A note on the lower bounds for the Hausdorff Dimension of the Geometric Lorenz Attractor

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Abstract. We present a lower bound for the Hausdorff Dimension of the Geometric Lorenz Attractor in the monotone homoclinic case. These lower bound only involves the eigenvalues of the singular point of the flow and the number of turns around of the homoclinic orbit.

Resumen. Se presenta un límite inferior para la dimensión de Hausdorff del Atractor Geométrico de Lorenz en el caso monótono homoclínico. Estos límites inferiores involucra sólo los auto–valores del punto singular del flujo y el número de vueltas alrededor de la órbita homoclínica.

1 Introduction

One hard problem in dynamics is getting lower bounds on the Hausdorff dimension for attractors in terms of dynamical features of the system. In [LM], this problem was addressed for the Geometric Model of The Lorenz Attractor Λ when the system presented a double-homoclinic loop. Let $\mathcal{L}$ be the flow used to describe the Geometric Model of the Lorenz Attractor, one of the main feature of this flow is that it has a unique singularity O such that the $D\mathcal{L}(O)$ has three eigenvalues λ_1, λ_2 and λ_3 which satisfy $\lambda_2 < \lambda_3 < 0 < \lambda_1$. When both branches of the unstable manifold of O meet the stable manifold of O we will be speaking about the *homoclinic case*.

In [LM] it was proved the following theorem.

Theorem 1 *For the homoclinic case, there exists a constant* $\gamma > 0$ *such that the Geometric Lorenz attractor of the flow* $\mathcal{L}$ *has*

$$dim_H(\Lambda) \geq 2 + \ln \rho(A) / \ln \left(\frac{1}{\gamma}\right) > 2,$$

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where $\rho(A)$ is the spectral radius of the matrix A, a $(0,1)$-matrix which describes the geometric distribution of Λ in the direction given by the eigenvector associated with λ_3.

The constant γ is given in terms of the eigenvalues λ_i and the number of turns of the homoclinic loop Γ, see [LM] for details. In this note we want to show a lower bound for $\rho(A)$ for the matrices A which appear in what we called the homoclinic monotone case. For us an orbit Γ should be a *monotone homoclinic orbit* for the system $\mathcal{L}$ if it is a homoclinic orbit associated with the singular point O such that the sequence obtained intersecting Γ with the Poincar section when is projected onto the direction given by the eigenvector associated to λ_2 turn out a finite monotone sequence. For a precise description see [LM]. Such a matrices A we will call *monotone homoclinic matrices* and we will denote the set of these matrices by $\mathcal{MH}$. It turns out that the lower bound for the spectral radius is given in terms of Littelwood Pisot numbers. A Pisot number is a real algebraic integer, all of whose conjugates lie strictly inside the open unit disk. A Littlewood polynomial is a polynomial all of whose coefficients are +1 or -1. A Littelwood Pisot number is a Pisot number whose minimal polynomial is a Littelwood polynomial.

Theorem 2 $\min_{A\in\mathcal{MH}}\rho(A) > (1+\sqrt{5})/2$.

As a corollary we get the following uniform estimative for the Geometric Lorenz attractor in the monotone homoclinic case.

Theorem 3 *In the monotone homoclinic case, the Geometric Lorenz attractor of the flow $\mathcal{L}$ has*

$$dim_H(\Lambda) \geq 2 + \ln\left((1+\sqrt{5})/2\right) / \ln\left(\frac{1}{\gamma}\right) > 2.$$

2 Monotone homoclinic matrices

Let the shift space $X_n \subset \{0,1\}^{\mathbb{N}}$ given by the restriction

$$\mathcal{F}_n = \{\overbrace{000...0}^{n-times}, \overbrace{111...1}^{n-times}\},$$

i.e., the set of sequences $\mathbf{i} = i_0 i_1 i_2 ...$, such that the above blocks never appear as a building block of this sequence. This set is invariant under the shift transformation σ given by $\sigma(\mathbf{i}) = \mathbf{j}$, where $j_s = i_{s+1}$. The following proposition in proved in [LM].

Proposition 1 *Let $X \subset \{0,1\}^{\mathbb{N}}$ be a shift of finite type, then there exists $\Phi : X \to Y$ a conjugation ($\Phi \circ \sigma = \sigma \circ \Phi$ and Φ a homeomorphism), where Y is a subshift of finite type over the alphabet built with the words not in $\mathcal{F}$.*

The space shift X_n is one of finite type. With this shift space we associate an alphabet given by

$$\mathcal{B}_n = \{ \overbrace{000...1}^{n-times}, \overbrace{00...01}^{n-times}, \overbrace{0...011}^{n-times}, \dots, \overbrace{1...110}^{n-times} \}.$$

We arrange this set accordingly to the lexicographical order relation. Its transition matrix $A_n = (a_{i\,j})$ is given by

$$a_{i\,j} = \begin{cases} 0 & if \quad i \leq (2^n-2)/2 - 1 \quad and \quad j \neq 2\,i, 2\,i+1 \\ 1 & if \quad i \leq (2^n-2)/2 \quad and \quad j = 2\,i, 2\,i+1 \\ 0 & if \quad i = (2^n-2)/2 \quad and \quad j \neq (2^n-2) \\ 1 & if \quad i = (2^n-2)/2 \quad and \quad j = (2^n-2) \\ 0 & if \quad i = (2^n-2)/2 + 1 \quad and \quad j \neq 1 \\ 1 & if \quad i = (2^n-2)/2 + 1 \quad and \quad j = 1 \\ A_{s\,j} & with \quad s = i - (2^n-2)/2 + 1 \quad if \quad i \geq (2^n-2)/2 + 2 \end{cases}$$

Example 1 *For $n = 3$, the size of the transition matrix is 6. A_3 is given by*

$$A_3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Example 2 *For $n = 4$, the size of the transition matrix is 14. A_4 is given by*

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The matrices A that we want to consider are the A_n above. We want to get a lower bound for the spectral radius of matrices A. In [LM] it is shown that matrices A are the corresponding matrices in Theorem 1 for the monotone homoclinic case.

In order to do that we recall some elementary facts from linear algebra.

Lemma 1 *If a polynomial q is a factor of the minimal polynomial of A, then*

$$\rho(A) \geq \max_{\lambda | q(\lambda)=0} \{|\lambda|\}.$$

Proof 1 *It is immediate.*

Lemma 2 *The polynomial $q_n(\lambda) = -1-\lambda-\lambda^2-\cdots-\lambda^{n-2}+\lambda^{n-1}$ is irreducible in $\mathbb{Z}[x]$, the ring of polynomial with integers coefficients.*

Proof 2 *See [B].*

Lemma 3 *Let A_n and $q_n(\lambda)$ be as above. Then q is a factor of the minimal polynomial of A_n. Then there is a vector v such that $q_n(A_n)v = 0$.*

Proof 3 *The matrix A_n has order $2^n - 2$. We denote the quantity $2^{(n-1)} - 1$ by l. We observe that $q(A_n)$ has as first and last row*

$$(\overbrace{-1,\ldots,-1}^{l\ times},\overbrace{1\ldots,1}^{l\ times}) \quad and \quad (\overbrace{1,\ldots,1}^{l\ times},\overbrace{-1\ldots,-1}^{l\ times}),$$

respectively. This is seen easily for $n = 3$, and for $n \geq 3$ is obtained by observing that in the case of the first row, the first row of A_n^s is obtained multiplying the first row of A_n^{s-1} by the columns of A_n. By induction we get the following pattern:

$$\begin{aligned}
\textit{first row of } A_n &: (0,1,1,\ldots),\\
\textit{first row of } A_n^2 &: (0,0,0,1,1,1,1,\ldots),\\
\vdots \quad &\quad \vdots\\
\textit{first row of } A_n^{n-2} &: (0,\ldots,0,\overbrace{1,\ldots,1}^{2^{(n-2)}},\overbrace{0,\ldots,0}^{l}),\\
\textit{first row of } A_n^{n-1} &:(0,\ldots,0,\overbrace{1,\ldots,1}).
\end{aligned}$$

For the last row, the argument is similar but now the pattern above begin from the right to left with the signs swapped.

Lemma 4 *The minimal polynomial of A_n has the polynomial q_n as a factor.*

Proof 4 *Since the characteristic polynomial of A_n and $B = A_n^t$ are the same it is enough to do the proof for matrix B. It is easy to see from the form of the first and last column of B that the vector $w = (1, 0, \dots, 0, 1)$ belongs to the kernel of $q_n(B)$. This implies that q belongs to the proper ideal*

$$I_w = \{p \in \mathbb{Z}[x] : p(A_n)w = 0\}.$$

By Lemma 2 we know that q_n is the generator of this ideal so it is a factor of the minimal polynomial of B and so of A_n.

From Lemma 1 we get

Corollary 1 *The spectral radius of A_n is bounded bellow by the Littelwood Pisot number whose minimal polynomial is q_n.*

Proof 5 (Proof of Theorem 2) *The polynomial $q_n(\lambda)$ in Lemma 2 are the minimal polynomial of the Littelwood Pisot numbers which form a strictly increasing sequence $\{\gamma_i\}_2^\infty$ with γ_i converging to 2. The least Littelwood Pisot number is the golden ratio $(1+\sqrt{5})/2$, see [M, Theorem 1]. So Theorem follows from the previous Corollary.*

References

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