LS-category of classifying spaces, II

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Abstract

Let X be a simply connected finite CW-complex. Denote by $B\,aut\,X$ the base of the universal fibration of fibre X. We show in this paper that the Lusternik-Schnirelmann category of $B\,aut\,X$ is not finite under some hypothesis on X.

1 Introduction

In this paper, X will denote a simply connected CW-complex of finite type. The LS-category of X, $cat(X)$, is the least integer n such that X can be covered by $(n+1)$ open subsets contractible in X, and is ∞ if no such n exists. The rational category of X, $cat_0(X)$, is the LS-category of its rationalization X_0 and satisfies $cat_0(X) \leq cat(X)$. One way to approach $cat(X)$ consists to compute $cat_0(X)$ by algebraic methods ([4]).

Recall first that the Sullivan minimal model of X is a free commutative cochain algebra $\left(\Lambda Z, d\right)$ such that $dZ \subset \Lambda^{\geq 2}Z$, with $Z^n \cong Hom_{\mathbb{Q}}(\pi_n(X),\, \mathbb{Q})$ ([10], [8]).

A differential graded Lie algebra (L, d) is called a Quillen model for X if there is a quasi-isomorphism between the Sullivan minimal model of X and the cochain algebra on (L, d).

The space X is called coformal if its Sullivan minimal model has the form $(\Lambda Z, d)$ where $dZ \subset \Lambda^2 Z$.

The n^{th} Gottlieb group of X, $G_n(X)$, is the subgroup of $\pi_n(X)$ defined as follows. The homotopy class of $\alpha : S^n \longrightarrow X$ belongs to $G_n(X)$ if the map $id \vee \alpha : X \vee S^n \longrightarrow X$ extends to $X \times S^n$.

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Fibrations whith fiber in the homotopy type of X are obtained, up to fiber homotopy equivalence, as pull back of the universal fibration $X \longrightarrow B\, aut^{\bullet} X \longrightarrow B\, aut\, X$ ([1],[3]); here $aut\, X$ denotes the monoid of self-homotopy equivalences of X, $aut^{\bullet} X$ is the monoid of pointed self-homotopy equivalences of X, and B is the Dold-Lashof functor ([2]).

Denote by $\tilde{B}\, aut\, X$ the universal covering of $B\, aut\, X$. We briefly recall the construction of a model of $\tilde{B}\, aut\, X$.

We derive from $\big(\Lambda Z, d\big)$ a connected differential graded Lie algebra of derivations, $(Der\, \Lambda Z, D)$ ([10]): in degree $k > 1$, we take the derivations of ΛZ decreasing degree by k. In degree one, we only consider the derivations θ which decrease degree by one and verify $[\, d,\, \theta] = 0$. The differential D is defined by $D\theta = [\, d,\, \theta\,] = d\theta - (-1)^{|\theta|}\theta d$ (see [10] for more details).

Theorem. ([10]) *The differential graded Lie algebra $(Der \Lambda Z, D)$ is a Quillen model for the universal covering $\tilde{B}\, aut\, X$ of $B\, aut\, X$.*

We know from ([6]) that $cat(\tilde{B}\, aut\, X)$ is not finite when the rational homotopy of X is finite dimensional or when X is a wedge of spheres. We shall use the model above to carry on whith the computation of the rational LS-category of $\tilde{B}\, aut\, X$. Since $cat(B\, aut\, X) \geq cat(\tilde{B}\, aut\, X) \geq cat_0(\tilde{B}\, aut\, X)$, whenever $cat_0(\tilde{B}\, aut\, X) = \infty$, the LS-category of $B\, aut\, X$ will also be infinite. We prove:

Theorem. *Let X be a simply connected finite CW-complex. The LS-category of $B\, aut\, X$ is infinite provided one of the following hypothesis is satisfied:*

(a) X is a coformal space and the Gottlieb group, $G(X_0)$, is non zero.

(b) $X = Y \times Z$, $G(Y_0) = G(Z_0) = 0$, $\pi_*(B\, aut\, Y) \otimes \mathbb{Q}$ is not finite dimensional and $\tilde{H}^*(Z, \mathbb{Q}) \neq 0$.

(c) $X = Y \vee Z$, $Z_0 \neq \vee S_0^i$, whith $\pi_*(Y) \otimes \mathbb{Q}$ infinite dimensional and Y, Z coformal spaces.

2 The classifying space spectral sequence

Let $(\Lambda Z, d)$ be the Sullivan minimal model of X. Define a filtration of the differential Lie algebra $Der(\Lambda Z, d)$ by the Lie differential sub algebras

$$F_p = \{\theta \in Der(\Lambda Z, d) \mid \theta(Z) \subset \Lambda^{\geq p} Z\}.$$

This filtration is compatible with the differential, and verifies $[\, F_p,\; F_q\,] \subset F_{p+q-1}$. Therefore, we obtain a spectral sequence of differential graded Lie algebras $(E^r(X),\, d^r)$ such that $E^2(X) \cong H_*(Der(\Lambda Z,\, d_2))$, where d_2 denotes the quadratic part of the differential d, and which converges to $H_*(Der(\Lambda\, Z,\, d))$. Moreover $E^{\infty}_{0,*}(X) \cong G(X_0)$.

Proposition 1 *Let X be a finite simply connected CW-complex, then $E^{\infty}_{q,*}(X) = 0$ for all q greater than the homological dimension of X.*

Proof. Let n be the homological dimension of X and $\theta \in F_p$, $p > n$, such that $[d, \theta] = 0$, we shall prove that θ is a boundary i.e. there is a derivation θ' such that $\theta = [d, \theta']$. For degree reasons, $\theta = 0$ on $Z^{\leq n+|\theta|}$, so that we put $\theta' = 0$ on $Z^{\leq n+|\theta|}$. Suppose now that that θ' is defined on $Z^{<r}, r > n + |\theta|$, such that $\theta = [d, \theta']$ on $Z^{<r}$. Let $x \in Z^r$, compute

$$\begin{aligned} d[\theta(x) + (-1)^{|\theta|}\theta'(dx)] &= d\theta(x) + (-1)^{|\theta|}d\theta'(dx) \\ &= (-1)^{|\theta|}\theta(dx) + (-1)^{|\theta|}d\theta'(dx) \\ &= (-1)^{|\theta|}(\theta - [\, d, \theta'\,])(dx) \\ &= 0. \end{aligned}$$

Therefore there is an element y in ΛZ such that $\theta(x) + (-1)^{|\theta|}\theta'(dx) = dy$. Then define $\theta'(x) = y$; the elements θ and $[\, d, \theta'\,]$ agree on $Z^{\leq r}$. ■

Corollary 2 *Let X be a finite simply connected CW-complex such that $G(X_0) = 0$, then $E^{\infty}_{\geq 2}(X)$ is a nilpotent ideal of $\pi_*(\Omega \tilde{B}\, aut\, X) \otimes \mathbb{Q}$.*

3 Proof of the theorem

3.1 Case (a)

Let n be an integer such that $G_n(X_0) \neq 0$, then according to ([4]), n is odd. Denote by $(\Lambda Z, d_2)$ the Sullivan minimal model of X. By definition ([4]), there is a derivation θ of ΛZ such that $\theta(z) = 1$ for some $z \in Z$, which commutes with the differential. Write $\theta = \theta_0 + \cdots \theta_i \cdots$ such that $\theta_i(Z) \subset \Lambda^i Z$. Since $[d_2, \theta_0] = 0$ and $\theta_0^2 = 0$, there is a KS-extension ([8]) $(\Lambda y, 0) \longrightarrow (\Lambda y \otimes \Lambda Z, d) \longrightarrow (\Lambda Z, \bar{d})$ such that $dx = d_2 x + y\theta_0(x)$ for $x \in \Lambda Z$. The spatial realization of this KS-extension is a fibration with fibre X_0 and basis $K(\mathbb{Q}, n+1)$. This provides a non trivial classifying map $f : K(\mathbb{Q}, n+1) \longrightarrow (\tilde{B}\, aut\, X)_0$. Therefore, applying the mapping theorem ([4]), we conclude that the category of $(\tilde{B}\, aut\, X)_0$ is not finite.

Remarks:

- A Gottlieb element of degree n of $(\Lambda Z, d)$ provides a derivation θ of ΛZ of degree $-n$ such that $\theta(z) = 1$ and $[d, \theta] = 0$ ([4]). As the homology class of θ is non zero in $\pi_{n+1}(\tilde{B}\, aut\, X) \otimes \mathbb{Q}$, there is a fibration $X_0 \longrightarrow E \longrightarrow S_0^{n+1}$ classified by $\theta : S_0^{n+1} \longrightarrow (\tilde{B}\, aut\, X)_0$. Moreover, if the hypothesis of case (a) are satisfied, this map factors through $K(\mathbb{Q}, n+1)$.

- The proof above holds for $S^n \times Y$ where Y is a finite CW-complex. Therefore the hypothesis on the coformalty of X is not necessary if we require $dz = 0$. In this case $(\Lambda Z, d) \cong (\Lambda z, 0) \otimes (\Lambda Y, d)$ ([9]).

3.2 Case (b)

Let $(\Lambda V,\ d)$ and $(\Lambda W,\ d')$ be Sullivan minimal models of Y and Z. There is an isomorphism

$$\Phi :\ Der((\Lambda V,\ d)\otimes(\Lambda W,\ d')) \xrightarrow{\cong} \left[(Der\Lambda V)\hat{\otimes}\Lambda W\right]\oplus\left[\Lambda V\hat{\otimes}(Der\Lambda W)\right],$$

where $\hat{\otimes}$ denotes the complete tensor product.

Let $\theta \in Der((\Lambda V,\ d)\otimes(\Lambda W, d'))$ be a derivation of degree n,

$$\Phi(\theta)=\sum_{i\geq 0}\theta_i\otimes b_i+\sum_{j\geq 0}a_j\otimes\psi_j$$

where $\theta_i \in Der\ \Lambda V,\ \psi_j \in Der\Lambda W,\ a_j \in \Lambda V,\ b_i \in \Lambda W$ are of respective degrees $n+i,\ n+j,\ ,-i,\ -j$, with

$$(\theta_i\otimes b_i)(v)=(-1)^{|v||b_i|}\theta_i(v).b_i,\ v\in V;\ \ (a_j\otimes\psi_j)(w)=a_j.\psi_j(w),\ w\in W.$$

The Lie bracket in $\left[(Der\Lambda V)\hat{\otimes}\Lambda W\right]\oplus\left[\Lambda V\hat{\otimes}(Der\Lambda W)\right]$ is defined by :

$$\begin{aligned}
[\,\theta_1\otimes w_1,\ \theta_2\otimes w_2] &= (-1)^{|\theta_1||w_1|}[\,\theta_1,\ \theta_2]\otimes w_1w_2,\ \theta_i\in Der\Lambda V,\ w_i\in W;\\
[\theta\otimes w,\ v\otimes\psi] &= (-1)^{|w||v\otimes\psi|+|\theta||v|}v\theta\otimes\psi(w)+(-1)^{|v||w|}\theta(v)\otimes w\psi,\\
&\quad \theta\in Der\Lambda V,\psi\in Der(\Lambda W),\ v\in V,\ w\in W;\\
[\,v_1\otimes\psi_1,v_2\otimes\psi_2\,] &= (-1)^{|v_2||\psi_1|}v_1v_2\otimes[\psi_1,\psi_2],\ \psi_i\in Der\Lambda W,\ v_i\in V.
\end{aligned}$$

Since $H^*(Y,\ \mathbb{Q})$ and $H^*(Z,\ \mathbb{Q})$ are finite dimensional, we can replace $(\Lambda V, d)$ and $(\Lambda W,\ d')$ by finite models, and in this case the complete tensor product is equivalent to the usual tensor product. Therefore

$$H_*(Der(\Lambda V\otimes\Lambda W))\cong[H_*(Der\Lambda V)\otimes H^*(Z)]\oplus[H^*(Y)\otimes H_*(Der\Lambda W)]\,.$$

Then $I=\left[H_*(Der\Lambda W)\otimes\tilde{H}^*(Z)\right]\oplus\left[\tilde{H}^*(Y)\otimes H_*(Der\Lambda W)\right]$ is an infinite nilpotent Lie ideal of $H_*(Der(\Lambda V\otimes\Lambda W))$, since $G(Y_0)=G(Z_0)=0$. Therefore, according to ([5]), the LS-category of $B\,aut\,X$ is not finite.

3.3 Case (c)

Let $(\Lambda V,\ d),\ (\Lambda W,\ d'),\ (\Lambda T,\ D)$ denote respectively minimal models of Y, Z and X. Let $\alpha\in H^*(Z,\ Q)$ be an element of the highest degree. It is represented by a cocycle $u\in(\Lambda W)^n$. Recall that $T=V\oplus W\oplus W'$ with

1. $DW'\subset(\Lambda^+V\otimes\Lambda^+W)\ \oplus(\Lambda^+W'\otimes\Lambda(V\oplus W))$.
2. The ideal $I=(\Lambda^+V\otimes\Lambda^+W)\ \oplus(\Lambda^+W'\otimes\Lambda(V\oplus W))$ is acyclic.

Choose an element $y\in V$ such that $|y|=k>n$. Define a map $f:\ Q.y\longrightarrow\Lambda T$ by $f(y)=u$, and extend it as a derivation θ of ΛT which commutes with the differential as follows:

– If $z \in W$, then $\theta(z) = 0$;

– Define θ on V by induction of the degree: $\theta = 0$ on the supplementary of $Q.y$ in $V^{\leq k}$. If $|z| > k$, suppose that θ had been defined on $V^{<|z|}$, then $\theta(Dz) \in \Lambda^{\geq 3}T$ is cocycle in I. As $\theta(Dz)$ is a coboundary, we choose $z' \in \Lambda^{\geq 2}T$ such that $\theta(Dz) = Dz'$, and define $\theta(z) = (-1)^{|\theta|}z'$.

– Similary for $z \in W'$, if $|z| \leq k$, define $\theta(z) = 0$, and extend it on W' such $\theta(W') \subset \Lambda^{\geq 2}T$.

The derivation θ is a cycle in $Der\, \Lambda T$ and it is not a boundary. In fact suppose that there is a derivation θ' in $Der\, \Lambda T$ such that $[D, \theta'] = \theta$. Therefore

$$u + (-1)^{|\theta'|}\theta'(Dy) = D\theta'(y),$$

thus u is cohomologuous to an element in the ideal generated by V, but this is impossible.

Since we have associated to each element $y \in V^{>n}$ an element $\theta \in E^{\infty}_{\geq 2}(X)$, $E^{\infty}_{\geq 2}(X)$ is not finite dimensional. According to corollary 2, $E^{\infty}_{\geq 2}(X)$ is a nilpotent ideal of $\pi_*(B\, aut\, X) \otimes Q$, therefore the category of $B\, aut\, X$ is not finite ([5]).

4 Examples

– We know from ([6]) that the category of $B aut X$ is infinite whenever $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional. The part (a) of the theorem above is a generalization of this result for coformal spaces. For instance, the LS-category of $BautX$ is not finite if X is a coformal space such that the center of $\pi_*(\Omega X) \otimes \mathbb{Q}$ is not trivial.

– Combining assertion (a) with (b) of the theorem, we obtain that the LS-category of $B\, aut\, X$ is not finite if X is a product of wedge of spheres or more generally if X is a product of coformal spaces X_i such that the center of $\pi_*(\Omega X_i) \otimes \mathbb{Q}$ is trivial.

– If X is a wedge of at least three coformal spaces, then the LS-category of $B\, aut\, X$ is not finite.

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