Subspace operations in affine Klingenberg spaces

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In two previous papers we introduced the notion of an Affine Klingenberg space $\mathcal{A}$ and presented a geometric description of its free subspaces. Presently, we consider the operations of join, intersection and parallelism on the free subspaces of $\mathcal{A}$.

As in the case of ordinary affine spaces, we obtain the Parallel Postulate. The situation with join and intersection is not that straightforward. In particular, the central problem is whether the join of two free subspaces is free?

We show that if $\mathcal{A}$ is not an ordinary affine space and dim $\mathcal{A} \geq 4$ then $\mathcal{A}$ has a subspace which is both not free and the join of two free subspaces. Thus, join and intersection do not possess the usual closure properties. We determine necessary and sufficient conditions under which the join of two free subspaces is free, and in such a case we verify the Dimension Formula.

The subspace operations are essential tools for establishing when $\mathcal{A}$ is desarguesian and when it can be embedded in a projective Klingenberg space.

1 Preliminaries

Let $\mathbb{P} = \{P, Q, ...\}$ be a set of points, $\mathbb{L} = \{\ell, m, ...\}$ be a set of lines and $\mathcal{A} = \{\mathbb{P}, \mathbb{L}, I, \|\}$ be an *incidence structure with parallelism*; that is, $\|$ is an equivalence relation on L such that for each $(P, \ell) \in \mathbb{P} \times \mathbb{L}$, there is a unique line $L(P, \ell)$ with $P\ I\ L(P, \ell) \| \ell$.

We call $\mathcal{A}$ an *affine Klingenberg space* (AK-space) if there is an equivalence relation $\sim$ *on* $\mathbb{P}$ (neighbour relation) such that $\mathcal{A} \ = \ <\mathbb{P}, \mathbb{L}, I, \|, \sim>$ $satisfies$ the axioms (A1) to (A7) below:

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(A1) *Any* $P \not\sim Q$ *are incident with a unique line* $P \vee Q$, *and any line is incident with at least two non-neighbouring points.*

We note from [1] that (A1) permits us to assume that the incidence is inclusion and so, lines are subsets of $\mathbb{P}$.

A subset $\mathbb{Q} \subseteq \mathbb{P}$ is a *subspace* of $\mathcal{A}$ if $\ell \cup \{P, Q\} \subseteq \mathbb{Q}$ and $P \not\sim Q$ imply that $P \vee Q \subseteq \mathbb{Q}$ and $L(P, \ell) \subseteq \mathbb{Q}$. Clearly, any point and any line is a subspace of $\mathcal{A}$ and $s(\mathcal{A})$, the set of all subpaces of $\mathcal{A}$, is closed under intersections. Let $\{\mathbb{Q}, \mathbb{R}\} \subset s(\mathcal{A})$. Then $\mathbb{Q}$ is a *neighbour* of $\mathbb{R}(\mathbb{Q} \approx \mathbb{R})$ if each point of $\mathbb{Q}$ is a neighbour to some point of $\mathbb{R}$.

Let $\Pi \subseteq \mathbb{P}$. Then $< \Pi > : = \cap\mathbb{Q}$, $\Pi \subseteq \mathbb{Q} \in s(\mathcal{A})$, is the *subspace generated by* Π. Next, Π is *independent* if $P \not\approx < \Pi \backslash \{P\} >$ *for* each $P \in \Pi$, otherwise, Π is *dependent.* Finally, Π is a *basis* of a subspace $\mathbb{Q}$ if Π is independent and $\mathbb{Q} = < \Pi >$. If Π is a basis of $\mathbb{Q} \in s(\mathcal{A})$ then $\mathbb{Q}$ is called a *free subspace* of $\mathcal{A}$.

Let $f(\mathcal{A})$ be the set of all free subspaces of $\mathcal{A}$. Then $\mathbb{Q} \in f(\mathcal{A})$ is a *plane* if it has a basis of cardinality three, and $\mathbb{H} \in f(\mathcal{A})$ is a *hyperplane* if there is a maximal independent subset $\{P_\lambda\}_\Lambda$ of $\mathcal{A}$ such that

$$\mathbb{H} = < \{P_\lambda\}_{\lambda \in \Lambda \backslash \{\lambda_0\}} > \text{ for some } \lambda_0 \in \Lambda.$$

We note from [1] that maximal independent subsets of $\mathcal{A}$ exist.

(A2) *If* $\mathbb{H}$ *is a hyperplane of* $\mathcal{A}$, $\ell \in \mathbb{L}$ *and* $\ell \not\approx \mathbb{H}$ *then* $|\ell \cap \mathbb{H}| \leq 1$.

(A3) *If* $\{P, Q, R\} \subset \mathbb{P}$ *and* $P \sim Q \not\sim R$ *then* $P \vee R \approx Q \vee R$.

We recall that for $\{p, q, r\} \subset \mathbb{L}$, $(p, q \not\approx r)$ means that p, q and r are distinct and mutually intersecting, $q \not\approx r$ and there exist $Q \in r \cap p$ and $R \in q \cap p$ such that $Q \notin q$ and $R \notin r$.

(A4) *If* $\{p, p', q, q', r, r'\} \subset \mathbb{L}$ *such that* $p \| p'$, $q \| q'$, $r \| r'$, $(p, q \not\approx r)$ *and* $q' \cap p' \neq \emptyset \neq r' \cap p'$ *then* $q' \not\approx r'$ *and* $q' \cap r' \neq \emptyset$.

(A5) $\mathbb{P}$ *contains an independent set of cardinality three.*

(A6) *Every line contains three mutually non-neighbouring points.*

For $\Pi \subset \mathbb{P}$, we call $\overline{\Pi} := \{P \in \mathbb{P} | P \sim X \text{ for some } X \in \Pi\}$, the *saturate* of Π.

(A7) *If* $\{\mathbb{Q}, \mathbb{R}\} \subset f(\mathcal{A})$ *and* $\mathbb{Q} \subseteq \mathbb{R} \subseteq \overline{\mathbb{Q}}$ *then* $\mathbb{Q} = \mathbb{R}$.

Henceforth, let $\mathcal{A} = < \mathbb{P}, \mathbb{L}, \in, \|, \sim >$ *be* a fixed AK-space. We list some essential properties of $\mathcal{A}$.

1.1 ([1], 3.3) $\approx$ *is* an equivalence relation of $\mathbb{L}$. (We write $\sim$ *for* $\approx$ *restricted* to $\mathbb{L}$.)

1.2 ([1], 3.7) Let $p \| q$. Then $p \sim q$ if and only if $P \sim Q$ for some $P \in p$ and $Q \in q$.

For points $P_1, ..., P_n$, we set $\{P_t\}_0^n = \{P_1, ..., P_n\}$, $\{\hat{P}_i\}_0^n = \{P_t\}_0^n \backslash \{P_i\}$, $< P_t >_0^n = < \{P_t\}_0^n >$ *and* $< \hat{P}_i >_0^n = < \{\hat{P}_i\}_0^n >$.

1.3 ([1],3.13) *Let* $\{P_t\}_0^n$ be independent, $n \geq 1$. Then $< P_t >_0^n$ has the following properties:

$(B)_n$ For each $X \in < P_t >_0^n$, $L(X, P_i \vee P_j)$ intersects $< \hat{P}_i >_0^n$ *and* $< \hat{P}_j >_0^n$, $0 \leq i \neq j \leq n$.

$(E)_n$ If $\{Q_t\}_0^n$ is independent and $P_t \sim Q_t$ for $0 \leq t \leq n$ then
$< P_t >_0^n \approx < Q_t >_0^n \approx < P_t >_0^n$.

$(F)_{n-1}$ If $Q \not\approx < \hat{P}_i >_0^n$ then $\{\hat{P}_i\}_0^n \cup \{Q\}$ is independent.

$(I)_n$ If $\{Q_t\}_0^m \subseteq < P_t >_0^n$ is independent then $m \leq n$ and there are $Q_{m+1}, ..., Q_n$ such that $< Q_t >_0^n = < P_t >_0^n$.

1.4 ([1],3.15 and 3.17) Let $\mathcal{A}^* =< \mathbb{P}^*, \mathbb{L}^*, I^*, \|^* >$ *be* the incidence structure with parallelism where $P^* := \{Q \in \mathbb{P} | Q \sim P\}$, $\ell^* := \{m \in \mathbb{L}\{m \approx \ell\}$ and

$\mathbb{P}^* := \mathbb{P}/\sim = \{P^* | P \in \mathbb{P}\}$,

$\mathbb{L}^* := \mathbb{L}/\sim = \{\ell^* \in \mathbb{L}\}$,

$P^* I^* \ell^* \iff$ *there* exist $Q \sim P$ and $m \sim \ell$ such that $P \in m$ and $Q \in \ell$,

$\ell^* \|^* m^* \iff$ *there* exist $\ell_1 \sim \ell$ and $m_1 \sim m$ such that $\ell_1 \| m_1$.

Then

a) $\mathcal{A}^*$ is an affine space,
b) $* : \mathcal{A} \to \mathcal{A}^*$ is an incidence preserving epimorphism,
c) if $\{P_\lambda\}_\Lambda$ is independent then $< P_\lambda >_\Lambda^* = < P_\lambda^* >_\Lambda$,
d) $\{P_\lambda\}_\Lambda$ is independent if and only if the following two conditions hold: (i) $\{P_\lambda^*\}_\Lambda$ is independent and (ii) $P_\alpha \neq P_\beta$ implies $P_\alpha^* \neq P_\beta^*$.
e) the cardinality of every maximal independent subset of $\mathcal{A}$ with one point removed is equal to the dimension of $\mathcal{A}^*$.

We call $\mathcal{A}^*$, the *underlying affine space* of $\mathcal{A}$. We note that $P^* = Q^*$ if and only if $\overline{P} = \overline{Q}$, and $\Pi^* = (\overline{\Pi})^*$ for any $\Pi \subseteq \mathbb{P}$.

For $\{\mathbb{Q}, \mathbb{R}\} \subset s(\mathcal{A})$, we set $\mathbb{Q} \vee \mathbb{R} =< \mathbb{Q} \cup \mathbb{R} >$ *and* call it the *join* of $\mathbb{Q}$ and $\mathbb{R}$.

1.5 ([2],1.8) Let $\{P_\lambda\}_\Lambda$ be independent and $X, Y \not\approx < P_\lambda >_\Lambda$. Then $\{X\} \cup \{P_\lambda\}_\Lambda$ and $\{Y\} \cup \{P_\lambda\}_\Lambda$ are independent, and if $Y \in X \vee < P_\lambda >_\Lambda$ then $Y \vee < P_\lambda >_\Lambda = X \vee < P_\lambda >_\Lambda$.

In [1] and [2], we determined that all maximal independent subsets of $\mathbb{Q} \in s(\mathcal{A})$ have the same cardinality. Accordingly, the *dimension* of $\mathbb{Q}$, $\dim(\mathbb{Q})$, is the cardinality of a maximal independent subset of $\mathbb{Q}$ with one point removed.

1.6 ([2],1.11) If $\mathbb{Q} \in s(\mathcal{A})$ then $\overline{\mathbb{Q}} \in s(\mathcal{A})$ and $\dim(\overline{\mathbb{Q}}) = \dim(\mathbb{Q})$.

1.7 ([2],1.12) Let $P \in \ell$. Then $|\ell \cap \overline{P}|$ is independent of the choice of P and ℓ . (We call $d(\mathcal{A}) = |\ell \cap \overline{P}|$, the *degree* of $\mathcal{A}$.)

1.8 ([2].2.5) Let $\mathbb{R} \in f(\mathcal{A})$.

a) If $\mathbb{R}$ contains a plane then it is an AK-space with the induced parallel and neighbour relations.

b) If $\ell \not\approx \mathbb{R}$ then $|\ell \cap \mathbb{R}| \leq 1$.

We set $f_n(\mathcal{A}) = \{\mathbb{Q} \in f(\mathcal{A}) | \dim(\mathbb{Q}) = n\}$, $n \geq 0$, and observe that the underlying affine space $\mathcal{A}^*$, with equality as the neighbour relation on $\mathbb{P}^*$, is also an AK-space.

1.9 Proposition. *If* $\mathbb{Q} \in f_n(\mathcal{A})$ *then* $\mathbb{Q}^* \in f_n(\mathcal{A}^*)$.

Proof. Let $\{Q_t\}_0^n$ be a basis of $\mathbb{Q}$. Then $\mathbb{Q}^* = (< Q_t >_0^n)^* =< Q_t^* >_0^n$ by 1.4 c), and $\{Q_t^*\}_0^n$ is independent by 1.4 d). Since $\mathcal{A}^*$ is an affine space, it and its subspaces are free. Thus $\mathbb{Q}^* \in f_n(\mathcal{A}^*)$. □

A *partial* AK-space is an incidence structure $\mathcal{A}' =< \mathbb{P}', \mathbb{L}', I', \|', \sim' >$ *with* parallelism $\|'$ and neighbour relation $\sim'$ which satisfies axioms (A1) to (A6). In [1] and [2], we called an AK-space a partial AK-space with a weakened axiom (A2) where a hyperplane $\mathbb{H}$ was replaced by a line h. We note that all the results preceding 1.8 are valid in such a weakened partial AK-space, and that 1.8 and the results in the next section are valid in a partial AK-space $\mathcal{A}'$.

The significance of (A7), as 1.10 below shows, is to ensure that any free subspace is generated by any of its maximal independent subsets.

1.10 ([2],2.4). Let $\mathcal{A}'$ be a partial AK-space. Then the following statements are equivalent.

(a) $\mathcal{A}'$ satisfies (A7); that is, $\mathcal{A}'$ is an AK-space.

(b) Every maximal independent subset of a free subspace $\mathbb{R}$ is a basis of $\mathbb{R}$.

(c) If $\mathbb{H}$ is a hyperplane of a free subspace $\mathbb{R}$, $X \in \mathbb{R}$ and $X \not\approx \mathbb{H}$ then $\mathbb{R} = \mathbb{H} \vee X$.

Let $\mathcal{A}'$ be a partial AK-space. If $\mathcal{A}'$ has finite dimension $n \geq 2$ then $(I)_n$ (see 1.3) and 1.10 imply that $\mathcal{A}'$ is an AK-space. However, if $\mathcal{A}'$ has infinite dimension then $\mathcal{A}'$ need not be an AK-space. Indeed, utilizing a module described by A. Kreuzer in [4], we can construct an infinite dimensional $\mathcal{A}'$, via the procedure delineated in [2], whose every infinite dimensional free subspace possesses a non-generating maximal independent subset.

2 Parallelism

Let $\{\mathbb{Q}, \mathbb{R}\} \subset f_n(\mathcal{A})$, $n \geq 2$. Then $\mathbb{Q}$ and $\mathbb{R}$ are *parallel* ($\mathbb{Q} \| \mathbb{R}$) if there exist bases, $\{Q_t\}_0^n$ of $\mathbb{Q}$ and $\{R_t\}_0^n$ of $\mathbb{R}$, such that $Q_0 \vee Q_i \| R_0 \vee R_i$ for $i = 1, ..., n$.

Our aim is to show that this definition of parallelism for free n-spaces satisfies the Parallel Postulate. As a first step, we note some results from [2].

2.1 ([2],1.15 and 1.16) Let $\{\mathbb{Q}, \mathbb{R}\} \subset f_n(\mathcal{A})$ and $\mathbb{Q} \| \mathbb{R}$, $n \geq 1$.

a) For each line $q \subseteq \mathbb{Q}$, there is a line $r \subseteq \mathbb{R}$ such that $q \| r$.

b) Let $Q \in \mathbb{Q}$, $X \not\approx \mathbb{Q}$ and $\ell \| Q \vee X$. Then $\ell \not\approx \mathbb{R}$.

2.2 Proposition. *Let $\{Q_t\}_0^n$ be independent, $n \geq 2$. For any point R_0, there exist points $R_j \in r_j = L(R_0, Q_0 \vee Q_j)$, $1 \leq j \leq n$, such that $\{R_t\}_0^n$ is independent and $R_i \vee R_j \| Q_i \vee Q_j$ for $0 \leq i \neq j \leq n$. In particular, $< Q_t >_0^n \, \| < R_t >_0^n$.*

Proof. Let $q_{ij} = Q_i \vee Q_j$ for $0 \leq i \neq j \leq n$.

Let $n = 2$. Then $< Q_0, Q_1, Q_2 > is$ a plane and the q_{ij} are mutually non-neighbouring. By (A1), there is a point $R_1 \in r_1$, such that $R_1 \not\sim R_0$. Next, (A4) yields that $r_2 \cap L(R_1, q_{12})$ is a point R_2 such that $R_2 \;\approx\; R_0 \vee R_1$. Then $\{R_t\}_0^2$ is independent by 1.5, and $R_i \vee R_j \| q_{ij}$ for $0 \leq i \neq j \leq 2$.

Let $n \geq 3$ and proceed by induction. Thus there exist points $R_j \in r_j$, $i \leq j \leq n-1$, such that $\{R_t\}_0^{n-1}$ is independent and

(1) $R_i \vee R_j \| q_{ij}$ for $0 \leq i \neq j \leq n-1$.

Since $< Q_0, Q_j, Q_n > is$ a plane for $1 \leq j \leq n-1$, it follows from the preceding that $r_n \cap L(R_j, q_{jn})$ is a point R_n^j such that $\{R_0, R_j, R_n^j\}$ is independent and

(2) $R_0 \vee R_n^j \| q_{0n}$ and $R_j \vee R_n^j \| q_{jn}$ for $1 \leq j \leq n-1$.

We claim that $R_n^1 = R_n^j$ for $j = 2, ..., n-1$. Since $< Q_1, Q_j, Q_n > is$ a plane, $(q_{ij}, q_{1n} \not\sim q_{jn})$ and $q_{1j} \| R_1 \vee R_j$, it follows from (2) and (A4) that $(R_1 \vee R_n^1) \cap (R_j \vee R_n^j)$ is a point Y such that (R_1, R_j, Y) is independent. We observe that $< Q_1, Q_j, Q_n >$ $\| < R_1, R_j, Y > and\ Q_0 \;\not\approx < Q_1, Q_j, Q_n >$. Thus $r_n \| Q_0 \vee Q_n$ and 2.1 *b*) yield that $r_n \;\not\approx < R_1, R_j, Y >$, and 1.8 *b*) implies that $|r_n \cap < R_1, R_j, Y > | \leq 1$. Since $\{R_n^1, R_n^j\} \subseteq r_n \cap < R_1, R_j, Y >, R_n^1 = R_n^j$.

Let $R_n = R_n^1$. Then $R_i \vee R_j \| q_{ij}$ for $0 \leq i \neq j \leq n$ by (1) and (2) and we need only to show that $\{R_t\}_0^n$ is independent. We recall that $\{R_t\}_0^{n-1}$ is independent, $< Q_t >_0^{n-1} \, \| < R_t >_0^{n-1}$ and $r_n \| Q_0 \vee Q_n$. Hence, as above, $r_n \;\not\approx < R_t >_0^{n-1}$. Since $r_n = R_0 \vee R_n$ and $R_0 \in < R_t >_0^{n-1}$, it follows readily from 1.4 *c*) and *d*) that $R_n \;\not\approx < R_t >_0^{n-1}$. Thus $\{R_t\}_0^n$ is independent by 1.5. □

2.3 Proposition. *Let $\{\mathbb{Q}, \mathbb{R}\} \subset f_n(\mathcal{A})$, $n \geq 1$. Then $\mathbb{Q} \| \mathbb{R}$ if and only if for each line $q \subseteq \mathbb{Q}$, there is a line $r \subseteq \mathbb{R}$ such that $q \| r$.*

Proof. The necessity follows from 2.1 a). For the sufficiency, let $\{Q_t\}_0^n$ be a basis for $\mathbb{Q}$ and choose a point $R_0 \in \mathbb{R}$. Then for $j = 1, ..., n$, $r_j = L(R_0, Q_0 \vee Q_j) \subseteq \mathbb{R}$. By 2.2, there is a point $R_j \in r_j$ such that $\{R_t\}_0^n$ is independent and $< R_t >_0^n \| \mathbb{Q}$. By $(I)_n$, $\mathbb{R} = < R_t >_0^n$. □

2.4 Proposition. *Parallelism is an equivalence relation on $f_n(\mathcal{A})$, $n \geq 1$.*

Proof. The reflexivity and symmetry follow from the definition. The transitivity follows from 1.1 and 2.3. □

2.5 The Parallel Postulate for $f_n(\mathcal{A})$, $n \geq 1$. *Let $\mathbb{Q} \in f_n(\mathcal{A})$ and $R \in \mathbb{P}$. Then there exists a unique $\mathbb{R} \in f_n(\mathcal{A})$ such that $R \in \mathbb{R} \| \mathbb{Q}$.*

Proof. As the assertion is true by assumption for $n = 1$, let $n \geq 2$. Then the existence of an $\mathbb{R} \in f_n(\mathcal{A})$ such that $R \in \mathbb{R} \| \mathbb{Q}$ follows from 2.2.

Let $\{\mathbb{R}, \mathbb{R}'\} \subset f_n(\mathcal{A})$ such that $R \in \mathbb{R} \cap \mathbb{R}'$ and $\mathbb{R} \| \mathbb{R}'$. Let $R = R_0$ and $\{R_t\}_0^n$ be a basis of $\mathbb{R}$. Then $R_0 \in \mathbb{R}'$, $\mathbb{R} \| \mathbb{R}'$ and 2.3 yield that $L(R_0, R_0 \vee R_j) \subset \mathbb{R}'$ for $j = 1, ..., n$. Since $R_0 \vee R_j \;=\; L(R_0, R_0 \vee R_j)$, we have that $\mathbb{R}' = < R_t >_0^n \;=\; \mathbb{R}$ by $(I)_n$. □

2.5.1 Corollary. *Let $\mathbb{Q}$ and $\mathbb{R}$ be parallel free n-spaces, $n \geq 1$. Then either $\mathbb{Q} = \mathbb{R}$ or $\mathbb{Q} \cap \mathbb{R} = \emptyset$.*

2.6 Proposition. *Let $\mathbb{Q}$ and $\mathbb{R}$ be parallel free n-spaces, $n \geq 1$. Then $\mathbb{Q} \approx \mathbb{R}$ if and only if $Q \sim R$ for some $Q \in \mathbb{Q}$ and $R \in \mathbb{R}$.*

Proof. As the assertion is true for $n = 1$ by 1.2, we assume that $n \geq 2$ and proceed by induction. Clearly, we need to verify only the sufficiency.

Let $Q_0 \in \mathbb{Q}$, $R_0 \in \mathbb{R}$ and $Q_0 \sim R_0$. By $(I)_n$ and $\mathbb{Q} \| \mathbb{R}$, there exist bases, $\{Q_t\}_0^n$ of $\mathbb{Q}$ and $\{R_t\}_0^n$ of $\mathbb{R}$, such that $Q_0 \vee Q_j \| R_0 \vee R_j$ for $j = 1, ..., n$. For each such j, $< \hat{Q}_j >_0^n \, \| < \hat{R}_j >_0^n$ and the induction hypothesis yield that $< \hat{Q}_j >_0^n \approx < \hat{R}_j >_0^n$.

Let $Q \in \mathbb{Q}$. By $(B)_n$, $L(Q, Q_0 \vee Q_1)$ intersects $< \hat{Q}_1 >_0^n$ at a point X. As $X \sim Y$ for some $Y \in < \hat{R}_1 >_0^n$, it follows from 1.2 that

$$L(Q, Q_0 \vee Q_1) = L(X, Q_0 \vee Q_1) \sim L(Y, Q_0 \vee Q_1) = L(Y, R_0 \vee R_1).$$

Since $\mathbb{R}$ is a subspace, $L(Y,\, R_0 \vee R_1) \subset \mathbb{R}$. Thus there is a point $R \in \mathbb{R}$ such that $Q \sim R$ and $\mathbb{Q} \approx \mathbb{R}$. □

Finally, we extend parallelism to $s(\mathcal{A})$. Let $\{\mathbb{C}, \mathbb{F}\} \subset s(\mathcal{A})$. Then $\mathbb{C}$ is *parallel* to $\mathbb{F}(\mathbb{C} \| \mathbb{F})$ if for any line $s \subseteq \mathbb{C}$, there is a line $u \subseteq \mathbb{F}$ such that $s || u$. We note also that if $\mathbb{Q} \,\epsilon f_m(\mathcal{A})$, $\mathbb{R} \in f_n(\mathcal{A})$, $m < n$, and $\mathbb{Q} || \mathbb{R}$ then it readily follows that $\mathbb{Q}$ is parallel to some free m-space in $\mathbb{R}$.

3 Free meets, free joins, and the dimension formula.

We recall that if the underlying affine space $\mathcal{A}^*$ of $\mathcal{A}$ has dimension at least three then $\mathcal{A}^*$ and all of its subspaces are free, it is desarguesian and can be coordinatized by a vector space over a skew field. Let $\mathbb{Q}^*$ and $\mathbb{R}^*$ be finite dimensional subspaces of $\mathcal{A}^*$ and $\mathbb{Q}^* \cap \mathbb{R}^* \neq \emptyset$. Then $\mathbb{Q}^* \vee \mathbb{R}^*$ is finite dimensional and with vector space techniques (cf. [3], pp. 15-19), one obtains the dimension formula:

(DF) $\dim(\mathbb{Q}^* \vee \mathbb{R}^*) + \dim(\mathbb{Q}^* \cap \mathbb{R}^*) \;=\; \dim(\mathbb{Q}^*) + \dim(\mathbb{R}^*)$.

With regard to the AK-space $\mathcal{A}$, we do not know if $\mathcal{A}$ is free or even if it is desarguesian when it has dimension at least three. Hence, before considering any generalization of (DF), we examine when $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R} \in f_n(\mathcal{A})$ imply that $\mathbb{Q} \vee \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{R}$ are free.

First, we need to assume that $\mathbb{Q} \not\approx \mathbb{R}$. For if $\mathbb{Q} \not\subseteq \mathbb{R}$ and $\mathbb{Q} \approx \mathbb{R}$ then $\mathbb{R} \neq \mathbb{Q} \vee \mathbb{R}$, $\mathbb{R} \subseteq \mathbb{Q} \vee \mathbb{R} \subseteq \overline{\mathbb{R}}$ and (A7) yield that $\mathbb{Q} \vee \mathbb{R}$ is not free.

Second, it is not sufficient to assume only that $\mathbb{Q} \not\approx \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{R} \neq \emptyset$.

3.1 Example. Let $\mathcal{A}$ be an AK-space with dimension at least four and degree at least two.

Let $\{P,\, Q_1,\, R_1,\, R_2,\, S\} \subseteq \mathbb{P}$ be independent.

Then $\mathbb{F} = \langle P,\, Q_1,\, R_1,\, R_2 \rangle \epsilon\, f_3(\mathcal{A})$ and $S \not\approx \mathbb{F}$. Let $\ell = R_2 \vee S$. Then $|\ell \cap \mathbb{F}| = 1$ by 1.8 a), and $|\ell \cap \overline{R}_2| = d(\mathcal{A}) \geq 2$. Hence, there is a point $Q_2 \in \overline{R}_2 \backslash \mathbb{F}$.

Since $Q_2 \sim R_2$, it follows that $\{P, Q_1, Q_2, R_1, S\}$ is independent, and

$$\mathbb{Q} \;=\; \langle P, Q_1, Q_2\rangle \text{ and } \mathbb{R} = \langle P, R_1, R_2\rangle$$

are planes such that $\mathbb{Q} \not\approx \mathbb{R}$ and $\mathbb{Q}\cap\mathbb{R} \neq \emptyset$. We note that

$$\mathbb{F} \subseteq \mathbb{Q}\vee\mathbb{R} \;=\; \langle P, Q_1, Q_2, R_1, R_2\rangle \;=\; \mathbb{F}\vee Q_2 \subseteq \overline{\mathbb{F}},$$

$R_2 \in (\mathbb{Q}\vee\mathbb{R})\backslash\mathbb{F}$ and (A7) yield that $\mathbb{Q}\vee\mathbb{R}$ is not free. □

Let $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R} \in f_n(\mathcal{A})$ such that $\mathbb{Q} \not\approx \mathbb{R}$ and $\mathbb{Q}\cap\mathbb{R} \neq \emptyset$, $1 \leq m \leq n$. In view of 3.1, our initial problem is to determine when $\mathbb{Q}\vee\mathbb{R}$ is free. In fact, as we shall see, that is the main difficulty in determining a dimension formula for $\mathcal{A}$.

To solve the initial problem, we return to $\mathcal{A}^*$. From 1.9, $\mathbb{Q}^* \in f_m(\mathcal{A}^*)$ and $\mathbb{R}^* \in f_n(\mathcal{A}^*)$. Thus $\mathbb{Q}^*\cap\mathbb{R}^* \neq \emptyset$ and (DF) yield that

$$\dim(\mathbb{Q}^*\vee\mathbb{R}^*) + \dim(\mathbb{Q}^*\cap\mathbb{R}^*) \;=\; n+m.$$

Of interest now is the relation between $(\mathbb{Q}\vee\mathbb{R})^*$ and $\mathbb{Q}^*\vee\mathbb{R}^*$, and the one between $(\mathbb{Q}\cap\mathbb{R})^*$ and $\mathbb{Q}^*\cap\mathbb{R}^*$. Clearly,

$$(\mathbb{Q}\cap\mathbb{R})^* \subseteq \mathbb{Q}^*\cap\mathbb{R}^* \text{ and } \mathbb{Q}^*\vee\mathbb{R}^* \subseteq (\mathbb{Q}\vee\mathbb{R})^*.$$

In order to determine these relations, we need to describe $\mathbb{Q}^*\vee\mathbb{R}^*$. But, as $\mathcal{A}^*$ is also an AK-space, we accomplish this by describing $\mathbb{Q}\vee\mathbb{R}$ when $\mathbb{Q}\vee\mathbb{R}$ is free.

Thus, in order to determine when $\mathbb{Q}\vee\mathbb{R}$ is free, we need to examine the consequences of $\mathbb{Q}\vee\mathbb{R}$ being free. We have already one such result.

3.2 ([2], 2.7) Let $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R} \in f_n(\mathcal{A})$ such that $\mathbb{Q} \not\approx \mathbb{R}$, $\mathbb{Q}\cap\mathbb{R} \neq \emptyset$ and $\mathbb{Q}\vee\mathbb{R} \in f_{n+1}(\mathcal{A})$, $1 \leq m \leq n$. Then $\mathbb{Q}\cap\mathbb{R} \in f_{m-1}(\mathcal{A})$.

We show that the converse of 3.2 is also valid, and then examine the general case.

3.3 Proposition. *Let $\{\mathbb{C}, \mathbb{F}\} \subset f(\mathcal{A})$, $\mathbb{C} \not\approx \mathbb{F}$. If $\mathbb{H} = \mathbb{C}\cap\mathbb{F}$ is a hyperplane of $\mathbb{C}$ then $\mathbb{C}\vee\mathbb{F} \in f(\mathcal{A})$ and $\mathbb{F}$ is a hyperplane of $\mathbb{C}\vee\mathbb{F}$.*

Proof. Since $\mathbb{C} \not\approx \mathbb{F}$, there is a point $S_0 \in \mathbb{C}$ such that $S_0 \not\approx \mathbb{F}$. If $\mathbb{H}$ is a hyperplane of $\mathbb{C}$ then $S_0 \not\approx \mathbb{F}$ and 1.10 *c*) imply that $\mathbb{C} = \mathbb{H}\vee S_0$. Hence,

$\mathbb{F}\vee S_0 \subseteq \mathbb{F}\vee\mathbb{C} = \mathbb{F}\vee[\mathbb{H}\vee S_0]$

$= [\mathbb{F}\vee\mathbb{H}]\vee S_0 = \mathbb{F}\vee S_0$.

Since $S_0 \not\approx \mathbb{F}$, $\mathbb{F}\vee\mathbb{C} = \mathbb{F}\vee S_0$ is free by 1.5. Clearly, $\mathbb{F}$ is a hyperplane of $\mathbb{F}\vee S_0$. □

3.3.1 Corollary. *Let $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R} \in f_n(\mathcal{A})$ such that $\mathbb{Q} \not\approx \mathbb{R}$ and $\mathbb{Q}\cap\mathbb{R} \neq \emptyset$, $1 \leq m \leq n$. Then $\mathbb{Q}\vee\mathbb{R} \in f_{n+1}(\mathcal{A})$ if and only if $\mathbb{Q}\cap\mathbb{R} \in f_{m-1}(\mathcal{A})$.*

3.4 Proposition. *Let $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R} \in f_n(\mathcal{A})$ such that $\mathbb{Q} \not\approx \mathbb{R}$ and $\mathbb{Q}\cap\mathbb{R} \neq \emptyset$, $1 \leq m \leq n$. Then there is an independent subset $\{Q_t\}_1^k$ of $\mathbb{Q}$, $1 \leq k \leq m$, such that $Q_1 \not\approx \mathbb{R}$, $Q_i \not\approx \mathbb{R}\vee\langle Q_t\rangle_1^{i-1}$ for $2 \leq i \leq k$ and $\mathbb{R}\vee\mathbb{Q} \subseteq \mathbb{R}\vee\langle Q_t\rangle_1^k$; moreover, if $\mathbb{R}\vee\mathbb{Q}$ is free then $\mathbb{R}\vee\mathbb{Q} = \mathbb{R}\vee\langle Q_t\rangle_1^k$ and $\dim(\mathbb{R}\vee\mathbb{Q}) = n+k$.*

Proof. Let $R_0 \in \mathbb{Q}\cap\mathbb{R}$. Then by $(I)_n$,there is a basis $\{R_t\}_0^n$ of $\mathbb{R}$. We note that if such a set $\{Q_t\}_1^k$ exists then $\{R_t\}_0^n \cup \{Q_t\}_1^k$ is independent by 1.5. Thus if $\mathbb{R}\vee\mathbb{Q}$ is free then

$\langle R_t\rangle_0^n \vee \langle Q_t\rangle_1^k \subseteq \mathbb{R} \vee \mathbb{Q} \subseteq \overline{\langle R_t\rangle_0^n \vee \langle Q_t\rangle_1^k}$
and (A7) yield that $\mathbb{R} \vee \mathbb{Q} = \langle R_t\rangle_0^n \vee \langle Q_t\rangle_1^k$.

Next, we determine the existence of $\{Q_t\}_1^k \subseteq \mathbb{Q}$. Since $\mathbb{Q} \not\approx \mathbb{R}$, there is a $Q_1 \in \mathbb{Q}$ such that $Q_1 \not\approx \mathbb{R} = \overline{\langle R_t\rangle_0^n}$. By 1.5, $\{R_t\}_0^n \cup \{Q_1\}$ is independent and $\mathbb{R} \vee Q_1 = \langle R_t\rangle_0^n \vee Q_1$. If $\mathbb{Q} \subseteq \overline{\mathbb{R} \vee Q_1}$ then the assertion is true with $k = 1$. If $\mathbb{Q} \not\subseteq \overline{\mathbb{R} \vee Q_1}$ then there is a $Q_2 \in \mathbb{Q}$ such that $Q_2 \not\approx \mathbb{R} \vee Q_1 = \{R_t\}_0^n \cup \{Q_1\}$, $\{R_t\}_0^n \cup \{Q_t\}_1^2$ is independent and $\mathbb{R} \vee \langle Q_t\rangle_1^2 = \langle R_t\rangle_0^n \vee \langle Q_t\rangle_1^2$. Again, either $\mathbb{Q} \subseteq \overline{\mathbb{R} \vee \langle Q_t\rangle_1^2}$ or $\mathbb{Q} \not\subseteq \overline{\mathbb{R} \vee \langle Q_t\rangle_1^2}$. Since $\mathbb{Q} \in f_m(\mathcal{A})$, it follows that there is a smallest $k \leq m$ such that $\{R_0\} \cup \{Q_t\}_1^k \subseteq \mathbb{Q}$ is independent and $\{Q_t\}_1^k$ has the required property. □

3.5 The Dimension Formula for AK-spaces. *Let $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R} \in f_n(\mathcal{A})$ such that $\mathbb{Q} \not\approx \mathbb{R}$, $\mathbb{Q} \cap \mathbb{R} \neq \emptyset$ and $\mathbb{Q} \vee \mathbb{R} \in f(\mathcal{A})$, $m \leq n$. Then for some integer k, $1 \leq k \leq m$,*

3.5.1 $\mathbb{Q} \vee \mathbb{R} \in f_{n+k}(\mathcal{A})$ and $\mathbb{Q} \cap \mathbb{R} \in f_{m-k}(\mathcal{A})$, and
3.5.2 $\dim(\mathbb{Q} \vee \mathbb{R}) + \dim(\mathbb{Q} \cap \mathbb{R}) = \dim(\mathbb{Q}) + \dim(\mathbb{R})$.

Proof. We note that 3.5.2 is an immediate consequence of 3.5.1, and that 3.5.1 has been verified for $k = 1$ in 3.3.1. Next, 3.4 yields that there is an independent subset $\{Q_t\}_1^k \subseteq \mathbb{Q}$ such that $Q_1 \not\approx \mathbb{R}$, $Q_i \not\approx \mathbb{R} \vee \langle Q_t\rangle_1^{i-1}$ for $2 \leq i \leq k$ and $\mathbb{R} \vee \mathbb{Q} = \mathbb{R} \vee \langle Q_t\rangle_1^k \in f_{n+k}(\mathcal{A})$. It follows that

$\mathbb{R}_i := \mathbb{R} \vee \langle Q_t\rangle_1^{k-i} \in f_{n+k-i}(\mathcal{A})$ for $i = 1, ..., k-1$.

We set $\mathbb{R}_k = \mathbb{R}$ and $\mathbb{R}_0 = \mathbb{R} \vee \mathbb{Q}$. Then $\mathbb{R}_{j+1}$ is a hyperplane of $\mathbb{R}_j$ for $j = 0, ..., k-1$.

Let us consider $\mathbb{Q}$ and $\mathbb{R}_1$. Clearly, $\mathbb{Q} \cap \mathbb{R}_1 \neq \emptyset$ and from $Q_k \in \mathbb{Q}$, it follows that $\mathbb{Q} \not\approx \mathbb{R}_1$. We note that

$$\mathbb{R}_0 = \mathbb{R} \vee \langle Q_t\rangle_1^k = \mathbb{R}_1 \vee Q_k \subseteq \mathbb{R}_1 \vee \mathbb{Q} \subseteq \mathbb{R}_0 .$$

Since $\mathbb{Q} \in f_m(\mathcal{A})$ and $\mathbb{R}_1$ is a hyperplane of $\mathbb{R}_1 \vee \mathbb{Q}$, it follows by 3.3.1 that

$$\mathbb{Q}_1 := \mathbb{Q} \cap \mathbb{R}_1 \in f_{m-1}(\mathcal{A}) .$$

Since $Q_{k-1} \in \mathbb{Q}_1$ and $\mathbb{Q} \cap \mathbb{R} \subseteq \mathbb{Q} \cap \mathbb{R}_2 = \mathbb{Q} \cap \mathbb{R}_2 \cap \mathbb{R}_1 = \mathbb{Q}_1 \cap \mathbb{R}_2$, we obtain that $\mathbb{Q}_1 \cap \mathbb{R}_2 \neq \emptyset$, $\mathbb{Q}_1 \not\approx \mathbb{R}_2$ and

$$\mathbb{R}_1 = \mathbb{R} \vee \langle Q_t\rangle_1^{k-1} = \mathbb{R}_2 \vee Q_{k-1} \subseteq \mathbb{R}_2 \vee \mathbb{Q}_1 \subseteq \mathbb{R}_1 .$$

Since $\mathbb{Q}_1 \in f_{m-1}(\mathcal{A})$ and $\mathbb{R}_2$ is a hyperplane of $\mathbb{R}_2 \vee \mathbb{Q}_1$, it follows that

$$\mathbb{Q}_2 = \mathbb{Q}_1 \cap \mathbb{R}_2 = (\mathbb{Q} \cap \mathbb{R}_1) \cap \mathbb{R}_2 = \mathbb{Q} \cap \mathbb{R}_2 \in f_{m-2}(\mathcal{A}) .$$

Repeating this argument $k-2$ more times yields that

$$\mathbb{Q}_k = \mathbb{Q} \cap \mathbb{R}_k = \mathbb{Q} \cap \mathbb{R} \in f_{m-k}(\mathcal{A}) .$$

□

Finally, we present a solution to the problem of when $\mathbb{Q} \vee \mathbb{R}$ is free.

3.6 Theorem. *Let* $\mathbb{Q} \in f_m(\mathcal{A})$ *and* $\mathbb{R} \in f_n(\mathcal{A})$ *such that* $\mathbb{Q} \not\approx \mathbb{R}$ *and* $\mathbb{Q} \cap \mathbb{R} \neq \emptyset$, $1 \leq m \leq n$.

3.6.1 If $\mathbb{Q} \vee \mathbb{R} \in f(\mathcal{A})$ then $\mathbb{Q}^* \vee \mathbb{R}^* = (\mathbb{Q} \vee \mathbb{R})^*$.
3.6.2 $\mathbb{Q} \vee \mathbb{R} \in f_{n+k}(\mathcal{A})$ if and only if $\mathbb{Q} \cap \mathbb{R} \in f_{m-k}(\mathcal{A})$ and $(\mathbb{Q} \cap \mathbb{R})^* = \mathbb{Q}^* \cap \mathbb{R}^*$, $1 \leq k \leq m$.

Proof. Let $\mathbb{Q} \vee \mathbb{R} \in f(\mathcal{A})$. Then by 3.4, $\mathbb{Q} \vee \mathbb{R} \in f_{n+k}(\mathcal{A})$ for some $1 \leq k \leq m$ and there is a basis $\{R_t\}_0^n \cup \{Q_t\}_1^k$ of $\mathbb{R} \vee \mathbb{Q}$ such that $\mathbb{R} = \langle R_t \rangle_0^n$ and $\langle Q_t \rangle_1^k \subseteq \mathbb{Q}$. From 1.4 *c*), 1.9 and (DF), we obtain that

$\mathbb{R}^* \vee \mathbb{Q}^* \subseteq (\mathbb{R} \vee \mathbb{Q})^* = \langle \{R_t\}_0^n \cup \{Q_t\}_1^k \rangle^*$
$= \langle R_t^* \rangle_0^n \vee \langle Q_t^* \rangle_1^k \subseteq \mathbb{R}^* \vee \mathbb{Q}^*$,

$\mathbb{Q}^* \vee \mathbb{R}^* = (\mathbb{Q} \vee \mathbb{R})^* \in f_{n+k}(\mathcal{A}^*)$ and $\mathbb{Q}^* \cap \mathbb{R}^* \in f_{m-k}(\mathcal{A}^*)$. By 3.5 and 1.9, $(\mathbb{Q} \cap \mathbb{R})^* \in f_{m-k}(\mathcal{A}^*)$. Since $(\mathbb{Q} \cap \mathbb{R})^* \subseteq \mathbb{Q}^* \cap \mathbb{R}^*$ and the two spaces are of equal dimension, $(\mathbb{Q} \cap \mathbb{R})^* = \mathbb{Q}^* \cap \mathbb{R}^*$ by $(I)_{m-k}$.

Conversely, let $\mathbb{Q} \cap \mathbb{R} \in f_{m-k}(\mathcal{A})$ and $(\mathbb{Q} \cap \mathbb{R})^* = \mathbb{Q}^* \cap \mathbb{R}^*$. Then $\mathbb{Q}^* \cap \mathbb{R}^* \in f_{m-k}(\mathcal{A}^*)$ and $\mathbb{Q}^* \vee \mathbb{R}^* \in f_{n+k}(\mathcal{A}^*)$ by 1.9 and (DF). Let $\{R_t\}_0^n$ be a basis of $\mathbb{R}$. By $(I)_n$, we may assume that $\{R_t\}_0^{m-k}$ is a basis of $\mathbb{Q} \cap \mathbb{R}$. We now apply 3.4 to $\mathcal{A}^*$ and $\mathbb{R}^* \vee \mathbb{Q}^*$. Thus there exist an independent subset $\{Q_t^*\}_1^k \subseteq \mathbb{Q}^*$ such that $Q_1^* \notin \mathbb{R}^*$, $Q_i^* \notin \mathbb{R}^* \vee \langle Q_t^* \rangle_0^{i-1}$ for $2 \leq i \leq k$ and $\mathbb{R}^* \vee \mathbb{Q}^* = \mathbb{R}^* \vee \langle Q_t^* \rangle_1^k$.

Clearly, we may choose $Q_1, ..., Q_k$ in $\mathbb{Q}$. Then $Q_1^* \notin \mathbb{R}^* = (\langle R_t \rangle_0^n)^*$ implies that $Q_1 \not\approx \langle R_t \rangle_0^n$. Hence $\{R_t\}_0^n \cup \{Q_1\}$ is independent by 1.5, and $\langle \{R_t\}_0^n \cup \{Q_1\} \rangle^* = \mathbb{R}^* \vee Q_1^*$ by 1.4 *c*). Similarly, $Q_i^* \notin \mathbb{R}^* \vee \langle Q_t^* \rangle_0^{i-1}$ yields that $\{R_t\}_0^n \cup \{Q_t\}_1^i$ is independent and $\langle \{R_t\}_0^n \cup \{Q_t\}_1^i \rangle^* = \mathbb{R}^* \vee \langle Q_t^* \rangle_1^i$, $2 \leq i \leq k$. Since $\mathbb{Q} \in f_m(\mathcal{A})$ and $\{R_t\}_0^{m-k} \cup \{Q_t\}_1^k \subseteq \mathbb{Q}$ is independent, it follows that $\mathbb{Q} = \langle R_t \rangle_0^{m-k} \vee \langle Q_t \rangle_1^k$ by $(I)_m$. Finally,

$\mathbb{R} \vee \langle Q_t \rangle_1^k \subseteq \mathbb{R} \vee \mathbb{Q} = \langle R_t \rangle_0^n \vee (\langle R_t \rangle_0^{m-k} \vee \langle Q_t \rangle_1^k)$
$= \langle R_t \rangle_0^n \vee \langle Q_t \rangle_1^k$
$= \mathbb{R} \vee \langle Q_t \rangle_1^k$

implies that $\mathbb{R} \vee \mathbb{Q} = \langle R_t \rangle_0^n \vee \langle Q_t \rangle_1^k \in f_{n+k}(\mathcal{A})$. □

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