\tilde{u}$ can be explicitly determined once C_1 and C_2 are chosen.

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