Gluing two affine spaces

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Summary. A construction is described in [2] by which, given two or more geometries of the same rank n, each equipped with a suitable parallelism giving rise to the same geometry at infinity, we can glue them together along their geometries at infinity, thus obtaininig a new geometry of rank $n+k-1$, k being the number of geometries we glue. In this paper we will examine a special case of that construction, namely the gluing of two affine spaces.

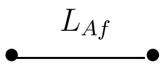
1 Introduction

In this section I recall some definitions and some basic results from [2], in order to make this paper as self-contained as possible. Gluings of two affine spaces will be studied in the other sections of this paper.

1.1 Some notation and terminology

I am going to use a number of basic notions of diagram geometry. I refer to [16] for them. The only difference between the notation used in this paper and that of [16] is the meaning of the symbol $Aut(\Gamma)$. In [16] that symbol denotes the full automorphism group of Γ, whereas in this paper (as in [2]) $Aut(\Gamma)$ means the group of type-preserving automorphisms of Γ (denoted by $Aut_s(\Gamma)$ in [16]).

As in [16], the symbols c and Af, when used as labels for diagrams, mean *circular spaces* (i.e., complete graphs) and *affine planes*, respectively. The labels c^* and Af^* have the meanings dual of the above. We introduce the symbols



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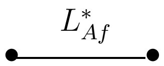

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to denote point-line systems of affine geometries and their duals, respectively.

In order to avoid any confusion between affine geometries and their point-line systems we state the following convention: by the name *affine geometry* we mean a 'full' affine geometry, consisting of points, lines, planes,..., hyperplanes. We keep the name *affine space* for the system of points and lines of an affine geometry.

1.2 Parallelism

In this section Γ is a geometry of rank $n > 1$, with set of types I and type function t. We denote the set of elements of Γ by X and, given a type $i \in I$, we set $X_i = t^{-1}(i)$. That is, X_i is the set of elements of Γ of type i. We denote the incidence relation of Γ by $*$. We distinguish an element $0 \in I$ and we call *points* the elements of type 0.

1.2.1 Definition

A *parallelism* (*with respect to* 0) is an equivalence relation $\|$ on $X \backslash X_0$ with the following properties (P1), (P2) and (P3).

(P1) Every equivalence class of $\|$ is contained in some fiber of t.

(P2) Given any two points a and b and an element x of the residue Γ_a of a, there is just one element $y \in \Gamma_b$ such that $y \| x$.

(P3) Given any two points a and b and elements $x, x' \in \Gamma_a$ and $y, y' \in \Gamma_b$ with $x \| y$ and $x' \| y'$, we have $x * x'$ if and only if $y * y'$.

When $x \| y$ we say that x and y are *parallel*. Thus, we can rephrase (P1) as follows: parallel elements have the same type. By (P2), distinct elements incident with some common point are never parallel. By (P2) and (P3), given any two points a and b, $\|$ induces an isomorphism between Γ_a and Γ_b.

Many examples of geometries with parallelism are described in [2]. I mention only three of them here: affine geometries and affine spaces, with their natural parallelism; nets (in particular, affine planes and grids); connected graphs admitting 1–factorizations (in particular, complete graphs with an even number of vertices [12] and complete bipartite graphs with classes of the same size [14]).

1.2.2 The geometry at infinity

Given a geometry Γ over the set of types I, let $0 \in I$ and let $\|$ be a parallelism of Γ with respect to 0. Given an element $x \in X \backslash X_0$, we denote by $\infty(x)$ the equivalence class of $\|$ containing x and we call it the *element at infinity* of x, also the *direction* of x.

By (P3), the incidence relation $*$ of Γ naturally induces an incidence relation among the directions of the elements of $X \backslash X_0$. Hence them form a geometry Γ^∞, which we call the *geometry at infinity* of $(\Gamma, \|)$ (the *line at infinity*, when Γ has rank 2). We take $I \backslash \{0\}$ as the set of types of Γ^∞, directions of elements of type i being given the type i. We have $\Gamma^\infty \cong \Gamma_a$ for every point a, by (P3).

1.2.3 Isomorphisms and automorphisms

Let Γ and Γ' be geometries over the same set of types I and let $\|$ and $\|'$ be parallelisms of Γ and Γ' respectively, with respect to the same type $0 \in I$. Each type-preserving isomorphism $\alpha : \Gamma \longrightarrow \Gamma'$ maps $\|$ onto a parallelism $\|_\alpha$ of Γ'. If $\|_\alpha = \|'$, then we say that α is an *isomorphism* from $(\Gamma, \|)$ to $(\Gamma', \|')$. Clearly, if $(\Gamma, \|) \cong (\Gamma', \|')$, then $\Gamma^\infty \cong \Gamma'^\infty$.

An *automorphism* of $(\Gamma, \|)$ is a type-preserving automorphism of Γ preserving $\|$. We denote the automorphism group of $(\Gamma, \|)$ by $Aut(\Gamma, \|)$.

The group $A = Aut(\Gamma, \|)$ induces on Γ^∞ a subgroup A^∞ of $Aut(\Gamma^\infty)$. The kernel of the action of A on Γ^∞ will be denoted by K^∞. By (P2), K^∞ acts semiregularly on X_0. Therefore, given a point a, the stabilizer A_a of a in A acts faithfully on Γ^∞.

If K^∞ is transitive on X_0, then we say that it is *point-transitive*. The following statements are proved in [2] (§2.5):

Proposition 1 *If K^∞ is point-transitive, then its orbits on $X \backslash X_0$ are just the classes of $\|$.*

Proposition 2 *Let K^∞ be point-transitive on Γ. Then A is the normalizer of K^∞ in $Aut(\Gamma)$.*

Proposition 3 *If K^∞ is point-transitive, then $A = K^\infty A_a$, for every point a.*

The following is an easy consequence of Proposition 3

Proposition 4 *Let K^∞ be point-transitive. Then A is flag-transitive on Γ if and only if A^∞ is flag-transitive on Γ^∞.*

1.3 Gluing

Gluings can be defined for any finite family of geometries with parallelism having 'the same' geometry at infinity (see [2]). However, I shall consider only gluings of two geometries in this paper.

1.3.1 The construction

Let I be a set of types of size at least 2 and let $0 \in I$. Let Γ_1 and Γ_2 be geometries over I, endowed with parallelisms $\|_1$ and $\|_2$ with respect to 0. Assume that $\Gamma_1^\infty \cong \Gamma_2^\infty$. Let α be a (possibly non type-preserving) isomorphism from Γ_2^∞ to Γ_1^∞ and let τ be the permutation induced by α on $I \backslash \{0\}$. We define the *gluing* $\Gamma = (\Gamma_1, \|_1) \circ_\alpha (\Gamma_2, \|_2)$ of $(\Gamma_1, \|_1)$ with $(\Gamma_2, \|_2)$ via α as follows.

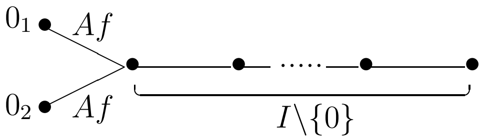
We take $(I \backslash \{0\}) \cup \{0_1, 0_2\}$ as the set of types of Γ. For $j = 1, 2$, the elements of Γ of type 0_j are the points of Γ_j. As elements of type $i \in I \backslash \{0\}$ we take the pairs (x_1, x_2) with x_j an element of Γ_j (for $j = 1, 2$), x_1 and x_2 of type i and $\tau^{-1}(i)$ respectively and $\alpha(\infty(x_2)) = \infty(x_1)$. We decide that all elements of type 0_1 are incident with all elements of type 0_2. For $j = 1, 2$, we decide that an element (x_1, x_2) and an element x of type 0_j are incident precisely when $x * x_j$ in Γ_j. Finally, we put $(x_1, x_2) * (y_1, y_2)$ if and only if $x_j * y_j$ in Γ_j, for $j = 1, 2$.

When we want to put emphasis on the fact that α induces τ on $I\backslash\{0\}$, we call Γ a *τ-gluing*. We say that the gluing Γ is *plain* when τ is the identity on $I\backslash\{0\}$ (that is, α is type-preserving). Otherwise, we say that Γ is a *twisted* gluing.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be diagrams for Γ_1 and Γ_2 respectively. A diagram for the glued geometry $\Gamma_1 \circ_\alpha \Gamma_2$ is obtained by pasting $\mathcal{D}_2$ with $\mathcal{D}_1$ on $I\backslash\{0\}$ via the permutation τ induced by α on $I\backslash\{0\}$.

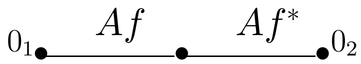
For instance, if $\Gamma_1 = \Gamma_2 = AG(n, K)$ and α is a (type-preserving) automorphism of $PG(n-1, K) = \Gamma_1^\infty = \Gamma_2^\infty$, then the glued geometry $\Gamma_1 \circ_\alpha \Gamma_2$ belongs to the following diagram of rank $n+1$:

$(\binom{Af}{Af}.A_{n-1})$



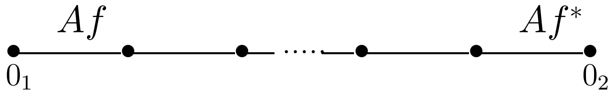
In particular, with $n = 2$ we get the following rank 3 diagram

$(Af.Af^*)$



When K is commutative and $n > 2$, we can also consider non type-preserving automorphisms (namely, correlations) of $PG(n-1, K)$. Let α be one of them. The (twisted) gluing $\Gamma_1 \circ_\alpha \Gamma_2$ belongs to the following diagram:

$(Af.A_{n-1}.Af^*)$



1.3.2 Automorphisms of glued geometries

Given Γ_1, Γ_2, $\|_1$, $\|_2$ and α be as in §1.3.1, we set $A_i = Aut(\Gamma_i, \|_i)$ for $i = 1, 2$. As in §1.2.2, K_i^∞ is the kernel of the action of A_i on Γ_i^∞ and $A_i^\infty \cong A_i/K_i^\infty$ is the subgroup induced by A_i in $Aut(\Gamma_i^\infty)$. We denote by $\alpha(A_2^\infty)$ the image of A_2^∞ in $Aut(\Gamma_1^\infty)$ via α. The following is proved in [2] (§3.4.2):

Proposition 5 *We have* $Aut(\Gamma_1 \circ_\alpha \Gamma_2) = (K_1^\infty \times K_2^\infty)(A_1^\infty \cap \alpha(A_2^\infty))$.

By this and Proposition 4 we get the following:

Proposition 6 *Let both K_1^∞ and K_2^∞ be point-transitive. Then the glued geometry $\Gamma_1 \circ_\alpha \Gamma_2$ is flag-transitive if and only if $A_1^\infty \cap \alpha(A_2^\infty)$ is flag-transitive on Γ_1^∞.*

1.3.3 Isomorphisms of gluings

Let Γ_1 and Γ_2 be as in §1.3.1. We keep the meaning stated in §1.3.2 for A_i, A_i^∞ and $\alpha(A_2^\infty)$, with α an isomorphism from Γ_2^∞ to Γ_1^∞. Furthermore, we assume that the following (very mild) condition holds in Γ_1 and Γ_2:

(O) no two distinct elements are incident with the same set of points.

Let α and β be two isomorphisms from Γ_1^∞ to Γ_2^∞ inducing the same permutation τ on $I\backslash\{0\}$. Then $\alpha\beta^{-1} \in Aut(\Gamma_1^\infty)$. The following is proved in [2] (Lemma 3.4):

Proposition 7 *We have* $\Gamma_1 \circ_\alpha \Gamma_2 \cong \Gamma_1 \circ_\beta \Gamma_2$ *if and only if* $\alpha\beta^{-1} \in \alpha(A_2^\infty)A_1^\infty$.

Therefore,

Proposition 8 *If* $A_1^\infty = Aut(\Gamma_1^\infty)$, *then (up to isomorphisms) there is a unique* τ*-gluing of* $(\Gamma_1, \|_1)$ *with* $(\Gamma_2, \|_2)$.

More generally, by modifying a bit an argument of [2] (§3.4.5) the following can be proved:

Proposition 9 *The isomorphism classes of* τ*-gluings of* $(\Gamma_1, \|_1)$ *with* $(\Gamma_2, \|_2)$ *are in one-to-one correspondence with the double cosets* $\alpha(A_2^\infty)gA_1^\infty$, *with* $g \in Aut(\Gamma_1^\infty)$, *where* α *is any given isomorphism from* Γ_2^∞ *to* Γ_1^∞ *inducing* τ *on* $I\backslash\{0\}$.

1.3.4 Canonical gluings

Assume $(\Gamma_2, \|_2) \cong (\Gamma_1, \|_1)$. A gluing $\Gamma_1 \circ_\alpha \Gamma_2$ is said to be *canonical* if α is induced by an isomorphism from $(\Gamma_2, \|_2)$ to $(\Gamma_1, \|_1)$. Note that only plain gluings can be said to be canonical, since, according to the definition stated in §1.2.2, isomorphisms of geometries with parallelism are type-preserving. (However, by modifying a bit the definitions of §1.2, one could also define canonical τ-gluings for any τ.)

It follows from Proposition 8 that all canonical gluings are pairwise isomorphic. In short, the canonical gluing is unique.

Let the gluing $\Gamma_1 \circ_\alpha \Gamma_2$ be canonical. Then $A_1^\infty = \alpha(A_2^\infty)$. Therefore

$$Aut(\Gamma_1 \circ_\alpha \Gamma_2) \; = \; (K_1^\infty \times K_2^\infty)A_1^\infty$$

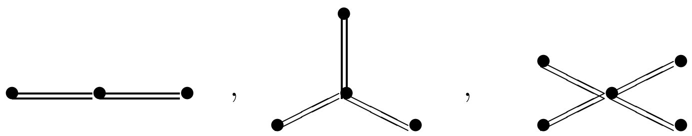
by Proposition 5. This is in fact the largest automorphism group for a gluing of $(\Gamma_1, \|_1)$ with $(\Gamma_2, \|_2)$ (see Proposition 5).

Let $\Gamma_1 \circ_\beta \Gamma_2$ be another plain gluing such that $\beta(A_2^\infty) = A_1^\infty$. Then $\alpha\beta^{-1}$ normalizes A_1^∞. Consequently, if A_1^∞ is its own normalizer in $Aut(\Gamma_1^\infty)$, then $\Gamma_1 \circ_\beta \Gamma_2 \cong \Gamma_1 \circ_\alpha \Gamma_2$, by Proposition 7. Thus, we have proved the following

Proposition 10 *Let* A_1^∞ *be its own normalizer in* $Aut(\Gamma_1^\infty)$. *Then the canonical gluing is the unique plain gluing* $\Gamma_1 \circ_\alpha \Gamma_2$ *for which* $\alpha(A_2^\infty) = A_1^\infty$.

1.3.5 A bit of 'history' and some applications

The earliest example of a construction that is clearly a gluing is due to Cameron [3], who glued generalized quadrangles admitting partitions of their set of lines into spreads, to obtain geometries of arbitary rank with diagrams as follows:



, , , etc.

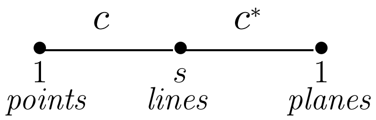
As Cameron says in [3], an idea by Kantor [11] is the 'ancestor' of his construction.

Independently of [3], examples of gluings have been discovered in [7] and [8] in the context of an investigation of geometries belonging to the diagram $Af.A_{n-1}.Af^*$ (in particular, $Af.Af^*$). A description of the minimal quotients of finite geometries belonging to the diagram $Af.A_{n-1}.Af^*$ is obtained in [8]. Those minimal quotients can only be of two types: either 'almost flat', or flat. The flat ones are in fact twisted gluings of two copies of $AG(n, q)$. When $n > 2$ there is just one twisted gluing of two copies of $AG(n, q)$ (see Proposition 8). This fact made it possible to accomplish the classification of all finite $Af.A_{n-1}.Af^*$ geometries with $n > 2$ (see [8]).

Gluings have been gaining in importance in other contexts, too. For instance, by exploiting the classification of 2-transitive groups preserving a 1-factorization of a complete graph, obtained by Cameron and Korchmaros [4], the following two theorems can be proved (see [1] for the first of them and [14] for the latter):

Theorem 11 *Let* Γ *be a flag-transitive geometry belonging to the following diagram*

$(c.c^*)$



$1 < s < \infty$

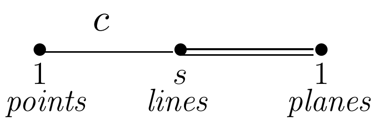
Assume also that Γ *is flat* (*that is, all points are incident with all planes*). *Then one of the following holds:*

(i) $s = 4$ *and* $Aut(\Gamma) = S_6$*;*

(ii) $s = 2^n - 2$ *for some* $n \geq 2$ *and* Γ *is a gluing of two copies of the* n*-dimensional affine space over* $GF(2)$. *Furthermore, either that gluing is the canonical one* (*in this case* $Aut(\Gamma) = 2^{2n}.L_n(2)$) *or* $Aut(\Gamma) = 2^{2n}X$ *with* $X \leq \Gamma L_1(2^n)$.

Theorem 12 *Let* Γ *be a flag-transitive geometry belonging to the following diagram*

$(c.C_2)$



$1 < s < \infty$

Assume furthermore that Γ *is flat. Then* $s = 2^n - 2$ *for some* $n \geq 2$ *and* Γ *is a gluing of the* n*-dimensional affine space over* $GF(2)$ *with a complete bipartite graph endowed with a suitable* 1*-factorization.*

2 Gluing two affine planes

Let Γ_1 and Γ_2 be two affine planes of the same order. We can assume that they have the same line at infinity $\Gamma^\infty = \Gamma_1^\infty = \Gamma_2^\infty$.

Γ^∞ is a geometry of rank 1. Hence any permutation of its elements is an automorphism of Γ^∞.

Let α be a permutation of Γ^∞. The glued geometry $\Gamma_1 \circ_\alpha \Gamma_2$ belongs to the diagram $Af.Af^*$ (see §1.3.1).

By Proposition 6, the geometry $\Gamma_1 \circ_\alpha \Gamma_2$ is flag-transitive if and only if both Γ_1 and Γ_2 are flag-transitive and $A_1^\infty \cap \alpha A_2^\infty \alpha^{-1}$ is transitive on the set Γ^∞. (Note that

A_2^∞ is a group of permutations of Γ^∞ and $\alpha(A_2^\infty) = \alpha A_2^\infty \alpha^{-1}$.) In particular, the canonical gluing of two copies of a flag-transitive affine plane is flag-transitive.

On the other hand, it might be that both Γ_1 and Γ_2 are flag-transitive but $\Gamma_1 \circ_\alpha \Gamma_2$ is not. An example of this kind will be given in §2.2.3, with $\Gamma_1 = \Gamma_2 = AG(2,7)$.

2.1 Gluing two copies of $AG(2,K)$

Let $\Gamma_1 = \Gamma_2 = AG(2,K)$, with K a division ring. Hence $\Gamma_1^\infty = \Gamma_2^\infty = PG(1,K)$ and $A_1^\infty = A_2^\infty = P\Gamma L_2(K)$. We denote $PG(1,K)$ by Γ^∞ and $P\Gamma L_2(K)$ by A^∞. Given a permutation α of the set Γ^∞, we write $AG(2,K) \circ_\alpha AG(2,K)$ for $\Gamma_1 \circ_\alpha \Gamma_2$, to remind ourselves of the assumption $\Gamma_1 = \Gamma_2 = AG(2,K)$.

By Proposition 7, a gluing $\Gamma_1 \circ_\alpha \Gamma_2$ is the canonical one if and only if $\alpha \in A^\infty$. Therefore, non-canonical gluings exist when $|K| > 4$.

It is well known that, if K is commutative, then $P\Gamma L_2(K)$ is its own normalizer in the group of all permutations of the set $PG(1,K) = \Gamma^\infty$ (see [10], Chapter II, §8, Exercise 14). By this and by Proposition 10, in the finite case we get the following:

Theorem 13 *A gluing $AG(2,q) \circ_\alpha AG(2,q)$ is the canonical one if and only if $Aut(AG(2,q) \circ_\alpha AG(2,q)) \cong p^{2h}.P\Gamma L_2(q)$ (where $p^h = q$, p prime).*

All other gluings of two copies of $AG(2,q)$ have automorphism groups smaller than $p^{2h}.P\Gamma L_2(q)$.

Problem. Can we generalize Theorem 13 to the case where K is an infinite commutative field ? Note that an infinite field might be isomorphic with some of its proper subfields. Hence, when K is infinite, the group $P\Gamma L_2(K)$ might be isomorphic with some of its proper subgroups.

2.2 Some examples of small order

2.2.1 The cases of $q = 2$, 3 or 4

Let $q \in \{2,3,4\}$. Then A^∞ is the full symmetric group on $q+1$ obiects. In these cases the canonical gluing is the unique gluing of $AG(2,q)$ with itself.

2.2.2 The case of $q = 5$

Let $q = 5$. Non-canonical gluing now exist. For instance, let α be the following permutation of $\Gamma^\infty = PG(1,5)$:

$$\alpha \;=\; (\infty)(0)(1)(2)(3,4)$$

It is straightforward to check that the stabilizer of the point ∞ of Γ^∞ in the group $X = A^\infty \cap \alpha A^\infty \alpha^{-1}$ is cyclic of order 4. Hence $|X| = 4t$ for some positive integer $t \leq 6$ and $X \neq A^\infty = PGL_2(5)$.

Therefore the gluing $AG(2,5) \circ_\alpha AG(2,5)$ is not the canonical one.

As $PGL_2(5) \cong S_5$, we have $|A^\infty| = 5!$ and the double coset $A^\infty \alpha A^\infty$ has size $(5!)^2/|X|$. Clearly,

$$6! \;\geq\; |A^\infty| + |A^\infty \alpha A^\infty| \;=\; 5! + \frac{(5!)^2}{4t}$$

This forces $t \geq 6$. On the other hand, $t \leq 6$, as we remarked above. Hence $t = 6$. Therefore X is transitive on Γ^∞. Thus, the (non-canonical) gluing $AG(2,5) \circ_\alpha AG(2,5)$ is flag-transitive.

As $t = 6$, we have $6! \;=\; |A^\infty| + |A^\infty \alpha A^\infty|$. Hence A^∞ admits only two double cosets in S_6, namely itself and $A^\infty \alpha A^\infty$. Consequently, by Proposition 9, there are only two ways of gluing $AG(2,5)$ with itself, namely the canonical one and the gluing we have described now. Both of them are flag-transitive.

2.2.3 The case of $q = 7$

Let $q = 7$. The following permutation of Γ^∞ is considered in [7]:

$$\alpha \;=\; (\infty)(0)(1)(2)(3,6,5,4)$$

It is straightforward to check that $A^\infty \cap \alpha A^\infty \alpha^{-1}$ contains the element $g \in A^\infty$ represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

which is in fact a Singer cycle on $PG(1,7)$. Therefore $A^\infty \cap \alpha A^\infty \alpha^{-1}$ is transitive on Γ^∞. Hence the gluing $AG(2,7) \circ_\alpha AG(2,7)$ is flag-transitive.

On the other hand, it is straightforward to check that no non-trivial element of $A^\infty \cap \alpha A^\infty \alpha^{-1}$ fixes any point of Γ^∞. That is, $A^\infty \cap \alpha A^\infty \alpha^{-1} \;=\; \langle g \rangle \;=\; Z_8$. Hence the gluing $AG(2,7) \circ_\alpha AG(2,7)$ is not the canonical one.

Non flag-transitive gluings of $AG(2,7)$ with itself also exist. For instance, let β be the following permutation of Γ^∞:

$$\beta \;=\; (\infty)(0)(1)(2)(3)(4,5,6)$$

It is straightforward to check that $A^\infty \cap \beta A^\infty \beta^{-1}$ does not contain any element mapping the point ∞ of Γ^∞ onto the point 0. Thus, the glued geometry $AG(2,7) \circ_\beta AG(2,7)$ is not flag-transitive.

3 Gluing two copies of $AG(n,K)$

From now on we shall denote the canonical gluing of two copies of $AG(n,K)$ by the symbol $AG(n,K) \circ AG(n,K)$. It belongs to the diagram $\binom{Af}{Af}.A_{n-1}$ (see §1.3.1) and it is flag-transitive.

Note that, by Proposition 9 and well known properties of affine and projective geometries, when $n > 2$ the canonical gluing $AG(n,K) \circ AG(n,K)$ is the unique plain gluing of two copies of $AG(n,K)$.

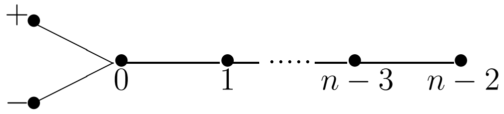
Keeping the hypothesis that $n > 2$, assume furthermore that K is commutative. Then $PG(n-1, K)$ admits correlations. Given a correlation α of $PG(n-1, K)$, we can construct the twisted gluing $AG(n, K) \circ_\alpha AG(n, K)$. It belongs to the diagram $Af.A_{n-1}.Af^*$ (see §1.3.1) and it is flag-transitive. Note that, by Proposition 9, and since all correlations of $PG(n-1, K)$ differ by elements of $P\Gamma L_2(n, K)$, the isomorphism type of $AG(n, K) \circ_\alpha AG(n, K)$ does not depend on the particular correlation α we have chosen.

It is proved in [8] that the twisted gluing of two copies of $AG(n, K)$ is the minimal quotient of the geometry obtained from $PG(n+1, K)$ by removing a hyperplane H and the residue of a point $p \in H$ (see [8]). In §3.2 I shall give an analogous of that result for the canonical gluing $AG(n, K) \circ AG(n, K)$. More precisely, we will prove that, if K is commutative, then $AG(n, K) \circ AG(n, K)$ is a quotient of a certain subgeometry of the building of type D_{n+1} over K.

3.1 Some subgeometries of D_{n+1}-buildings

3.1.1 Removing two hyperplanes from a D_{n+1}-building

Let K be a commutative field and let Δ be the building of type D_{n+1} over K, $n \geq 2$. I allow $n = 2$, with the convention that the symbols D_3 and A_3 mean the same. According to this convention, $PG(3, K)$ may be called a building of type D_3. I take $+$, $-$, 0, 1,..., $n-2$ as types, as follows:



Let us write ε to denote any of the two types $+$ or $-$. For every element x of Δ, let $\sigma^\varepsilon(x)$ be the set of elements of Δ of type ε incident to x.

For $\varepsilon = +$ or $-$, let Δ^ε be the half–spin geometry relative to the type ε (see [19]). That is, Δ^ε is the geometry of rank 2 having the elements of Δ of type ε as points and those of type 0 as lines, with the incidence inherited from Δ. As the Intersection Property holds in Δ, the geometry Δ^ε is a partial plane. In particular, distinct lines of Δ^ε are incident with distinct sets of points. Hence, the lines of Δ^ε can be viewed as distinguished sets of elements of type ε.

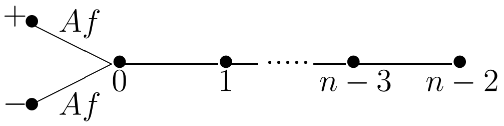
A proper subset H of the set of points of Δ^ε is said to be a *geometric hyperplane* of Δ^ε (a *hyperplane*, for short) if every line of Δ^ε not contained in H meets H in precisely one point (see [19]).

Given hyperplanes H^+ and H^- of Δ^+ and Δ^- respectively, we can construct a new geometry $\overline{\Delta}$ as follows.

The elements of $\overline{\Delta}$ are the elements x of Δ such that $\sigma^\varepsilon(x) \not\subseteq H^\varepsilon$ for $\varepsilon = +$ or $-$. Two elements x, y of $\overline{\Delta}$ are said to be incident in $\overline{\Delta}$ if they are incident in Δ and, furthermore, $\sigma^\varepsilon(x) \cap \sigma^\varepsilon(y) \not\subseteq H^\varepsilon$, for $\varepsilon = +, -$.

I call $\overline{\Delta}$ *the geometry obtained from* Δ *by removing* H^+ and H^-. It is straightforward to prove that $\overline{\Delta}$ is indeed a geometry (this amounts to prove that it is residually connected).

Let b be an element of $\overline{\Delta}$. Then $b \notin H^-$. The residue Δ_b of b in Δ is a projective geometry isomorphic to $PG(n, K)$. We take $\sigma^+(b)$ as the set of points of that projective geometry. Then $H^+ \cap \sigma^+(b)$ is a hyperplane of $\Delta_b = PG(n, K)$. When we remove H^+ from Δ, we are forced to remove $H^+ \cap \sigma^+(b)$ from Δ_b. What remains is isomorphic with $AG(n, K)$. Removing H^- gives no effect on Δ_b, since $\sigma^-(x) \not\subseteq H^-$ for every $x \in \Delta_b$ (indeed $b \in \sigma^-(x)$ for every such x, and $b \notin H^+$). Therefore, the residue of b in $\overline{\Delta}$ is isomorphic to $AG(n, K)$. It is now clear that $\overline{\Delta}$ belongs to the diagram $\binom{Af}{Af}.A_{n-1}$:



3.1.2 A particular choice of H^+ and H^-

Keeping the notation of the previous paragraph, let a^+ and a^- be incident elements of Δ of type $+$ and $-$ respectively.

If n is even, then we define H^+ as the set of elements of Δ of type $+$ having distance $< n/2$ from some element of $\sigma^+(a^-)$ in the collinearity graph of Δ^+.

If n is odd, then we define H^+ as the set of elements of type $+$ having distance $< (n+1)/2$ from a^+ in the collinearity graph of Δ^+.

The hyperplane H^- of Δ^- is defined just as H^+, but interchanging the roles of $+$ and $-$. The following is a special case of Theorem 2.4(ii) of [19]:

Lemma 14 *The sets H^+ and H^- are hyperplanes of Δ^+ and Δ^-, respectively.*

It is worthwhile to examine the case of $n = 2$ closer. Let $n = 2$. Then $\Delta = PG(3, K)$. Chosen the elements of type $+$ as points of $PG(3, K)$, a^- is a plane and H^+ is the set of its points. The point a^+ is one them and H^- is the set of the planes incident with it. Thus, removing H^+ and H^- from Δ amounts to remove from $PG(3, K)$ a plane and the star of one of its points.

3.2 From $\overline{\Delta}$ to $AG(n, K) \circ AG(n, K)$

Let H^+ and H^- be the hyperplanes defined in §3.1.2 and let $\overline{\Delta}$ be the geometry obtained from Δ by removing H^+ and H^-, as in §3.1.1. Let G be the stabilizer of a^+ and a^- in $Aut(\Delta)$ and let N be the elementwise stabilizer of $H^+ \cup H^-$ in G. It is straightforward to check that N defines a quotient of $\overline{\Delta}$, which is flag-transitive, since N is normal in $Aut(\overline{\Delta})$ and $Aut(\overline{\Delta})$ is flag-transitive.

Theorem 15 *We have $\overline{\Delta}/N = AG(n, K) \circ AG(n, K)$.*

Proof. If n is even (odd) then an orbit of N on the set of elements of $\overline{\Delta}$ of type ε is the set of elements of $\overline{\Delta}$ of type ε incident with some element of type $n-2$ incident with a^ε but not with a^η (with a^η but not with a^ε) for $\{\varepsilon, \eta\} = \{+, -\}$.

Let Γ^+ (respectively, Γ^-) be the geometry obtained from the residue of a^+ (of a^-) in Δ by removing the elements incident to a^- (to a^+). Both Γ^+ and Γ^- are copies of $AG(n,K)$. We can take the residue in Δ of the flag $\{a^+, a^-\}$ as the (common) geometry at infinity of Γ^+ and Γ^-. Let us denote this residue by Γ^∞.

Let σ be the shadow operator in Δ with respect to the type $n-2$. By the Intersection Property in Δ, for every element x of $\overline{\Delta}$ there is just one element x^ε of Δ incident with a^ε and such that $\sigma(x) \cap \sigma(a^\varepsilon) = \sigma(x^\varepsilon) \cap \sigma(a^\varepsilon)$, for $\varepsilon = \pm$. Since x belongs to $\overline{\Delta}$, it has maximal distance in Δ from both a^+ and a^-. Hence x^+ and x^- have the same type in Δ. Furthermore,

$$\sigma(x^\varepsilon) \cap \sigma(a^\eta) \;=\; \sigma(a^+) \cap \sigma(a^-) \cap \sigma(x), \qquad (\text{for}\{\varepsilon, \eta\} = \{+, -\})$$

by the definition of x^ε. Hence x^+ and x^-, viewed as elements of the affine geometries Γ^+ and Γ^- respectively, have the same element at infinity in Γ^∞.

Let us consider the natural embedding of Δ in the lattice of linear subspaces of a $(2n+2)$-dimensional vector space $V(2n+2,K)$ over K. With a suitable choice of the basis of $V(2n+2,K)$, it is not difficult to compute the matrices of $O^+_{2n+2}(K)$ that represent elements of N. Thus, by straightforward calculations one can prove that two elements x, y of $\overline{\Delta}$ belong to the same orbit of N if and only if $x^\varepsilon = y^\varepsilon$ for $\varepsilon = +, -$. Therefore $\overline{\Delta}/K$ is a plain gluing of Γ^+ with Γ^-.

When $n > 2$, the above is enough to prove that $\overline{\Delta}/K \cong AG(n,K) \circ AG(n,K)$, by the uniqueness of the plain gluing of two copies of $AG(n,K)$ with $n > 2$.

Let $n = 2$. Thus $\Delta = PG(3,K)$ and $\overline{\Delta}$ is obtained from $PG(3,K)$ by removing the plane H^+ and the star of the point $a^+ \in H^+$. Also Γ^∞ is the bundle of lines of H^+ through a^+. The affine plane Γ^+ is the complement of Γ^∞ in the star of a^+, whereas by removing the lines of Γ^- and the point a^+ from H^+ we get the dual of the affine plane Γ^-. Two lines of $\overline{\Delta}$ belong to the same orbit of N if and only if they are coplanar with a^+ and intersect H^+ in the same point. The orbits of N on the set of lines of $\overline{\Delta}$ can be represented by the pairs (S,p), where S is a plane of $PG(3,K)$ passing through a^+ and distinct from H^+ and $p \in H^+ \cap S$, with $p \neq a^+$.

Thus, in order to prove that $\overline{\Delta}/N$ is the canonical gluing of Γ^+ with Γ^-, we need to find an isomorphism α from Γ^+ to Γ^- such that $\alpha(S) \in S$ for every line S of Γ^+ (I recall that the lines of Γ^+ are planes of $PG(3,K)$ on a^+, whereas the lines of Γ^- are points of H^+).

Since K is commutative, $PG(3,K)$ admits a symplectic polarity π. We can always assume to have chosen π in such a way that H^+ is the polar plane of a^+ with respect to π. Then π induces an isomorphism α from Γ^+ to Γ^- with the property that $\alpha(S) \in S$ for every line S of Γ^+, as we wanted. □

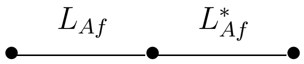
Remark. When $n = 2$ and $K = GF(q)$, the isomorphism between $\overline{\Delta}/N$ and $AG(2,q) \circ AG(2,q)$ can also be obtained as a consequence of Theorem 13 (see [7]).

4 Gluing two affine spaces

When $n > 2$, the affine space of points and lines of $AG(n,K)$ is a proper subgeometry of $AG(n,K)$. We denote it by $AS(n,K)$, to avoid any confusion between it and $AG(n,K)$. More precisely, $AS(n,K)$ is the affine space of points and lines of $AG(n,K)$, equipped with the parallelism $\|$ inherited from $AG(n,K)$. (Note that, when $K \neq GF(2)$, $\|$ can be recovered from the incidence structure of $AS(n,K)$.) We denote by Γ^{∞} the set of points of the geometry at infinity $PG(n-1,K)$ of $AG(n,K)$. That is, Γ^{∞} is the line at infinity of $AS(n,K)$.

A gluing of two copies of $AS(n,K)$ belongs to the following diagram

$(L_{Af}.L^{*}_{Af})$



The line at infinity Γ^{∞} of $AS(n,K)$ is just a set. Thus, for every permutation α of Γ^{∞}, we can glue $AS(n,K)$ with itself via α.

4.1 Canonical gluings

The symbol $AS(n,K) \circ AS(n,K)$ will denote the canonical gluing of two copies of $AS(n,K)$. When $n > 2$, $AS(n,K) \circ AS(n,K)$ is a truncation of the (unique) plain gluing of two copies of $AG(n,K)$. Hence it is a quotient of a truncation of the geometry $\overline{\Delta}$ defined in §3.3, by Theorem 15.

It is well known that when K is commutative $P\Gamma L_n(K)$ is its own normalizer in the group of all permutations of the set Γ^{∞} of points of $PG(n-1,K)$. By this and by Proposition 10, in the finite case we get the following:

Theorem 16 *A gluing $AS(n,q) \circ_{\alpha} AS(n,q)$ is the canonical one if and only if $Aut(AS(n,q) \circ_{\alpha} AS(n,q)) \cong p^{nh}.P\Gamma L_n(q)$, where $p^h = q$, p prime.*

That is, the gluing $AS(n,q) \circ_{\alpha} AS(n,q)$ is canonical if and only if its automorphism group is as large as possible. We can say more:

Theorem 17 *Let $(n,q) \neq (3,2), (3,8)$. Then the gluing $AS(n,q) \circ_{\alpha} AS(n,q)$ is the canonical one if and only if $P\Gamma L_n(q) \cap \alpha P\Gamma L_n(q)\alpha^{-1}$ is flag-transitive on $PG(n-1,q)$.*

Proof. The "only if" claim is obvious. Let us prove the "if" statement. Let $G = P\Gamma L_n(q) \cap \alpha P\Gamma L_n(q)\alpha^{-1}$ be flag transitive on $PG(n-1,q)$. By a theorem of Higman [9], one of the following occurs:

(1) $G \geq L_n(q)$;
(2) $n = 4$, $q = 2$ and $G = A_7$;
(3) $n = 3$, $q = 2$ and $G = Frob(21)$;
(4) $n = 3$, $q = 8$ and $G = Frob(9 \cdot 73)$.

In case (1) α normalizes the socle $L_n(q)$ of $P\Gamma L_n(q)$. Hence it also normalizes $P\Gamma L_n(q)$. Therefore $\alpha \in P\Gamma L_n(q)$ because $P\Gamma L_n(q)$ is its own normalizer in the group of all permutations of Γ^{∞}. Hence the gluing $AS(n,q) \circ_{\alpha} AS(n,q)$ is canonical.

Let (2) occur. Then there are two subgroups X, Y of $L_4(2)$, both isomorphic with A_7 and such that α maps X onto Y, and $Y = L_4(2) \cap \alpha L_4(2)\alpha^{-1}$. However, $L_4(2)$ has just one conjugacy class of subgroups isomorphic with A_7. Therefore, by multiplying α by a suitable element of $L_4(2)$ if necessary, we can always assume that $X = Y$. That is, α normalizes X.

The stabilizers in X of the lines of $PG(3,2)$ form one conjugacy class of subgroups of X. They have index 35 in X and all subgroups of X with that index belong to that conjugacy class (see [6]). Therefore α permutes those subgroups of X. Hence it permutes their orbits on $PG(3,2)$. On the other hand, if H is the stabilizer in X of a line L of $PG(3,2)$, X has just two orbits on the set Γ^∞, namely L and its complement in Γ^∞. It is now clear that α permutes the lines of $PG(3,2)$. Hence $\alpha \in L_4(2)$. Thus, case (2) is impossible.

Cases (3) and (4) are the two exceptions mentioned in the statement of the theorem. □

4.2 Two exceptional examples

The assumption that $(n,q) \neq (3,2),(3,8)$ is essential in Theorem 17. Indeed, let $n = 3$ and $q = 2$, for instance, and let $G = Frob(21) \leq L_3(2)$, flag-transitive on the projective plane $PG(2,2)$ (see [9]).

For every point a of $PG(2,2)$, the stabilizer G_a of a in G fixes a unique line L_a of $PG(2,2)$. Given a line $L = \{a,b,c\}$ of $PG(2,2)$, the lines L_a, L_b, L_c form a triangle. Let us denote by L' the set of vertices of that triangle. Let $\mathcal{L}$ be the set of lines of $PG(2,2)$ and define $\mathcal{L}' = \{L' \mid L \in \mathcal{L}\}$.

Then $\mathcal{L}'$ is the set of lines of a model Π of $PG(2,2)$ and $\alpha(PG(2,2)) = \Pi$ for some permutation α of the set of points of $PG(2,2)$. Let α be such a permutation. Then $L_3(2) \cap \alpha L_3(2)\alpha^{-1} = Frob(21)$. Therefore, the gluing $AS(3,2) \circ_\alpha AS(3,2)$ is not the canonical one. Nevertheless $L_3(2) \cap \alpha L_3(2)\alpha^{-1}$ is flag-transitive in $PG(2,2)$.

A similar argument works when $n = 3$ and $q = 8$, with $Frob(9 \cdot 73)$ instead of $Frob(21)$. Thus, a non canonical gluing $AS(3,8) \circ_\alpha AS(3,8)$ also exists, with $L_3(8) \cap \alpha L_3(8)\alpha^{-1} = Frob(9.73)$, flag-transitive on $PG(2,8)$.

4.3 A problem

Let $X = P\Gamma L_n(q)$ and $Y = \alpha X \alpha^{-1}$ for a permutation α of the $(q^n-1)/(q-1)$ points of $PG(n-1,q)$. Is it true that $X \cap Y$ is transitive on the set of points of $PG(n-1,q)$ only if it contains a Singer cycle ?

Assume that $X \cap Y$ contains a Singer cycle S and that $X \neq Y$. Is it true that, if q is large enough (say, $q > 5$) then $X \cap Y$ is contained in the normalizer of S in X ?

5 Universal covers

In this section we investigate the universal covers of $AS(n,K) \circ AS(n,K)$ and $AG(n,K) \circ AG(n,K)$, with K a commutative field. We shall focus on the cases of $n = 2$ and of $K = GF(2)$.

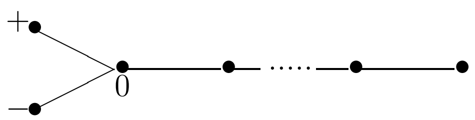
5.1 The case of $n = 2$

Let $\overline{\Delta}$ be the geometry obtained from $PG(3,K)$ by removing a plane π and the star of a point $p \in \pi$ (compare §3.1). It follows from [13] that $\overline{\Delta}$ is simply connected (see also [8]). This together with Theorem 15 imply the following:

Theorem 18 *Let K be commutative. Then the geometry $\overline{\Delta}$ is the universal cover of $AG(2,K) \circ AG(2,K)$.*

5.2 The case of $K = GF(2)$

Henceforth we denote by Γ_m the Coxeter complex of type D_m and by $Tr(\Gamma_m)$ the $\{+,0,-\}$-truncation of Γ_m, that is the subgeometry of Γ_m formed by the elements of type $+$, 0 and $-$, where $+$, 0 and $-$ are as follows:



The following result, proved in [1], is a completion of Theorem 11:

Lemma 19 *The geometry $AS(n,2) \circ AS(n,2)$ is a quotient of $Tr(\Gamma_m)$, with $m = 2^n$.*

By Theorem 1 of [15] and since Coxeter complexes are simply connected, $Tr(\Gamma_m)$ is simply connected. Thus, Lemma 19 implies the following:

Theorem 20 *The universal cover of $AS(n,2) \circ AS(n,2)$ is $Tr(\Gamma_m)$, with $m = 2^n$.*

5.3 An unexpected consequence of Theorem 20

The universal cover of $AG(2,2) \circ AG(2,2)$ is the geometry $\overline{\Delta}$ mentioned in §5.1, with $K = GF(2)$. Actually, that geometry is isomorphic with $Tr(\Gamma_4)$.

When $n > 2$ things look more intriguing. Let $\overline{\Delta}$ be as in §3.1, with $K = GF(2)$ and $n > 2$, and let $Tr(\overline{\Delta})$ be its $\{+,0,-\}$-truncation. Let $m = 2^n$. By theorems 20 and 15, $Tr(\Gamma_m)$ is the universal cover of $Tr(\overline{\Delta})$. Hovever, $Tr(\overline{\Delta})$ contains less elements than $Tr(\Gamma_m)$. Hence $Tr(\overline{\Delta})$ is a proper quotient of $Tr(\Gamma_m)$.

By Theorem 1 of [15], the $\{-,0,+\}$-truncation of the universal cover of $\overline{\Delta}$ is the universal cover of $Tr(\overline{\Delta})$. This has the following (surprising) consquence:

Theorem 21 *When $n > 2$ and $K = GF(2)$, the geometry $Tr(\Gamma_m)$ (with $m = 2^n$) is the $\{+,0,-\}$-truncation of the universal cover of $\overline{\Delta}$.*

Let Ξ be the universal cover of $\overline{\Delta}$. All elements of Γ_m of type $+$ belong to Ξ, by Theorem 21. The number of these elements is

$$2^{m-1} \ = \ 2^{2^n-1}$$

whereas, denoted by ν the number of elements of $\overline{\Delta}$ of type $+$, we have

$$\nu \ < \ \prod_{i=1}^{n}(2^i+1) \ < \ 2^{(n+2)(n+1)/2}$$

Hence $\nu < 2^{m-1}$ whenever $n > 2$. Therefore

Corollary 22 *When $n > 2$ and $K = GF(2)$, the geometry $\overline{\Delta}$ is not simply connected.*

5.4 Problems

1. Describe the universal cover Ξ of $\overline{\Delta}$ when $n > 2$ and $K = GF(2)$. Note that

$$\frac{2^{m-1}}{\nu} > 2^{2^n - (n^2+3n+4)/2}$$

and the latter goes to infinity with the same speed as 2^{2^n}. Thus, Ξ very soon becomes huge in comparison with $\overline{\Delta}$.

2. Is $\overline{\Delta}$ simply connected when $K \neq GF(2)$ and $n > 2$?

3. Given a non-commutative field K, let $\overline{\Delta}$ be the geometry obtained from $PG(3, K)$ be removing a plane π and the star of a point $p \in \Pi$.

Let Θ be the equivalence relation defined on the set of elements of $\overline{\Delta}$ as follows: two points (planes) correspond by Θ if they are collinear with p (respectively, if they meet π in the same line); two lines correspond by Θ if they are coplanar with p and meet π in the same point.

Then Θ defines a quotient of $\overline{\Delta}$. It is not difficult to check that $\overline{\Delta}/\Theta$ is a gluing of $AG(2, K)$ with $AG(2, K^{op})$, where K^{op} is the dual of K. Characterize these gluings.

4. Which is the universal cover of $AG(n, K) \circ AG(n, K)$ when K is non-commutative ?

5. What about non-canonical gluings of two copies of $AG(2, K)$? Are they simply connected ? And what about gluings of two copies of a non-desarguesian affine plane, or gluings of two non-isomorphic affine planes ?

6. What about non-canonical gluings of two copies of $AS(n, K)$? Are they simply connected ?

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