An intrinsic characterization of the shape of paracompacta by means of non-continuous signle-valued maps

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Abstract

In this paper we give a new description of the shape of paracompact spaces. We introduce the intrinsic notion of an approximate net between two spaces and we prove that two paracompacta have the same shape if and only if they are isomorphic objects in the category HAN_p whose objects are paracompact spaces and whose morphisms are homotopy classes of approximate nets.

Introduction

K. Borsuk introduced shape theory in 1968 as a way to study the global properties of compact metric spaces. His basic notion to replace the notion of mapping in homotopy theory was that of a fundamental sequence between two compact metric spaces defined in terms of the neighborhoods of these compacta embedded in a convenient AR-space (e.g. the Hilbert cube). Shape theory was later developed for general topological spaces, the main approach being the Mardešić-Segal method of using inverse systems of ANR-spaces. See [2] for details.

A new line of approach to shape theory consists in getting rid of these external elements (ambient AR-s, ANR-expansions) by dropping the requirement of full continuity of the maps involved in its description. Felt [1] established a relationship between the concepts of ε-continuity and shape for compacta and Sanjurjo [4] gave

Received by the editors November 1993
Communicated by Y. Félix
AMS Mathematics Subject Classification : 54C56, 55P55
Keywords : $\mathcal{V}-$continuity, $\mathcal{V}-$homotopy, approximate net, multi-net.

the first complete internal description of the shape category of compacta. His key notion to replace Borsuk's fundamental sequence was the notion of a proximate net (some kind of sequence of ε-continuous functions). Later, in [5], he gave another description using multi-nets. This multivalued approach was generalized by Morón and Ruiz del Portal [3] (using normal open coverings) to obtain an internal description of the shape category of paracompact spaces. This paper combines the original single-valued approach of [4] and some techniques of [3] to get a simpler description of the shape of paracompact spaces. The main new tools are the concepts of $\mathcal{V}$-continuity ($\mathcal{V}$ a normal open covering) and of approximate net.

Acknowledgement

The author wishes to express his warmest thanks to Professor José M.R. Sanjurjo for many helpful discussions and for useful comments on an earlier version of this paper.

1 $\mathcal{V}$-continuity and $\mathcal{V}$-homotopy

All the spaces considered here are topological spaces. By $f : X \to Y$ we denote a (not necessarily continuous) function from a space X into a space Y.

Let $\mathcal{V}$ denote a normal open covering of Y (notation $\mathcal{V} \in \mathcal{NC}(Y)$ as in [3]).

Definition 1.1. $f : X \to Y$ is *$\mathcal{V}$-continuous at $x \in X$* provided that there exist an open subset U of X containing x, and an element V of $\mathcal{V}$ such that $f(U) \subset V$.

Definition 1.2. $f : X \to Y$ is *$\mathcal{V}$-continuous* provided that f is $\mathcal{V}$-continuous at each $x \in X$ or, equivalently, provided that there exists an open covering $\mathcal{U}$ of X which refines the covering $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$.

Notice that we do not require that for every $V \in \mathcal{V}$, containing $f(x)$ there exists an open U such that $x \in U$ and $f(U) \subset V$. Our apparently weaker notion of $\mathcal{V}$-continuity (inspired by Felt's paper [1]) will show to be suitable for our purposes. Moreover it generalizes ε-continuity for functions into a metric space (Y, d) (see [4]) in the following way: f ε-continuous at $x \Rightarrow f$ B_ε-continuous at $x \Rightarrow f$ 2ε-continuous at x. (Here $B_\varepsilon = \{B_d(y, \varepsilon) \mid y \in Y\}$)

The following properties of $\mathcal{V}$-continuity are easily proved.

Proposition 1.3.

(i) $f : X \to Y$ *continuous at* $x \in X \Rightarrow f$ *$\mathcal{V}$-continuous at x for every* $\mathcal{V} \in \mathcal{NC}(Y)$.

(ii) $f : X \to Y$ *$\mathcal{V}$-continuous at* $x \in X$ *for every* $\mathcal{V} \in \mathcal{NC}(Y)$ *and Y normal* $\Rightarrow f$ *continuous at x.*

(iii) $f : X \to Y$ *$\mathcal{V}$-continuous at* $x \in X$ *and* $\mathcal{V} \geq \mathcal{W} \in \mathcal{NC}(Y)$ *(i.e. $\mathcal{V}$ refines $\mathcal{W}$)* $\Rightarrow f$ *$\mathcal{W}$-continuous at x.*

(iv) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ *and* $\mathcal{V} \in \mathcal{NC}(Y), \mathcal{W} \in \mathcal{NC}(Z)$ *such that* $\mathcal{V} \geq g^{-1}(\mathcal{W})$. *Then f $\mathcal{V}$-continuous at* $x \in X \Rightarrow g \circ f$ *$\mathcal{W}$-continuous at x.*

Now we introduce $\mathcal{V}$-homotopy in the expected way.

Definition 1.4. $f, g : X \to Y$ are said to be $\mathcal{V}$-*homotopic* (notation $f \simeq_{\mathcal{V}} g$) provided that there exists a $\mathcal{V}$-continuous function $H : X \times I \to Y$ such that $H(.,0) = f$ and $H(.,1) = g$.

H is then said to be a $\mathcal{V}$-*homotopy* from f to g.

Notice that $\mathcal{V}$-homotopy is an equivalence relation in
$\mathcal{C}_{\mathcal{V}}(X,Y) = \{f : X \to Y \mid f\ \mathcal{V}\text{-continuous}\}$

An easy application of prop. 1.3. (iv) shows that:

Proposition 1.5. *Let* $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{g} Z$ *and* $\mathcal{V} \in \mathcal{NC}(Y)$,$\mathcal{W} \in \mathcal{NC}(Z)$ *such that* $\mathcal{V} \geq g^{-1}(\mathcal{W})$. *Then* $f_1 \simeq_{\mathcal{V}} f_2 \Rightarrow g \circ f_1 \simeq_{\mathcal{W}} g \circ f_2$.

The situation $X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$ seems to be more implicated. However we can prove (using the essential idea from lemma 1 of [3]:

Proposition 1.6. *Let* $g \simeq_{\mathcal{W}} h : Y \to Z$ *with* $\mathcal{W} \in \mathcal{NC}(Z)$ *and* Y *paracompact. Then there exists a* $\mathcal{V} \in \mathcal{NC}(Y)$,$\mathcal{V} \geq\ g^{-1}(\mathcal{W})$, $\mathcal{V} \geq h^{-1}(\mathcal{W})$ *such that, for each* $f \in \mathcal{C}_{\mathcal{V}}(X,Y)$,
$g \circ f \simeq_{\mathcal{W}} h \circ f$.

Proof: Let $G : Y \times I \to Z$ be a $\mathcal{W}$-homotopy from g to h. Then there exists , for each $(y,t) \in Y \times I$, an open $V_y^{(t)} \ni y$, an open $J_t^{(y)} \ni t$ and a $W = W_{(y,t)} \in \mathcal{W}$ such that $G\left(V_y^{(t)} \times J_t^{(y)}\right) \subset W$.

By compactness of $I : I \subset J_{t_1}^{(y)} \cup \ldots \cup J_{t_n}^{(y)}$.

Let $V_y = \cap_{j=1}^{n} V_y^{(t_j)}$ where $G\left(V_y^{(t_j)} \times J_{t_j}^{(y)}\right) \subset W_{(y,t_j)}$.

It follows that $G\left(V_y \times J_{t_j}^{(y)}\right) \subset W_{(y,t_j)}$ for all $j \in \{1,\ldots,n\}$.

$\mathcal{V} = \{V_y \mid y \in Y\}$ is an open covering of Y, hence $\mathcal{V} \in \mathcal{NC}(Y)$ since Y is paracompact ([2] p.325). Each $(y,0)$ belongs to some $V_y \times J_{t_{i_0}}^{(y)}$.

Hence $g(V_y) = G(V_y \times \{0\}) \subset G\left(V_y \times J_{t_{i_0}}^{(y)}\right) \subset W_{(y,t_{i_0})} \in \mathcal{W}$, and thus $\mathcal{V} \geq g^{-1}(\mathcal{W})$.

Similarly $\mathcal{V} \geq h^{-1}(\mathcal{W})$.

Now let $f \in \mathcal{C}_{\mathcal{V}}(X,Y)$ and let $F = G \circ (f \times 1_I) : X \times I \to Z$.

Then $F(.,0) = g \circ f$, $F(.,1) = h \circ f$ and, for each $(x,t) \in X \times I$, there exist an open $U \ni x$ and a $V_y \in \mathcal{V}$ such that $f(U) \subset V_y$. Hence, for $J_{t_j}^{(y)} \ni t$, we have that

$F\left(U \times J_{t_j}^{(y)}\right) = G\left(f(U) \times J_{t_j}^{(y)}\right) \subset G\left(V_y \times J_{t_j}^{(y)}\right) \subset W_{(y,t_j)} \in \mathcal{W}$, which shows the $\mathcal{W}$-continuity of F. Hence $F : g \circ f \simeq_{\mathcal{W}} h \circ f$.

The following theorem will be very useful in showing that two functions are $\mathcal{V}$-homotopic.

Theorem 1.7. *Let $\mathcal{V}, \mathcal{W} \in \mathcal{NC}(Y)$ such that $\mathcal{W} \geq^* \mathcal{V}$ (i.e. $\mathcal{W}$ star-refines $\mathcal{V}$). Let $f, g : X \to Y$ be $\mathcal{W}$-continuous and $\mathcal{W}$-close. Then $f \simeq_{\mathcal{V}} g$.*

Proof: Define $H : X \times I \to Y$ by $H(x,t) = f(x)$ for $(x,t) \in X \times [0,1[$ and by $H(x,t) = g(x)$ for $(x,t) \in X \times \{1\}$.

We shall show that H is $\mathcal{V}$-continuous at each point (x,t) of $X \times I$.

First case: $(x,t) \in X \times [0,1[$.

f is $\mathcal{W}$-continuous at x, hence also $\mathcal{V}$-continuous at x by prop. 1.3. (iii). This means that there exist an open $U \ni x$ and a $V \in \mathcal{V}$ such that $f(U) \subset V$. Then $U \times [0,1[$ is open in $X \times I$ and contains (x,t). Since $H(U \times [0,1[) = f(U) \subset V$ it follows that H is $\mathcal{V}$-continuous at (x,t).

Second case: $(x,t) = (x,1) \in X \times 1$.

By the $\mathcal{W}$-continuity of f and g at x there exist open subsets U_1, U_2 of X containing x and elements W_1, W_2 of $\mathcal{W}$ such that $f(U_1) \subset W_1, g(U_2) \subset W_2$. f and g being $\mathcal{W}$-close, there exists a $W \in \mathcal{W}$ such that $\{f(x), g(x)\} \subset W$.

Let $U = U_1 \cap U_2 \ni x$. Then $H(U \times I) = f(U) \cup g(U) \subset W_1 \cup W_2 \subset St(W, \mathcal{W})$ which is a subset of some $V \in \mathcal{V}$ since $\mathcal{W} \geq^* \mathcal{V}$. Hence H is $\mathcal{V}$-continuous at $(x,1)$.

The following approximation lemma and its corollary are similar to corollaries 1 and 2 of [3]. We include the proofs since our definition of $\mathcal{V}$-continuity and $\mathcal{V}$-homotopy is not the same as in [3] (restricted to single-valued maps).

Lemma 1.8. Let Y be an ANR for metric spaces (see [2]). For every $\mathcal{V} \in \mathcal{NC}(Y)$ there exists a $\mathcal{V}' \in \mathcal{NC}(Y)$ such that for every paracompact space X and every $f \in \mathcal{C}_{\mathcal{V}'}(X,Y)$ there exists a continuous $g : X \to Y$ such that f and g are $\mathcal{V}$-close.

Proof: Let $r : A \to Y$ be a continuous retraction of some open neighborhood A of Y in a convex subset R of a normed space L (see [2] p.35).

Let $\mathcal{W}_1, \mathcal{W}_2 \in \mathcal{NC}(A)$ such that $\mathcal{W}_2 \geq^* \mathcal{W}_1 \geq r^{-1}(\mathcal{V})$ and such that $\mathcal{W}_1$ is also a convex covering. Let $\mathcal{V}' = \mathcal{W}_2 \cap Y$ and let $f \in \mathcal{C}_{\mathcal{V}'}(X,Y)$ with X paracompact.

For each $x \in X$ there exist an open $U_x \ni x$ and a $V' \in \mathcal{V}'$ such that $f(U_x) \subset V'$. Let $\overline{f} = i \circ f : X \to A$ $(i : Y \subset A)$.

It follows that $\overline{f}$ has the following property:

$(*)$ For each $x \in X$ there exist an open $U_x \ni x$ and a $W_2^x \in \mathcal{W}_2$ such that $\overline{f}(U_x) \subset W_2^x$.

Let $\mathcal{U} = \{U_j \mid j \in J\}$ be a locally finite refinement of $\{U_x \mid x \in X\}$ (by paracompactness of X), and let $\{\lambda_j : X \to I \mid j \in J\}$ be a partition of unity subordinated to $\mathcal{U}$.

Choose a fixed point x_j in each U_j and define $\overline{g} : X \to R : x \to \sum_{j \in J} \lambda_j(x)\, f(x_j)$ (which is an element of R by convexity of R).

$\overline{g}$ is continuous (since all λ_j are continuous and the sum in the definition of $\overline{g}$ is locally finite).

We shall now prove that (i) $\overline{g}(X) \subset A$, (ii) $\overline{f}$ and $\overline{g}$ are $\mathcal{W}_1$-close,

(iii) $g = r \circ \overline{g}$ is the required continuous approximation.

(i) Let $x \in X$. Let $U_{j_0} \in \mathcal{U}$ such that $x \in U_{j_0}$. Since $\mathcal{U} \geq \{U_x \mid x \in X\}, U_{j_0}$ is contained in some $U_{x_0'}$. Hence $U_{j_0} \subset \overline{f}^{-1}(W_2^{x_0'})$ (see (*))

Let $U_1 = U_{j_0}, U_2, \ldots, U_k$ denote all elements of $\mathcal{U}$ such that $x \in U_j$ for $j \in \{1, \ldots, k\}$. Then $\overline{g}(x) = \sum_{j=1}^{k} \lambda_j(x) f(x_j)$.

Each of these $f(x_j) \in \overline{f}(U_j) \subset \overline{f}(U_{x_j'}) \subset W_2^{x_j'} \in \mathcal{W}_2$, hence $\{f(x_1), \ldots, f(x_k)\}$ $\subset W_2^{x_1'} \cup \ldots \cup W_2^{x_k'} \subset St(W_2^{x_0'}, \mathcal{W}_2)$ since $W_2^{x_j'} \cap W_2^{x_0'} \supset \overline{f}(U_{x_j'}) \cap \overline{f}(U_{x_0'}) \supset \overline{f}(U_j) \cap$ $\overline{f}(U_1) \supset \overline{f}(U_j \cap U_1) \neq \phi$.

From $\mathcal{W}_2 \geq^* \mathcal{W}_1$ it follows that there exists a $W_1 \in \mathcal{W}_1$ such that $\{f(x_1), \ldots, f(x_k)\} \subset W_1$.

W_1 is convex and $\sum_{j=1}^{k} \lambda_j(x) = \sum_{j \in J} \lambda_j(x) = 1$, hence we have that $\overline{g}(x) \in W_1 \subset A$.

(ii) Let $x \in X$. Using notations as in the proof of (i), we see that $\overline{f}(x) \in \overline{f}(U_{x_0'}) \subset W_2^{x_0'} \subset St\left(W_2^{x_0'}, \mathcal{W}_2\right) \subset W_1 \in \mathcal{W}_1$ and $\overline{g}(x) \in W_1$. Hence $\overline{f}$ and $\overline{g}$ are $\mathcal{W}_1$-close.

(iii) $g = r \circ \overline{g} : X \to Y$ is continuous (since r and $\overline{g}$ are continuous). $f = r \circ i \circ f = r \circ \overline{f}$. From (ii) and $\mathcal{W}_1 \geq r^{-1}(\mathcal{V})$ it follows that g and f are $\mathcal{V}$-close, which finishes the proof.

Corollary 1.9. Let Y be an ANR. There exists a $\mathcal{V} \in \mathcal{NC}(Y)$ such that for every paracompact space X we have that $f, g \in \mathcal{C}(X, Y)$ and $f \simeq_{\mathcal{V}} g$ imply that $f \simeq g$.

(Note: this corollary also generalizes lemma 3 of [1]).

Proof: Y being ANR, there exists a $\mathcal{W} \in \mathcal{NC}(Y)$ such that any pair of $\mathcal{W}$-close continuous mappings into Y is homotopic (see [2] p.39). Then, using lemma 1.8, we can find a $\mathcal{V} \in \mathcal{NC}(Y)$ with the property that for every $h \in \mathcal{C}_{\mathcal{V}}(X, Y)$ (X paracompact) there exists a $k \in \mathcal{C}(X, Y)$ such that h and k are $\mathcal{W}$-close.

Now let $f, g \in \mathcal{C}(X, Y)$ such that $f \simeq_{\mathcal{V}} g$. Let $H \in \mathcal{C}_{\mathcal{V}}(X \times I, Y)$ be a $\mathcal{V}$-homotopy from f to g.

Since $X \times I$ is also paracompact, there exists a $K \in \mathcal{C}(X \times I, Y)$ which is $\mathcal{W}$-close to H. Then $f = H_0$ and K_0 are $\mathcal{W}$-close continuous mappings into Y, hence $f \simeq K_0$. Similarly, $g \simeq K_1$.

Consequently, $f \simeq g$ which finishes the proof.

2 Approximate nets and the category HAN_p

We now introduce the concept of approximate net from X to Y which is some sort of "single-valued version" of the notion of multi-net (see [5] and [3]). In the compact metric case it relates to the notion of proximate net in [4].

Definition 2.1. An *approximate net from X to Y* is a family $\underline{f} = \{f_\mathcal{V} \in \mathcal{C}_\mathcal{V}(X, Y) \mid \mathcal{V} \in \mathcal{NC}(Y)\}$ such that $f_\mathcal{W} \simeq_\mathcal{V} f_\mathcal{V}$ whenever $\mathcal{W} \geq \mathcal{V}$.

Definition 2.2. Two approximate nets $\underline{f}$ and $\underline{g}$ from X to Y are said to be *homotopic*, notation $\underline{f} \simeq \underline{g}$, if $f_\mathcal{V} \simeq_\mathcal{V} g_\mathcal{V}$ for every $\mathcal{V} \in \mathcal{NC}(Y)$.

This homotopy relation is an equivalence relation in $AN(X, Y) = \{\underline{f} \mid \underline{f}$ approximate net from X to $Y\}$.

(Note: for transitivity, use the fact that any two normal coverings of Y admit a common refinement which is again normal, see [2] p.325).

We shall denote by $[\underline{f}]$ the homotopy class of $\underline{f}$ and by $\underline{\hat{f}}$ the *"constant" approximate net generated by* a $f \in \mathcal{C}(X, Y)$, i.e. $\hat{f}_\mathcal{V} = f$ for every $\mathcal{V} \in \mathcal{NC}(Y)$.

The following result is analogous to a result of Borsuk for fundamental sequences.

Theorem 2.3. *Let $\underline{f}$ be an approximate net from a paracompact space X to an ANR Y. Then there exists a unique homotopy class $[g]$, with $g \in \mathcal{C}(X, Y)$, such that $[\underline{f}] = [\underline{\hat{g}}]$.*

Proof: By corollary 1.9. there exists a $\mathcal{V}_0 \in \mathcal{NC}(Y)$ such that $g, h \in \mathcal{C}(X, Y)$ and $g \simeq_{\mathcal{V}_0} h$ imply that $g \simeq h$. Let $\mathcal{V}" \geq^* \mathcal{V}_0$ and $\mathcal{V}' \geq \mathcal{V}"$ such that every $\alpha \in \mathcal{C}_{\mathcal{V}'}(X, Y)$ has a continuous $\mathcal{V}"$-approximation β (lemma 1.8).

In particular, $f_{\mathcal{V}'} \in \mathcal{C}_{\mathcal{V}'}(X, Y)$ has a continuous $\mathcal{V}"$-approximation g. By prop. 1.3.(i) and (iii) we have that $f_{\mathcal{V}'}, g \in \mathcal{C}_{\mathcal{V}"}(X, Y)$ and, using theorem 1.7, it follows that $f_{\mathcal{V}'} \simeq_{\mathcal{V}_0} g$. $\underline{f}$ being an approximate net, we also have that $f_{\mathcal{V}'} \simeq_{\mathcal{V}_0} f_{\mathcal{V}_0}$ (since $\mathcal{V}' \geq \mathcal{V}_0$).

Hence $f_{\mathcal{V}_0} \simeq_{\mathcal{V}_0} g$.

Now let $\mathcal{V} \in \mathcal{NC}(Y)$ be arbitrary. Let $\mathcal{W} \in \mathcal{NC}(Y)$ be a common refinement of $\mathcal{V}$ and $\mathcal{V}_0$. Then $f_\mathcal{W} \simeq_\mathcal{V} f_\mathcal{V}$.

On the other hand, using a similar argument for $f_\mathcal{W}$ instead of $f_{\mathcal{V}_0}$, there exists an $h \in \mathcal{C}(X, Y)$ such that $f_\mathcal{W} \simeq_\mathcal{W} h$. Since $f_\mathcal{W} \simeq_{\mathcal{V}_0} f_{\mathcal{V}_0}$, it follows that $g \simeq_{\mathcal{V}_0} h$, hence that $g \simeq h$. This implies that $g \simeq_\mathcal{W} h$, hence that $f_\mathcal{W} \simeq_\mathcal{W} g$. Since $\mathcal{W} \geq \mathcal{V}$ it follows that $f_\mathcal{V} \simeq_\mathcal{V} f_\mathcal{W} \simeq_\mathcal{V} g$, which proves that $[\underline{f}] = [\underline{\hat{g}}]$.

In order to prove uniqueness of $[g]$, let $g' \in \mathcal{C}(X, Y)$ also satisfy $[\underline{f}] = [\underline{\hat{g}}']$. Then $g \simeq_{\mathcal{V}_0} f_{\mathcal{V}_0} \simeq_{\mathcal{V}_0} g'$, hence $g \simeq g'$ and thus $[g] = [g']$, which finishes the proof.

We shall now introduce a notion of composition of (h-classes of) approximate nets $\underline{f}$ from X to Y and $\underline{g}$ from Y to Z. In order to do this we need to assume that Y is paracompact.

Let $\mathcal{W} \in \mathcal{NC}(Z)$. Since $g_{\mathcal{W}}$ is $\mathcal{W}$-continuous, there exists an open covering $\mathcal{V}$ of Y which refines $g_{\mathcal{W}}^{-1}(\mathcal{W})$ (def. 1.2). $\mathcal{V}$ is normal by the paracompactness of Y.

Define $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} : X \to Z$, then $h_{\mathcal{W}}$ is $\mathcal{W}$-continuous by prop. 1.3.(iv).

This definition depends on the choice of $\mathcal{V}$. However, we shall prove immediately that another choice of $\mathcal{V} \geq g_{\mathcal{W}}^{-1}(\mathcal{W})$ does not affect the $\mathcal{W}$-homotopy class of $h_{\mathcal{W}}$.

Lemma 2.4. Let $\mathcal{V}, \mathcal{V}' \in \mathcal{NC}(Y)$ such that $\mathcal{V}, \mathcal{V}' \geq g_{\mathcal{W}}^{-1}(\mathcal{W})$ $(\mathcal{W} \in \mathcal{NC}(Z))$. Then $g_{\mathcal{W}} \circ f_{\mathcal{V}} \simeq_{\mathcal{W}} g_{\mathcal{W}} \circ f_{\mathcal{V}'}$.

Proof: Let $\mathcal{V}" \in \mathcal{NC}(Y)$ be a common refinement of $\mathcal{V}$ and $\mathcal{V}'$.

Then $f_{\mathcal{V}"} \simeq_{\mathcal{V}} f_{\mathcal{V}}$ and $f_{\mathcal{V}"} \simeq_{\mathcal{V}} f_{\mathcal{V}'}$, since $\underline{f}$ is an approximate net. By prop. 1.5 it follows that $g_{\mathcal{W}} \circ f_{\mathcal{V}"} \simeq_{\mathcal{W}} g_{\mathcal{W}} \circ f_{\mathcal{V}}$ and $g_{\mathcal{W}} \circ f_{\mathcal{V}"} \simeq_{\mathcal{W}} g_{\mathcal{W}} \circ f_{\mathcal{V}'}$, hence $g_{\mathcal{W}} \circ f_{\mathcal{V}} \simeq_{\mathcal{W}} g_{\mathcal{W}} \circ f_{\mathcal{V}'}$.

We shall now prove the main result about $\underline{h} = \{h_{\mathcal{W}} \mid \mathcal{W} \in \mathcal{NC}(Z)\}$

Theorem 2.5. *$\underline{h}$ is an approximate net from X to Z. Its homotopy class $[\underline{h}]$ depends only on $[\underline{f}]$ and $[\underline{g}]$, and is called the composition of $[\underline{f}]$ and $[\underline{g}]$.*
(Notation:$[\underline{h}] = [\underline{g}] \circ [\underline{f}]$).

Proof: We already showed that $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ $(\mathcal{V} \geq g_{\mathcal{W}}^{-1}(\mathcal{W}))$ is $\mathcal{W}$-continuous. In order to finish the proof that $\underline{h}$ is an approximate net, we have to show that $\mathcal{W}_1 \geq \mathcal{W}_2$ implies $h_{\mathcal{W}_1} \simeq_{\mathcal{W}_2} h_{\mathcal{W}_2}$.

Since $\underline{g} = \{g_{\mathcal{W}} : Y \to Z \mid \mathcal{W} \in \mathcal{NC}(Z)\}$ is an approximate net we know that $g_{\mathcal{W}_1} \simeq_{\mathcal{W}_2} g_{\mathcal{W}_2}$.

Using prop. 1.6, we can find a $\mathcal{V} \in \mathcal{NC}(Y)$, refining both $g_{\mathcal{W}_1}^{-1}(\mathcal{W}_2)$ and $g_{\mathcal{W}_2}^{-1}(\mathcal{W}_2)$ such that $g_{\mathcal{W}_1} \circ f_{\mathcal{V}} \simeq_{\mathcal{W}_2} g_{\mathcal{W}_2} \circ f_{\mathcal{V}}$.

We know that $h_{\mathcal{W}_1} = g_{\mathcal{W}_1} \circ f_{\mathcal{V}_1}$ and $h_{\mathcal{W}_2} = g_{\mathcal{W}_2} \circ f_{\mathcal{V}_2}$ for some $\mathcal{V}_1 \in \mathcal{NC}(Y), \mathcal{V}_1 \geq g_{\mathcal{W}_1}^{-1}(\mathcal{W}_1)$ and $\mathcal{V}_2 \in \mathcal{NC}(Y), \mathcal{V}_2 \geq g_{\mathcal{W}_2}^{-1}(\mathcal{W}_2)$.

Lemma 2.4 implies that $h_{\mathcal{W}_2} \simeq_{\mathcal{W}_2} g_{\mathcal{W}_2} \circ f_{\mathcal{V}}$.

To deal with $h_{\mathcal{W}_1}$, consider a common refinement $\mathcal{V}'$ of $\mathcal{V}_1$ and $\mathcal{V}$.

Lemma 2.4 implies that $h_{\mathcal{W}_1} \simeq_{\mathcal{W}_1} g_{\mathcal{W}_1} \circ f_{\mathcal{V}'}$, hence also that $h_{\mathcal{W}_1} \simeq_{\mathcal{W}_2} g_{\mathcal{W}_1} \circ f_{\mathcal{V}'}$. But also $f_{\mathcal{V}'} \simeq_{\mathcal{V}} f_{\mathcal{V}}$ ($\underline{f}$ approximate net) and $\mathcal{V} \geq g_{\mathcal{W}_1}^{-1}(\mathcal{W}_2)$ imply that $g_{\mathcal{W}_1} \circ f_{\mathcal{V}'} \simeq_{\mathcal{W}_2} g_{\mathcal{W}_1} \circ f_{\mathcal{V}}$ (prop. 1.5). So $h_{\mathcal{W}_1} \simeq_{\mathcal{W}_2} g_{\mathcal{W}_1} \circ f_{\mathcal{V}}$ and finally $h_{\mathcal{W}_1} \simeq_{\mathcal{W}_2} h_{\mathcal{W}_2}$ as required.

Next we have to show that $\underline{f} \simeq \underline{f}'$ and $\underline{g} \simeq \underline{g}'$ imply $\underline{h} \simeq \underline{h}'$.

For every $\mathcal{W} \in \mathcal{NC}(Z)$ we have that $g_{\mathcal{W}} \simeq_{\mathcal{W}} g'_{\mathcal{W}}$ and, by prop. 1.6, there exists a $\mathcal{U} \in \mathcal{NC}(Y), \mathcal{U} \geq g_{\mathcal{W}}^{-1}(\mathcal{W}), \mathcal{U} \geq (g'_{\mathcal{W}})^{-1}(\mathcal{W})$ such that $g_{\mathcal{W}} \circ f_{\mathcal{U}} \simeq_{\mathcal{W}} g'_{\mathcal{W}} \circ f_{\mathcal{U}}$.

As before, $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}}$ and $h'_{\mathcal{W}} = g'_{\mathcal{W}} \circ f'_{\mathcal{V}}$ for some $\mathcal{V}$ and $\mathcal{V}' \in \mathcal{NC}(Y)$ satisfying $\mathcal{V} \geq g_{\mathcal{W}}^{-1}(\mathcal{W})$ and $\mathcal{V}' \geq (g'_{\mathcal{W}})^{-1}(\mathcal{W})$.

Using lemma 2.4, prop. 1.5 and the fact that $f_{\mathcal{U}} \simeq_{\mathcal{U}} f'_{\mathcal{U}}$ (since $\underline{f} \simeq \underline{f}'$), we obtain that $h_{\mathcal{W}} = g_{\mathcal{W}} \circ f_{\mathcal{V}} \simeq_{\mathcal{W}} g_{\mathcal{W}} \circ f_{\mathcal{U}} \simeq_{\mathcal{W}} g'_{\mathcal{W}} \circ f_{\mathcal{U}} \simeq_{\mathcal{W}} g'_{\mathcal{W}} \circ f'_{\mathcal{U}} \simeq_{\mathcal{W}} g'_{\mathcal{W}} \circ f'_{\mathcal{V}'} = h'_{\mathcal{W}}$.

Hence $\underline{h} \simeq \underline{h}'$ as required.

Corollary 2.6. If we consider the class of all paracompacta and the homotopy classes of approximate nets between them, with the notion of composition as in theorem 2.5, we obtain a category, denoted by HAN_p.

3 The main theorem

This section is entirely devoted to the proof of the following theorem which characterizes the shape of paracompacta.

Theorem 3.1. *Let X, Y be paracompact spaces. Then $Sh\ X = Sh\ Y$ iff X and Y are isomorphic objects of the category HAN_p, i.e. iff there exist approximate nets $\underline{f}$ from X to Y and $\underline{g}$ from Y to X such that $[\underline{g}] \circ [\underline{f}] = [\underline{\hat{1}}_X]$ and $[\underline{f}] \circ [\underline{g}] = [\underline{\hat{1}}_Y]$.*

One part of this theorem will be proved using the characterization of the shape of paracompacta as developed in [3], i.e. using the notion of multi-net. We shall first explain how to construct an approximate net associated with a multi-net.

Let $\overline{F} = \{F_\mathcal{V}\}_{\mathcal{V} \in \mathcal{NC}(Y)}$ be a multi-net from X to Y (see [3]), i.e.

i) each $F_\mathcal{V} : X \to Y$ is a $\mathcal{V}$-multivalued map, i.e. it is an upper semi-continuous multivalued map which is $\mathcal{V}$-small (i.e. for every $x \in X$ there exists a $V_x \in \mathcal{V}$ such that $F_\mathcal{V}(x) \subset V_x$)

ii) $\mathcal{W}, \mathcal{V} \in \mathcal{NC}(Y), \mathcal{W} \geq \mathcal{V}$ implies that $F_\mathcal{W} \simeq^\mathcal{V} F_\mathcal{V}$, i.e. there exists a $\mathcal{V}$-multivalued map $H : X \times I$ such that $H_0 = F_\mathcal{W}$ and $H_1 = F_\mathcal{V}$.

Now, for each $x \in X$ we choose an element $y \in F_\mathcal{V}(x)$.

In this way we obtain a single-valued map $f_\mathcal{V} : X \to Y : x \to y$. We can do this for every $\mathcal{V} \in \mathcal{NC}(Y)$, thus obtaining a family $\underline{f} = \{f_\mathcal{V} : X \to Y \mid \mathcal{V} \in \mathcal{NC}(Y)\}$.

Lemma 3.2. $\underline{f}$ is an approximate net from X to Y. (We say that $\underline{f}$ is *associated with the multi-net* $\overline{F}$).

Proof: (i) Each $f_\mathcal{V}$ is $\mathcal{V}$-continuous.

Let $x \in X$. $F_\mathcal{V}$ being $\mathcal{V}$-small, there exists a $V \in \mathcal{V}$ such that $F_\mathcal{V}(x) \subset V$. $F_\mathcal{V}$ being upper semi-continuous, there exists for this V an open neighborhood U of x such that $F_\mathcal{V}(U) \subset V$. It follows that $f_\mathcal{V}(U) \subset V$, which shows that $f_\mathcal{V}$ is $\mathcal{V}$-continuous at x.

(ii) $\mathcal{W}, \mathcal{V} \in \mathcal{NC}(Y), \mathcal{W} \geq \mathcal{V}$ implies that $f_\mathcal{W} \simeq_\mathcal{V} f_\mathcal{V}$.

We already know that $F_\mathcal{W} \simeq^\mathcal{V} F_\mathcal{V}$, i.e. there exists a $\mathcal{V}$-multivalued map $H : X \times I \to Y$ such that $H_0 = F_\mathcal{W}$ and $H_1 = F_\mathcal{V}$. Let $G : X \times I \to Y$ be the single-valued map defined by $G(x, 0) = f_\mathcal{W}(x), G(x, 1) = f_\mathcal{V}(x)$ and $G(x, t) \in H(x, t)$ (arbitrarily chosen) for $t \in]0, 1[$.

One easily verifies that G is $\mathcal{V}$-continuous (compare the proof of (i)), hence $G : f_\mathcal{W} \simeq_\mathcal{V} f_\mathcal{V}$.

Of course, the definition of $\underline{f}$ depends on the choice of each $f_\mathcal{V}(x)$ as an element of $F_\mathcal{V}(x)$. However we shall prove immediately that the homotopy class $[\underline{f}]$ does not depend on the choices made. More precisely we prove:

Lemma 3.3. $[\underline{f}]$ depends only on $[\overline{F}]$.

Proof: Let $\overline{F}$ and $\overline{G}$ be homotopic multi-nets from X to Y (see[3]). This means that $F_{\mathcal{V}} \simeq^{\mathcal{V}} G_{\mathcal{V}}$ for every $\mathcal{V} \in \mathcal{NC}(Y)$, i.e. there exists a $\mathcal{V}$-multivalued map $H : X \times I \to Y$ such that $H_0 = F_{\mathcal{V}}, H_1 = G_{\mathcal{V}}$.

Let $\underline{f}$ be any approximate net associated with $\overline{F}$, i.e. $f_{\mathcal{V}}(x) \in F_{\mathcal{V}}(x)$ for every $x \in X$ and every $\mathcal{V} \in \mathcal{NC}(Y)$ Let $\underline{g}$ be any approximate net associated with $\overline{G}$.

Then we define a single-valued map $K : X \times I \to Y$ by $K(x, 0) = f_{\mathcal{V}}(x), K(x, 1) = g(x)$ and $K(x, t) \in H(x, t)$ for $t \in]0, 1[$.

As in the proof of lemma 3.2.(ii), it follows that $K : f_{\mathcal{V}} \simeq_{\mathcal{V}} g_{\mathcal{V}}$. Hence $\underline{f} \simeq \underline{g}$.

Our definition of composition of approximate nets agrees with that of multi-nets as defined in [3] in such a way that one can easily verify:

Lemma 3.4. Let $\underline{f}$ be an approximate net from X to Y associated with a multi-net $\overline{F}$, and let $\underline{g}$ be an approximate net from Y to Z associated with a multi-net $\overline{G}$. (Y paracompact). Then for every composition $\overline{H}$ of $\overline{F}$ and $\overline{G}$, there exists an associated composition $\underline{h}$ of $\underline{f}$ and $\underline{g}$.

Corollary 3.5. There exists a functor Ψ from the category HM_p (of paracompacta and homotopy classes of multi-nets, see [3]) to the category HAN_p which is the identity on objects and which sends each homotopy class $[\overline{F}]$ onto the homotopy class $[\underline{f}]$ where $\underline{f}$ is associated with $\overline{F}$.

Proof of the "only if"-part of theorem 3.1 :

Let $ShX = ShY$ (X and Y paracompact). Using the fact that the shape category of paracompacta is isomorphic to HM_p (main result of [3]) it follows that X and Y are isomorphic objects of HM_p. By application of corollary 3.5, it then follows that $\Psi(X) = X$ and $\Psi(Y) = Y$ are isomorphic objects of HAN_p, which finishes this part of the proof.

In order to prove the converse implication of theorem 3.1, we shall not use the category HM_p but instead the category $\sum_p$ whose objects are paracompact spaces and whose morphisms $X \to Y$ are the natural transformations from $HTop(Y, -)$ to $HTop(X, -)$. Here $HTop(Y, -)$ is the covariant functor from $HPol$ (the homotopy category of polyhedra, or homotopy category of metric ANR's, see [2], theorem 1 of page 45) to Set which assigns to an ANR A the set $HTop(Y, A)$ of homotopy classes of continuous maps from Y to A and to $[v] \in HPol(A', A)$ the functor $[v]_Y$: $HTop(Y, A') \to HTop(Y, A) : [k'] \to [v \circ k']$

In theorem 7 of page 31 of [2] it is shown that Σ_p is isomorphic to Sh_p.

So the first problem is to associate with each approximate net $\underline{f} = \{f_{\mathcal{V}} : X \to Y \mid \mathcal{V} \in \mathcal{NC}(Y)\}$ a natural transformation F: $HTop(Y, -) \to HTop(X, -)$.

Let A be an ANR and let $[k] \in HTop(Y, A)$.

Let $\mathcal{W} \in \mathcal{NC}(A)$. k being continuous, it follows that $k^{-1}(\mathcal{W}) \in \mathcal{NC}(Y)$. Define $h_{\mathcal{W}} = k \circ f_{k^{-1}(\mathcal{W})} : X \to A$ and let $\underline{h} = \{h_{\mathcal{W}} : X \to A \mid \mathcal{W} \in \mathcal{NC}(A)\}$.

Lemma 3.6. $\underline{h}$ is an approximate net from X to A. Its homotopy class only depends on $[\underline{f}]$ and $[k]$.

Proof: It suffices to observe that $\underline{h}$ can be regarded as a composition of the approximate nets $\underline{f}$ and $\underline{\hat{k}}$ (constant net generated by k). From theorem 2.5, it follows that $[\underline{h}] = [\underline{\hat{k}}] \circ [\underline{f}]$. Now observe that $k \simeq k' \Rightarrow \underline{\hat{k}} \simeq \underline{\hat{k}}'$ (trivially) and that $\underline{\hat{k}} \simeq \underline{\hat{k}}' \Rightarrow k \simeq k'$ (use the fact that A is an ANR and corollary 1 on page 39 of [2]).
Hence $[\underline{h}]$ only depends on $[\underline{f}]$ and $[k]$ and the proof is finished.

Now we define $F_A : HTop(Y, A) \to HTop(X, A)$ by $F_A([k]) = [l]$ where $[l]$ is the unique homotopy class with $l \in \mathcal{C}(X, A)$, such that $[\underline{h}] = [\underline{\hat{l}}]$ (theorem 2.3)

We still have to verify the naturality condition, i.e. that

$$\begin{array}{ccc} HTop(Y, A') & \stackrel{F_{A'}}{\to} & HTop(X, A') \\ [v]_Y \downarrow & & \downarrow [v]_X \\ HTop(Y, A) & \stackrel{F_A}{\to} & HTop(X, A) \end{array}$$

commutes for each $[v] \in HPol(A', A)$. This is very easy.

On the one hand , $([v]_X \circ F_{A'})\,[k'] = [v]_X[l'] = [v \circ l']$ where $[l']$ is the unique homotopy class, with $l' \in \mathcal{C}(X, A')$, such that $[\underline{\hat{k}}'] \circ [\underline{f}] = [\underline{\hat{l}}']$ (1)

On the other hand $(F_A \circ [v]_Y)\,[\underline{\hat{k}}'] = F_A[v \circ k'] = [l]$ where $[l]$ is the unique homotopy class, with $l \in \mathcal{C}(X, A)$, such that $[\underline{v \hat{\circ} k'}] \circ [\underline{f}] = [\underline{\hat{l}}]$ (2)

From (1) and (2) it follows that $[\underline{v \hat{\circ} l'}] = [\underline{\hat{v}}] \circ [\underline{\hat{l}'}] = [\underline{\hat{v}}] \circ [\underline{\hat{k}'}] \circ [\underline{f}] = [\underline{\hat{l}}]$ hence that $[v \circ l'] = [l]$ by theorem 2.3, which finishes the proof that F $= (F_A)_{A \in ANR}$ is a natural transformation (which only depends on $[\underline{f}] \in HAN_p(X, Y)$!)

Corollary 3.7. There exists a functor Φ from HAN_p to Σ_p which is the identity on objects and which sends each homotopy class $[\underline{f}]$ of approximate nets from X to Y onto the natural transformation $F : HTop(Y, -) \to HTop(X, -)$ defined above.

Proof: The reader will easily verify that $\Phi[\underline{\hat{1}_X}]$ is the identity transformation and that $\Phi[\underline{f}] = F, \Phi[\underline{g}] = G$ imply that $\Phi\left([\underline{g}] \circ [\underline{f}]\right) = G \circ F$ $([\underline{f}] \in HAN_p(X, Y)$, $[\underline{g}] \in HAN_p(Y, Z))$.

Proof of the "if"-part of theorem 3.1. :

Let X and Y be isomorphic objects of HAN_p. Then $\Phi(X) = X$ and $\Phi(Y) = Y$ are isomorphic objects of Σ_p, hence of Sh_p, which means that $ShX = ShY$ and the proof is finished!

A similar use of the functors that we introduced, yields the following characterization of shape domination:

Theorem 3.8. *Let X and Y be paracompact spaces. Then $ShX \leq ShY$ iff there exist approximate nets $\underline{f}$ from X to Y, and $\underline{g}$ from Y to X such that $[\underline{g}] \circ [\underline{f}] = [\hat{\underline{1}}_X]$.*

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