DIVULGACIÓN MATEMÁTICA

A simple application of the implicit function theorem*

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Abstract. In this note we show that the roots of a polynomial are C^∞ depend of the coefficients. The main tool to show this is the Implicit Function Theorem.

Resumen. En esta nota se muestra el hecho que las raíces de un polinomio son C^∞, depende de los coeficientes. La herramienta principal para mostrar este resultado es el teorema de la función implícita.

1 Motivation

Quadratic equations appear already in antiquity (Babylon, 2000 BC) and the solution of the general equation, even when it appears in different forms in the Euclid's Elements treated by geometric arguments, is obtained algebraically by the Arab mathematician al-Khowarizmi. He is credited with being the first to solve the quadratic equation

$$ax^2 + bx + c = 0, \tag{1}$$

where a, b and c are real numbers with $a \neq 0$.

Denoting by $\Delta = b^2 - 4ac$, we have the the zeros of (1) are given by

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}. \tag{2}$$

Thus, we have
$i)$ distinct roots if $\Delta > 0$
$ii)$ equals roots if $\Delta = 0$ and

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iii) complex roots if $\Delta < 0$.

In the case i) we say that the roots of (1) are simple and in the case ii) we say that the root of (1) is multiple.

A natural question that arises is:

Question 1.1 *If we change slightly the coefficients a, b and c in* (1), *what happens with the root x given by* (1.2)*?*

First, let's look at two simple examples:

Example 1.2 *Find the roots of the following polynomial p given by*

$$p(x) = x^2 + x - 6. \tag{3}$$

Using the formula (2) *we obtain $x = 2$ and $x = -3$. This is the polynomial p has real and distinct roots.*

Example 1.3 *Find the roots of the following polynomial p given by*

$$p(x) = (1,01)x^2 + 0,99x - 6,01. \tag{4}$$

Using the formula (2) *we obtain $x = 1,99$ and $x = -2,97$. This is the polynomial p has real and distinct roots.*

From (3) and (4) we can see that if we change slightly the coefficients of the quadratic function then the roots will also change slightly

With the intention of answering this question, we look to the right side of equation (2) as a function that depends on the coefficients a, b and c. This is

$$x = x(a, b, c). \tag{5}$$

Thus, if we show that x as a function of the coefficients a, b and c is a continuous function, this answer our question posed above.

More generally consider a polynomial function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^{n} a_i x^i, \tag{6}$$

where $a_j \in \mathbb{R}$, $j = 0, 1, \cdots, n$.

The aim of this short note is to show that the simple real zeros of the equation (6) depend C^∞ of the parameter $\lambda = (a_0, a_1, \cdots, a_n) \in \mathbb{R}^{n+1}$ formed by the coefficients of p.

Definition 1.1 *We say that $c \in \mathbb{R}$ is a zero of p if $p(c) = 0$.*

Theorem 1.1 *If c is a zero of p then there exists a polinomial function q such that*

$$p(x) = (x - c)q(x). \tag{7}$$

Proof. If c is a zero of p then for all x we have

$$p(x) = p(x) - p(c) = \sum_{i=0}^{n} a_i(x^i - c^i) = (x - c)q(x).$$

□

Definition 1.2 *We say that c is a simple zero of p if $q(c) \neq 0$.*

If $q(c) = 0$ then $q(x) = (x - c)r(x)$, thus $p(x) = (x - c)^2 r(x)$, where r is another polinomial function.

Deriving the equality (7) we have

$$p'(x) = q(x) + (x - c)q'(x). \tag{8}$$

The following result is easy to prove

Theorem 1.2 *The number c is simple zero of p if only if $p'(c) \neq 0$.*

Proof. Follows from (8). □

2 C^∞ dependence of the zeros of polynomials with respect to the coefficients

The main result of this short note is the following result which is the positive answer to the Question 1.1

Theorem 2.1 *If c is a simple zero of p then c is a function of class C^∞ of the coeficients $a_0, a_1, \cdots, a_n$ of the polinomial p.*

Proof. Let us define the following function

$$\begin{aligned} f \quad : \mathbb{R} \times \mathbb{R}^{n+1} \quad &\to \quad \mathbb{R} \\ (x, \lambda) \quad &\to \quad f(x, \lambda) \quad = \sum_{i=0}^{n} a_i x^i \end{aligned}$$

where $\lambda = (a_0, \cdots, a_n)$.

Clearly we see that f is C^∞ class in the variables x and λ.

Suppose c_0 is a simple zero of polynomial function p^0 corresponding to the particular case $\lambda_0 = (a_0^0, a_1^0, \cdots, a_n^0) \in \mathbb{R}^{n+1}$ then

- $f(c_0, \lambda_0) = \sum\limits_{i=0}^{n} a_i^0 (c_0)^i = p^0(c_0) = 0$ and
- $\dfrac{\partial f}{\partial x}(c_0, \lambda_0) = \dfrac{\partial}{\partial x}\big(\sum\limits_{i=0}^{n} a_i x^i\big)\big|_{(c_0,\lambda_0)} = (p^0)'(c_0) \neq 0.$

By the implicit function theorem (Theorem 3.1) there are open neighborhoods $U \subset \mathbb{R}$ of c_0 and $V \subset \mathbb{R}^{n+1}$ of λ_0 such that for all $\lambda \in V$ there exists an unique $c \in U$ with $f(c, \lambda) = 0$.

Thus, we have a unique map

$$\begin{aligned} c : V &\to U \\ \lambda &\mapsto c(\lambda) \end{aligned}$$

such that $f(c(\lambda), \lambda) = 0$. This tells us that the polynomial function p corresponding to the value of the parameter $\lambda = (a_0, \cdots, a_n) \in \mathbb{R}^{n+1}$ near λ_0, also has exactly a simple zero $c(\lambda) \in \mathbb{R}$ near to c_0.

Since $f \in C^k$ for all $k \in \mathbb{N}$, the function $c(\cdot) \in C^k$ for all $k \in \mathbb{N}$. This tells us that the simple zeros of p are C^∞ dependent of the coefficients. Moreover, the following formula holds

$$\frac{\partial c}{\partial \lambda}(\lambda) = -\left[\frac{\partial f}{\partial c}(c(\lambda), \lambda)\right]^{-1} \cdot \frac{\partial f}{\partial \lambda}(c(\lambda), \lambda).$$

□

Remark 2.1 *We believe that this simple application of the implicit function theorem is appropriate for the undergraduate classroom or homework. Also, using this same theorem we can conclude that the eigenvalues of a matrix* $A = (a_{ij})_{n\times n}$ *depend continuously of the coefficients* a_{ij}.

Remark 2.2 *This positive result should be contrasted with Abel's theorem, which in algebra states that there is not a formula for the zeros of a polynomial of degree* $n \geqslant 5$ *in terms of coefficients.*

Remark 2.3 *For polynomials with more than one variable we have analytical dependence of the zeros, for more details see [2], p.* 362.

3 Implicit function theorem

For completeness of this note we states here the Implicit function theorem. More information about the history, theory and applications of this theorem can be found in [1].

Denoting by

$$\mathcal{L}(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n; T \text{ is linear}\}$$
$$\mathrm{Aut}(\mathbb{R}^n) = \{T \in \mathcal{L}(\mathbb{R}^n) : T \text{ is a bijection}\}.$$

Theorem 3.1 (Implicit Theorem) *Let $W \subset \mathbb{R}^n \times \mathbb{R}^p$ a open set and $f \in C^k(W, \mathbb{R}^n)$. Suppose that for some $(x_0, y_0) \in W$, $f(x_0, y_0) = 0$ and $D_x f(x_0, y_0) \in \mathrm{Aut}(\mathbb{R}^n)$. Then there are open neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^p$ of y_0 such that for all $y \in V$ there exists a unique $x \in U$ with $f(x, y) = 0$.*

Thus we have a unique map

$$\begin{aligned} \psi : V &\to U \\ y &\mapsto \psi(y) \end{aligned}$$

with $\psi(y) = x$ and $f(\psi(y), y) = 0$. Moreover, the derivative of ψ in y, $D\psi(y) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$ is given by

$$D\psi(y) = -[D_x f(\psi(y), y)]^{-1} \circ D_y f(\psi(y), y), \quad y \in V.$$

Proof. See [1]. □

References

[1] Krantz, Steven G.; Parks, Harold R. *The Implicit function theorem. History, theory, and applications.* Birkhäuser Boston, Inc., Boston, MA, 2002

[2] Hörmander, Lars, *The analysis of linear partial differential operators.* II. *Differential operators with constant coefficients.* Reprint of the 1983 original. Classics in Mathematics. Springer-Verlag, 2005.

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