Spectral asymptotics and bifurcation for nonlinear multiparameter elliptic eigenvalue problems

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Abstract

This paper is concerned with the nonlinear multiparameter elliptic eigenvalue problem

$$
\begin{aligned}
&u''(r) + \frac{N-1}{r}u'(r) + \mu u(r) - \sum_{i=1}^{k} \lambda_i f_i(u(r)) = 0, \quad 0 < r < 1, \\
&u(r) > 0, \quad 0 \le r < 1, \\
&u'(0) = 0, u(1) = 0,
\end{aligned}
$$

where $N \geq 1, k \in N$ and $\mu, \lambda_i \geq 0$ $(1 \leq i \leq k)$ are parameters. The aim of this paper is to study the asymptotic properties of eigencurve $\mu(\lambda, \alpha) = \mu(\lambda_1, \lambda_2, \cdots, \lambda_k, \alpha)$ with emphasis on the phenomenon of bifurcation from the first eigenvalue μ_1 of $-\triangle|_D$ and on gaining a clearer picture of the bifurcation diagram. Here, $\alpha > 0$ is a normalizing parameter of eigenfunction associated with $\mu(\lambda, \alpha)$. To this end, we shall establish asymptotic formulas of $\mu(\lambda, \alpha)$ as $|\lambda| \to \infty, 0$.

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1 Introduction.

We consider the following nonlinear multiparameter eigenvalue problem

$$\begin{aligned} & u''(r) + \frac{N-1}{r}u'(r) + \mu u(r) - \sum_{i=1}^{k} \lambda_i f_i(u(r)) = 0, \quad 0 < r < 1, \\ & u(r) > 0, \quad 0 \le r < 1, \\ & u'(0) = 0, u(1) = 0, \end{aligned} \tag{1.1}$$

We assume the following conditions (A.1)-(A.2) on f_i:
(A.1) $f_i : R_+ \to R$ is C^1, $f_i(0) = 0, f_i'(0) = 0$.
(A.2) The mapping $u \mapsto \dfrac{f_i(u)}{u}$ (prolonged by 0 at $u = 0$) is strictly increasing for $u \ge 0$. Furthermore, $\lim_{u\to\infty} \dfrac{f_i(u)}{u} = \infty$.

This problem arises from the investigation of a positive radially symmetric solution of the following elliptic eigenvalue problems:

$$\begin{aligned} -\triangle u + \sum_{i=1}^{k} \lambda_i f_i(u) = \mu u \ \text{ in } \ B = \{x \in R^N : |x| < 1\}, \\ u > 0 \ \text{ in } \ B, \\ u = 0 \ \text{ on } \ \partial B, \end{aligned}$$

In fact, it is known by Gidas, Ni and Nirenberg [7] that a positive solution of the above equation is radially symmetric.

The aim of this paper is to study the asymptotic properties of eigencurve $\mu(\lambda, \alpha)$ with emphasis on the phenomenon of bifurcation from μ_1 and on gaining a clearer picture of the bifurcation diagram. Here μ_1 is the first eigenvalue of $-\triangle$ with Dirichlet 0 boundary condition. To this end, we shall establish asymptotic formulas of eigenvalue $\mu = \mu(\lambda, \alpha) = \mu(\lambda_1, \lambda_1, \cdots, \lambda_k, \alpha)$ as $|\lambda| \to \infty, 0$. It is known by Berestycki [3] that for given $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)(\lambda_i \ge 0), \alpha > 0$, there uniquely exists an eigenvalue $\mu = \mu(\lambda, \alpha) > \mu_1$ associated with eigenfunction $u_\lambda(\alpha, x) > 0$ satisfying $\|u_\lambda\|_2 = \alpha$.

In order to motivate our problem, let us briefly recall some known facts concerning multiparameter eigenvalue problems. Multiparameter linear spectral theory began with the oscillation theory and there are many works. We refer to Binding [4] and Binding and Browne [5], for example. We also refer to Faierman [6] and the references cited therein for further information in this direction. However, few results have been given for nonlinear multiparameter problems.

As the first step to treat nonlinear multiparameter eigenvalue problems, we restrict our attention here to the equation (1.1) in the unit ball of R^N. As for the local properties of $\mu(\lambda, \alpha)$, we shall show that $\mu(\lambda, \alpha)$ is continuous in $\lambda \in R_+^k \backslash \{0\}$ $(R_+ := [0, \infty))$ and bifurcation from μ_1 occurs, that is, $|\mu(\lambda, \alpha) - \mu_1| \to 0$ as $|\lambda| \to 0$ (as expected) (Theorem 2.6). Furthermore, in order to understand the bifurcation diagram globally, we shall investigate the asymptotic behavior of $\mu(\lambda, \alpha)$ as $|\lambda| \to \infty$;

we begin with the simple case $k = 1$ and study the asymptotic behavior of u_λ and $\mu(\lambda, \alpha)$ as $|\lambda| \to \infty$ (Theorem 2.1, Theorem 2.2). By using these results, we shall establish an asymptotic formula of $\mu(\lambda, \alpha)$ as $|\lambda| \to \infty$ for the general case $k \geq 2$ (Theorem 2.3). In particular, the typical case $f_i(u) = u^{p_i}$ $(p_i > 1)$ is dealt with and more precise asymptotic formula of $\mu(\lambda, \alpha)$ as $|\lambda| \to \infty$ will be established (Theorem 2.4).

2 Main Results.

We explain notations before stating our results. Let

$$\|u\|_X^2 = \int_0^1 r^{N-1} u'(r)^2 dr, \|u\|_s^s = \int_0^1 r^{N-1} |u(r)|^s dr \;\; for \;\; s > 1, \tag{2.1}$$

$$\|u\|_\infty = \sup_{0 \leq r \leq 1} |u(r)|. \tag{2.2}$$

We fix $\alpha > 0$. For a given $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in R_+^k \setminus \{0\}$, let $(\mu(\lambda, \alpha), u_\lambda(r))$ be the unique solution of (1.1) with $\|u_\lambda\|_2 = \alpha$. Now we state our results.

Theorem 2.1. *Assume (A.1)-(A.2). Furthermore, assume that* $k = 1$. *Then* $u_\lambda(r) \to \sqrt{N}\alpha$ *and* $u'_\lambda(r) \to 0$ *uniformly on any compact subsets in* $[0, 1)$ *as* $\lambda_1 \to \infty$.

Theorem 2.2. *Assume (A.1)-(A.2). Furthermore, assume that* $k = 1$. *Then the following asymptotic formula holds as* $\lambda_1 \to \infty$*:*

$$\mu(\lambda, \alpha) = \frac{f_1(\sqrt{N}\alpha)}{\sqrt{N}\alpha}\lambda_1 + o(\lambda_1). \tag{2.3}$$

In order to consider the general case $k \geq 2$, we assume (A.3):

(A.3) Assume that there exist j $(1 \leq j \leq k)$ and constants $K_i \geq 0$ such that for $\lambda_j \gg 1$

$$0 \leq \frac{\tau_i}{\lambda_j} := \frac{\lambda_i}{\lambda_j} - K_i \to 0. \tag{2.4}$$

Theorem 2.3. *Assume (A.1)-(A.3). Then the following asymptotic formula holds as* $\lambda_j \to \infty$*:*

$$\mu(\lambda, \alpha) = \sum_{i=1}^k K_i \frac{f_i(\sqrt{N}\alpha)}{\sqrt{N}\alpha}\lambda_j + o(\lambda_j). \tag{2.5}$$

In the following special situation, more precise remainder estimate can be obtained:

Theorem 2.4. *Let* $f_i(u) = u^{p_i}(p_i > 1)$. *Furthermore, assume (A.3) with* $K_i = 0$ *for all* $i \neq j$. *Then there exist constants* $C_1, C_2 > 0$ *such that for* $\lambda_j \gg 1$*:*

$$N^{\frac{p-1}{2}}\alpha^{p-1}\lambda_j + C_1\alpha^{\frac{p-1}{2}}\lambda_j^{\frac{1}{2}} \leq \mu(\lambda, \alpha) \leq N^{\frac{p-1}{2}}\alpha^{p-1}\lambda_j + C_2\alpha^{\frac{p-1}{2}}\lambda_j^{\frac{1}{2}} + C_2\sum_{i \neq j}\lambda_i. \tag{2.6}$$

For the case $N = 1$ in Theorem 2.4, we can obtain more general result: let $\mu_n(\lambda, \alpha)(n \in N)$ denote the eigenvalue of (1.1) associated with eigenfunction $u_{\lambda,n}(r)$ with $n-1$ exact interior zeros satisfying $\|u_{\lambda,n}\|_2 = \alpha$. We know from Heinz [8] that for a given $\lambda_i \geq 0$ and $\alpha > 0$, there uniquely exists $\mu = \mu_n(\lambda, \alpha)$ for $n \in N$.

Corollary 2.5. *Let $N = 1$. Assume the conditions imposed in Theorem 2.4. Then for $n \in N$, the formula (2.6) holds for $\mu = \mu_n(\lambda, \alpha)$.*

Theorem 2.6. *Assume (A.1)-(A.2). Then $\mu(\lambda, \alpha)$ is continuous in $\lambda \in R^k_+ \setminus \{0\}$. Furthermore, the following asymptotic formula holds as $|\lambda| \to 0$:*

$$0 < \mu(\lambda, \alpha) - \mu_1 \leq C_3 \sum_{i=1}^{k} \lambda_i. \tag{2.7}$$

The remainder of this paper is organized as follows. In Section 3 we shall prove Theorem 2.1 and Theorem 2.2. Section 4 is devoted to the proof of Theorem 2.3. The proofs of Theorem 2.4 and Corollary 2.5 will be given in Section 5. Finally, we shall prove Theorem 2.6 in Section 6.

3 Proof of Theorem 2.1 and Theorem 2.2.

In what follows, we denote $\mu(\lambda) = \mu(\lambda, \alpha)$ for simplicity. At first, we shall recall some fundamental properties of u_λ. Let $\sigma_\lambda := \max_{0 \leq r \leq 1} u_\lambda(r)$. Since $u'_\lambda(r) \leq 0$ for $r \in [0, 1]$ by [7], $\sigma_\lambda = u_\lambda(0)$. Furthermore, let $g_\lambda(u) := \dfrac{\sum_{i=1}^{k} \lambda_i f_i(u)}{u}$. By (A.2), there exists $g_\lambda^{-1}(u)$ for $u \geq 0$. Then we know from Berestycki [3, Remarque 2.1] that

$$g_\lambda^{-1}(\mu(\lambda) - \mu_1)\phi(r) \leq u_\lambda(r) \leq g_\lambda^{-1}(\mu(\lambda)), \tag{3.1}$$

where ϕ is the positive first eigenfunction associated with μ_1 satisfying $\|\phi\|_\infty = 1$. In particular, we have by putting $r = 0$ in (3.1)

$$g_\lambda^{-1}(\mu(\lambda) - \mu_1) \leq \sigma_\lambda \leq g_\lambda^{-1}(\mu(\lambda)); \tag{3.2}$$

this is equivalent to

$$\mu(\lambda) - \mu_1 \leq \sum_{i=1}^{k} \lambda_i \frac{f_i(\sigma_\lambda)}{\sigma_\lambda} \leq \mu(\lambda). \tag{3.3}$$

Since $k = 1$ in this section, we denote $\lambda = \lambda_1, f(u) = f_1(u)$, and consider the following equation:

$$\begin{aligned} & u''(r) + \frac{N-1}{r} u'(r) + \mu u(r) - \lambda f(u(r)) = 0, \ \ 0 < r < 1, \\ & u(r) > 0, \ \ 0 \leq r < 1, \\ & u'(0) = 0, u(1) = 0. \end{aligned} \tag{3.4}$$

Lemma 3.1. *There exists a constant $C_4 > 0$ such that for $\lambda \gg 1$*

$$C_4^{-1}\lambda \le \mu(\lambda) \le C_4\lambda. \tag{3.5}$$

Proof. Let $s_1 = \|\phi\|_2$ and $\alpha_1 = \dfrac{\alpha}{s_1}$. Then we obtain by (3.1) that

$$g_\lambda^{-1}(\mu(\lambda) - \mu_1)s_1 \le \alpha \le \frac{g_\lambda^{-1}(\mu(\lambda))}{\sqrt{N}};$$

this implies that

$$\mu(\lambda) - \mu_1 \le g_\lambda(\alpha_1) = \lambda\frac{f(\alpha_1)}{\alpha_1}, \lambda\frac{f(\sqrt{N}\alpha)}{\sqrt{N}\alpha} \le \mu(\lambda);$$

this implies (3.5) for $\lambda \gg 1$. ∎

The following lemma is a direct consequence of (A.2), (3.3) and Lemma 3.1.

Lemma 3.2. *There exists a constant $C_5 > 0$ such that $C_5^{-1} \le \sigma_\lambda \le C_5$ for $\lambda \gg 1$.*

Let $F(t) := \int_0^t f(s)ds$ and $G_\lambda(t) := \dfrac{1}{2}\mu(\lambda)t^2 - \lambda F(t)$ for $t \ge 0$.

Lemma 3.3. *The following equality holds for $r \in [0,1]$.*

$$\begin{aligned} &\frac{1}{2}u_\lambda'(r)^2 + \int_0^r \frac{N-1}{s}u_\lambda'(s)^2 ds + G_\lambda(u_\lambda(r)) \\ &= G_\lambda(\sigma_\lambda) = \frac{1}{2}u_\lambda'(1)^2 + \int_0^1 \frac{N-1}{s}u_\lambda'(s)^2 ds. \end{aligned} \tag{3.7}$$

Proof. Multiply (1.1) by u_λ' to obtain

$$u_\lambda''(r)u_\lambda'(r) + \frac{N-1}{r}u_\lambda'(r)^2 + \mu(\lambda)u_\lambda(r)u_\lambda'(r) - \lambda f(u_\lambda(r))u_\lambda'(r) = 0;$$

this implies that

$$\frac{d}{dr}\left\{\frac{1}{2}u_\lambda'(r)^2 + \int_0^r \frac{N-1}{s}u_\lambda'(s)^2 ds + G_\lambda(u_\lambda(r))\right\} = 0.$$

Hence, for $r \in [0,1]$, by putting $r = 0$ and $r = 1$ we obtain

$$\begin{aligned} &\frac{1}{2}u_\lambda'(r)^2 + \int_0^r \frac{N-1}{s}u_\lambda'(s)^2 ds + G_\lambda(u_\lambda(r)) \equiv \textit{ constant } = G_\lambda(\sigma_\lambda) \\ &= \frac{1}{2}u_\lambda'(1)^2 + \int_0^1 \frac{N-1}{s}u_\lambda'(s)^2 ds. \end{aligned} \tag{3.8}$$

Thus the proof is complete. ∎

Lemma 3.4. *$G_\lambda(t)$ is increasing for $0 \le t \le \sigma_\lambda$.*

Proof. Since $G_\lambda'(t) = \mu(\lambda)t - \lambda f(t)$, we obtain by (A.2) that there uniquely exists $t_\lambda > 0$ such that $G_\lambda'(t) > 0$ for $0 < t < t_\lambda$ and $G_\lambda'(t) < 0$ for $t_\lambda < t$. Then since $G_\lambda'(\sigma_\lambda) \ge 0$ by (3.3), we find that $\sigma_\lambda \le t_\lambda$. Hence, we obtain our conclusion. ∎

Lemma 3.5. *Let $J := [r_0, r_1]$ $(0 < r_0 < r_1 < 1)$ be an arbitrary compact interval. Then there exists a constant $C_J > 0$ such that $|u_\lambda'(r)| \le C_J$ for $r \in J$ and $\lambda \gg 1$.*

Proof. We know from (3.4) that for $r \in (0,1)$

$$(r^{N-1}u_\lambda'(r))' = r^{N-1}(\lambda f(u_\lambda(r)) - \mu(\lambda)u_\lambda(r)) \le 0; \tag{3.9}$$

this implies that for $r_0 \le r \le r_1$

$$\left(\frac{r_0}{r}\right)^{N-1} |u_\lambda'(r_0)| \le |u_\lambda'(r)| \le \left(\frac{r_1}{r}\right)^{N-1} |u_\lambda'(r_1)|. \tag{3.10}$$

We fix $r_2 > 0$ such that $r_1 < r_2 < 1$. Then by the same argument just above, we obtain for $r_1 \le r \le r_2$

$$\left(\frac{r_1}{r_2}\right)^{N-1} |u_\lambda'(r_1)| \le |u_\lambda'(r)| \le \left(\frac{r_2}{r_1}\right)^{N-1} |u_\lambda'(r_2)|. \tag{3.11}$$

If $|u_\lambda'(r_1)| \to \infty$ as $\lambda \to \infty$, then by (3.11) and Lemma 3.2 we obtain

$$\begin{aligned} 2C_5 \ge 2\sigma_\lambda \ge u_\lambda(r_1) - u_\lambda(r_2) &= \int_{r_1}^{r_2} |u_\lambda'(r)|dr \\ &\ge (r_2 - r_1)\left(\frac{r_1}{r_2}\right)^{N-1} |u_\lambda'(r_1)| \to \infty. \end{aligned} \tag{3.12}$$

This is a contradiction. Hence, $|u_\lambda'(r_1)|$ is bounded for $\lambda \gg 1$. Now our assertion follows from (3.10). ∎

Lemma 3.6. *Let $[0, r_0] \subset [0, 1)$ be an arbitrary compact interval. Then $|u_\lambda(r) - \sigma_\lambda| \to 0$ as $\lambda \to \infty$ uniformly for $r \in [0, r_0]$.*

Proof. Assume that there exist $r_0 \in (0,1), 0 < \delta < 1$ and a subsequence $(\lambda_q)_{q \in N}$ such that $\lambda_q \to \infty$ as $q \to \infty$ and

$$u_0 := u_{\lambda_q}(r_0) \le \sigma_{\lambda_q}(1 - \delta) \tag{3.13}$$

and derive a contradiction. We denote $\lambda = \lambda_q$ for simplicity. We define z_λ by $u_\lambda(r) = \sigma_\lambda(1 - z_\lambda(r))$. Then by (3.13) we have $\delta \le z_\lambda(r_0)$. Furthermore, by (3.1) and (3.3)

$$\begin{aligned} \lambda\frac{f(\sigma_\lambda)}{\sigma_\lambda} - \mu_1 \le \mu(\lambda) - \mu_1 &\le g_\lambda\left(\frac{u_\lambda(r_0)}{\phi(r_0)}\right) \\ &= \lambda f\left(\frac{\sigma_\lambda(1 - z_\lambda(r_0))}{\phi(r_0)}\right) \Big/ \left(\frac{\sigma_\lambda(1 - z_\lambda(r_0))}{\phi(r_0)}\right). \end{aligned} \tag{3.14}$$

Therefore, by (A.2) we obtain that $1 - z_\lambda(r_0) \ge \phi(r_0)(1-\delta)$, that is, $z_\lambda(r_0) \le 1 - \phi(r_0)(1-\delta)$. Hence, by choosing a subsequence if necessary, we may assume that $z_\lambda(r_0) \to z_0$ as $\lambda \to \infty$, where $\delta \le z_0 \le 1 - \phi(r_0)(1-\delta)$. Hence, for fixed $0 < \epsilon \ll 1$ we have $1 - z_0 - \epsilon \le 1 - z_\lambda(r_0) \le 1 - z_0 + \epsilon$ for $\lambda \gg 1$. Let $r_{1,\lambda}$ satisfy $u_1 := u_\lambda(r_{1,\lambda}) = \sigma_\lambda(1 - z_0 + 2\epsilon)$. We shall show that $r_{1,\lambda} \to 0$ as $\lambda \to \infty$. We obtain by mean value theorem

$$\begin{aligned} G(u_\lambda(r_1)) - G(u_\lambda(r_0)) &= G'(u_0 + \theta(u_1 - u_0))(u_1 - u_0) \\ &= G'(u_0 + \theta(u_1 - u_0))\sigma_\lambda(1 - z_0 + 2\epsilon - 1 + z_\lambda(r_0)) \\ &\ge G'(u_0 + \theta(u_1 - u_0))\sigma_\lambda(2\epsilon - |z_0 - z_\lambda(r_0)|) \\ &\ge G'(u_0 + \theta(u_1 - u_0))\sigma_\lambda\epsilon, \end{aligned} \tag{3.15}$$

where $0 \le \theta \le 1$. Since $G'(t) = \mu(\lambda)t - \lambda f(t)$, we see from Lemma 3.4 that there uniquely exists $t_\lambda \ge \sigma_\lambda > 0$ such that $G'(t_\lambda) = 0$ and $G'(t) > 0$ for $0 < t < t_\lambda$ and $G'(t) < 0$ for $t > t_\lambda$. Furthermore, by (A.2) and Lemma 3.1, we see that there exists constant $C_6, C_7 > 0$ such that $C_6 \le t_\lambda \le C_7$. Let $0 < \eta \ll 1$ be fixed. Furthermore, let $t_\eta \in [\eta, t_\lambda - \eta]$ satisfy $G'(t_\eta) = \min_{\eta \le t \le t_\lambda - \eta} G'(t)$. Since $\mu(\lambda)/\lambda = f(t_\lambda)/t_\lambda$, we obtain by (A.2) and Lemma 3.1 that

$$\begin{aligned} G'(t_\eta) &= \lambda t_\eta \left(\frac{\mu(\lambda)}{\lambda} - \frac{f(t_\eta)}{t_\eta} \right) \ge \lambda\eta \left(\frac{\mu(\lambda)}{\lambda} - \frac{f(t_\eta - \eta)}{t_\eta - \eta} \right) \\ &\ge \lambda\eta \left(\frac{f(t_\lambda)}{t_\lambda} - \frac{f(t_\lambda - \eta)}{t_\lambda - \eta} \right) \ge \lambda\eta \min_{C_6 \le t \le C_7} \left(\frac{f(t)}{t} - \frac{f(t-\eta)}{t-\eta} \right) \ge C_8\eta\lambda. \end{aligned} \tag{3.16}$$

By definition of u_0 and u_1 and by using (3.1), we can choose $0 < \eta \ll 1$ such that $u_0 + \theta(u_1 - u_0) \in [\eta, t_\lambda - \eta]$. We obtain by (3.15) and (3.16) that

$$G(u_1) - G(u_0) \ge C_5^{-1} C_8 \epsilon\eta\lambda \longrightarrow \infty. \tag{3.17}$$

On the other hand, if there exists a compact interval $J \subset (0,1)$ such that $[r_{1,\lambda}, r_0] \subset J$ for $\lambda \gg 1$, then we have by (3.7) and Lemma 3.5 that

$$G(u_1) - G(u_0) = \frac{1}{2}u_\lambda'(r_0)^2 + \int_{r_{1,\lambda}}^{r_0} \frac{N-1}{s} u_\lambda'(s)^2 ds - \frac{1}{2}u_\lambda'(r_{1,\lambda})^2 \le C_J \tag{3.18}$$

This contradicts (3.17). Hence, $r_{1,\lambda} \to 0$ as $\lambda \to \infty$.

Finally, let $r_\epsilon \in (0,1)$ satisfy $\phi(r_\epsilon) = (1+\epsilon)^{-1}$. Since $r_{1,\lambda} < r_\epsilon$ for $\lambda \gg 1$, we obtain by (3.1) and (A.2) that

$$\lambda \frac{f(\sigma_\lambda)}{\sigma_\lambda} - \mu_1 \le g_\lambda \left(\frac{u_\lambda(r_\epsilon)}{\phi(r_\epsilon)} \right) \le g_\lambda \left(\frac{u_\lambda(r_{1,\lambda})}{\phi(r_\epsilon)} \right) = \lambda \frac{f((1+\epsilon)\sigma_\lambda(1 - z_0 + 2\epsilon))}{(1+\epsilon)\sigma_\lambda(1 - z_0 + 2\epsilon)}. \tag{3.19}$$

However, we find from (A.2) that this is impossible, since we obtain by Lemma 3.1 and Lemma 3.2 that for $0 < \epsilon \ll 1$

$$\begin{aligned} &\sigma_\lambda - \sigma_\lambda(1+\epsilon)(1 - z_0 + 2\epsilon) = \sigma_\lambda(z_0 - 3\epsilon + \epsilon z_0 - 2\epsilon^2) \\ &\quad \ge C_5^{-1}(z_0 - 3\epsilon + \epsilon z_0 - 2\epsilon^2) \ge C_9 z_0 \ge C_9\delta. \end{aligned} \tag{3.20}$$

Hence, (3.13) is impossible and we obtain that for $r \in [0, r_0]$, as $\lambda \to \infty$

$$|u_\lambda(r) - \sigma_\lambda| \le |u_\lambda(r_0) - \sigma_\lambda| \to 0. \tag{3.21}$$

Thus the proof is complete. ■

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $K \subset [0, 1)$ be an arbitrary compact set. Let $0 < \delta \ll 1$ satisfying $K \subset J = [0, 1 - \delta]$. Then by Lemma 3.6 $|u_\lambda(r) - \sigma_\lambda| \le \delta$ for $\lambda \gg 1$ and $r \in J$. Then

$$\alpha^2 = \|u_\lambda\|_2^2 = \int_0^{1-\delta} r^{N-1}\sigma_\lambda^2 dr + \int_0^{1-\delta} r^{N-1}(u_\lambda(r)^2 - \sigma_\lambda^2)dr + \int_{1-\delta}^1 r^{N-1}u_\lambda(r)^2 dr; \tag{3.22}$$

this along with Lemma 3.6 implies that for $\lambda \gg 1$

$$|\alpha^2 - \frac{1}{N}\sigma_\lambda^2| \le C_{10}\delta. \tag{3.23}$$

Then for $r \in K$ and $\lambda \gg 1$, we obtain by Lemma 3.6 and (3.23) that

$$|N\alpha^2 - u_\lambda(r)^2| \le C_{11}\delta. \tag{3.24}$$

Furthermore, by (3.7) we obtain

$$\begin{aligned} r^{N-1}u_\lambda'(r)^2 &\le 2r^{N-1}(G_\lambda(\sigma_\lambda) - G_\lambda(u_\lambda(r))) \\ &= r^{N-1}\left\{(\sigma_\lambda^2 - u_\lambda(r)^2) - 2\lambda(F(\sigma_\lambda) - F(u_\lambda))\right\} \\ &\le r^{N-1}(\sigma_\lambda^2 - u_\lambda(r)^2). \end{aligned} \tag{3.25}$$

Now, Theorem 2.1 follows from Lemma 3.6, (3.24) and (3.25). ■

Proof of Theorem 2.2. Since $\sigma_\lambda \to \sqrt{N}\alpha$ as $\lambda \to \infty$ by Theorem 2.1, we obtain Theorem 2.2 by (3.3). ■

4 Proof of Theorem 2.3.

In this section, we shall prove Theorem 2.3 by using Theorem 2.1. We use the same notations as those of Section 3. Without loss of generality, we may assume that (A.3) holds for $j = 1$. Since $\lambda_i = K_i\lambda_1 + \tau_i$, we obtain by (3.4) that

$$\begin{aligned} &u''(r) + \frac{N-1}{r}u'(r) + \mu u(r) - \lambda_1\left(\sum_{i=1}^k K_i f_i(u(r)) + \sum_{i=1}^k \frac{\tau_i}{\lambda_1} f_i(u(r))\right) \\ &\quad = 0, \quad 0 < r < 1, \\ &u(r) > 0, \quad 0 \le r < 1, \\ &u'(0) = 0, u(1) = 0. \end{aligned} \tag{4.1}$$

Lemma 4.1. *There exists a constant* $C_{12} > 0$ *such that*

$$C_{12}^{-1}\lambda_1 \le \mu(\lambda) \le C_{12}\lambda_1. \tag{4.2}$$

Proof. We know from (3.2) that for

$$g_\lambda(u) = \lambda_1 \left(\sum_{i=1}^{k} K_i \frac{f_i(u)}{u} + \sum_{i=1}^{k} \frac{\tau_i}{\lambda_1} \frac{f_i(u)}{u} \right)$$

$$g_\lambda^{-1}(\mu(\lambda) - \mu_1)\phi_1 \le u_\lambda \le g_\lambda^{-1}(\mu(\lambda)). \tag{4.3}$$

Then we obtain by (4.3) that

$$g_\lambda^{-1}(\mu(\lambda) - \mu_1)s_1 \le \alpha \le \frac{g_\lambda^{-1}(\mu(\lambda))}{\sqrt{N}}, \tag{4.4}$$

where $s_1 = \|\phi_1\|_2$. Then for $\alpha_1 = \alpha/s_1$

$$\begin{aligned} \mu(\lambda) - \mu_1 &\le \lambda_1 \left(\sum_{i=1}^{k} K_i \frac{f_i(\alpha_1)}{\alpha_1} + \sum_{i=1}^{k} \frac{\tau_i}{\lambda_1} \frac{f_i(\alpha_1)}{\alpha_1} \right), \\ \mu(\lambda) &\ge \lambda_1 \left(\sum_{i=1}^{k} K_i \frac{f_i(\sqrt{N}\alpha)}{\sqrt{N}\alpha} + \sum_{i=1}^{k} \frac{\tau_i}{\lambda_1} \frac{f_i(\sqrt{N}\alpha)}{\sqrt{N}\alpha} \right). \end{aligned} \tag{4.5}$$

By using (A.3) and (4.5), we obtain (4.2). ■

Proof of Theorem 2.3. satisfying (A.3), by repeating the same arguments as those in Section 3, we can show that the properties of Lemma 3.2 and Lemma 3.6 also hold in our situation. Consequently, by (3.3) we obtain

$$\mu(\lambda) - \mu_1 \le \lambda_1 \left(\sum_{i=1}^{k} K_i \frac{f_i(\sigma_\lambda)}{\sigma_\lambda} + \sum_{i=1}^{k} \frac{\tau_i}{\lambda_1} \frac{f_i(\sigma_\lambda)}{\sigma_\lambda} \right) \le \mu(\lambda). \tag{4.6}$$

Since $\sigma_\lambda \to \sqrt{N}\alpha$ as $\lambda_1 \to \infty$, we obtain our conclusion by using (A.3). Thus the proof is complete. ■

5 Proof of Theorem 2.4 and Corollary 2.5.

We begin with the definition of subsolution and supersolution. For the equation

$$\begin{aligned} -\triangle u = h(u) \quad &\text{in } B, \\ u = 0 \quad &\text{on } \partial B \end{aligned} \tag{5.1}$$

$\tilde{u}$ is called subsolution of (5.1) if $\tilde{u}$ satisfies

$$\begin{aligned} -\triangle \tilde{u} \le h(\tilde{u}) \quad &\text{in } B, \\ \tilde{u} \le 0 \quad &\text{on } \partial B. \end{aligned} \tag{5.2}$$

Furthermore, $\bar{u}$ is called supersolution of (5.1) if $\bar{u}$ satisfies

$$\begin{aligned} -\triangle \bar{u} &\geq h(\bar{u}) \quad \text{in } B, \\ \bar{u} &\geq 0 \quad \text{on } \partial B. \end{aligned} \tag{5.3}$$

We know from Amann [1] that if $\tilde{u} \leq \bar{u}$ in B, then there exists a solution u of (5.1) such that $\tilde{u} \leq u \leq \bar{u}$ in B.

In this section, we may assume without loss of generality that (A.3) holds for $j = 1$. We put $p = p_1, v_\lambda = \lambda_1^{\frac{1}{p-1}} u_\lambda$. Then it follows from (3.4) that v_λ satisfies

$$\begin{aligned} &v_\lambda''(r) + \frac{N-1}{r} v_\lambda'(r) + \mu(\lambda) v_\lambda(r) - H_\lambda(v_\lambda(r)) = 0, \ \ 0 < r < 1, \\ &v_\lambda(r) > 0, \ \ 0 \leq r < 1, \\ &v_\lambda'(0) = 0, v_\lambda(1) = 0, \end{aligned} \tag{5.4}$$

where $H_\lambda(v) := v^p + \sum_{i=2}^k \lambda_i \lambda_1^{-\frac{p_i - 1}{p-1}} v^{p_i}$.

Lemma 5.1. *Let* $\tau_\lambda = \sum_{i=2}^k \lambda_i C_{12}^{\frac{p_i-1}{p-1}}$. *Then* $\varphi_\lambda(r) = (\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \phi$ *is a subsolution of (5.4) for* $\lambda_1 \gg 1$.

Proof. We see from Lemma 4.1 that $\mu(\lambda) - \mu_1 - \tau_\lambda > 0$ for $\lambda_1 \gg 1$. Since $\phi(r)^{p_i} \leq \phi(r)$, we obtain

$$\begin{aligned} &\varphi_\lambda''(r) + \frac{N-1}{r} \varphi_\lambda'(r) + \mu(\lambda) \varphi_\lambda(r) - H_\lambda(\varphi_\lambda(r)) \\ &\geq -(\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \mu_1 \phi + \mu(\lambda)(\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \phi \\ &- \left((\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{p}{p-1}} \phi + \sum_{i=2}^k \lambda_i \lambda_1^{-\frac{p_i-1}{p-1}} (\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{p_i}{p-1}} \phi \right) \\ &= (\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \left(\tau_\lambda - \sum_{i=2}^k \lambda_i \left(\frac{\mu(\lambda) - \mu_1 - \tau_\lambda}{\lambda_1} \right)^{\frac{p_i-1}{p-1}} \right) \phi \\ &\geq (\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \left(\tau_\lambda - \sum_{i=2}^k \lambda_i \left(\frac{\mu(\lambda)}{\lambda_1} \right)^{\frac{p_i-1}{p-1}} \right) \phi \\ &\geq (\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \left(\tau_\lambda - \sum_{i=2}^k \lambda_i C_{12}^{\frac{p_i-1}{p-1}} \right) \phi = 0. \end{aligned} \tag{5.5}$$

Thus the proof is complete. ∎

The following lemma is obvious:

Lemma 5.2. $\Phi_\lambda(r) := \mu(\lambda)^{\frac{1}{p-1}}$ *is a supersolution of (5.4).*

Since $\varphi_\lambda(r) \le \Phi_\lambda(r)$ and $v_\lambda(r)$ is the unique solution of (5.4), we obtain by Lemma 5.1 and Lemma 5.2 that for $0 \le r \le 1$

$$\varphi_\lambda(r) \le v_\lambda(r) \le \Phi_\lambda(r). \tag{5.6}$$

Especially, by putting $r = 0$ in (5.6), we obtain

$$(\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \le \sigma_\lambda \le \mu(\lambda)^{\frac{1}{p-1}}. \tag{5.7}$$

Lemma 5.3. *v_λ is a supersolution of*

$$\begin{aligned} &w_\lambda''(r) + \frac{N-1}{r} w_\lambda'(r) + (\mu(\lambda) - \tau_\lambda) w_\lambda(r) - w_\lambda(r)^p = 0, \ \ 0 < r < 1, \\ &w_\lambda(r) > 0, \ \ 0 \le r < 1, \\ &w_\lambda'(0) = 0, w_\lambda(1) = 0, \end{aligned} \tag{5.8}$$

Proof. By (5.4), (5.7) and Lemma 4.1 we obtain

$$\begin{aligned} &v_\lambda''(r) + \frac{N-1}{r} v_\lambda'(r) + (\mu(\lambda) - \tau_\lambda) v_\lambda(r) - v_\lambda(r)^p \\ &\quad = \left(\sum_{i=2}^k \lambda_i \lambda_1^{-\frac{p_i-1}{p-1}} v_\lambda(r)^{p_i-1} - \sum_{i=2}^k \lambda_i C_{12}^{\frac{p_i-1}{p-1}} \right) v_\lambda(r) \\ &\quad \le \left(\sum_{i=2}^k \lambda_i \lambda_1^{-\frac{p_i-1}{p-1}} \mu(\lambda)^{\frac{p_i-1}{p-1}} - \sum_{i=2}^k \lambda_i C_{12}^{\frac{p_i-1}{p-1}} \right) v_\lambda(r) \le 0. \end{aligned} \tag{5.9}$$

Thus the proof is complete. ■

Lemma 5.4. *$\varphi_\lambda(r)$ is a subsolution of (5.8).*

Proof.

$$\begin{aligned} &\varphi_\lambda''(r) + \frac{N-1}{r} \varphi_\lambda'(r) + (\mu(\lambda) - \tau_\lambda) \varphi_\lambda(r) - \varphi_\lambda(r)^p \\ &\quad \ge -(\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \mu_1 \phi + (\mu(\lambda) - \tau_\lambda)(\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{1}{p-1}} \phi \\ &\quad - (\mu(\lambda) - \mu_1 - \tau_\lambda)^{\frac{p}{p-1}} \phi = 0. \end{aligned} \tag{5.10}$$

Thus the proof is complete. ■

Let w_λ be the unique positive solution of (5.8). Then we derive from (5.6) and Lemma 5.4 that

$$\varphi_\lambda(r) \le w_\lambda(r) \le v_\lambda(r). \tag{5.11}$$

We note here that by Lemma 4.1 and (A.3), $\mu(\lambda) - \tau_\lambda \to \infty$ as $\lambda_1 \to \infty$.

Lemma 5.5. *9, Theorem Let $\eta > \mu_1$. Furthermore, let z_η be the unique positive solution of the following equation:*

$$\begin{aligned} &z''(r) + \frac{N-1}{r}z'(r) + \eta z(r) - z(r)^p = 0, \quad 0 < r < 1, \\ &z(r) > 0, \quad 0 \le r < 1, \\ &z'(0) = 0, z(1) = 0, \end{aligned} \tag{5.12}$$

Then there exist constants $C_{13}, C_{14} > 0$ such that for $\eta \gg 1$

$$\|z_\eta\|_2^{p-1} + C_{13}\|z_\eta\|_2^{\frac{p-1}{2}} \le \eta \le \|z_\eta\|_2^{p-1} + C_{14}\|z_\eta\|_2^{\frac{p-1}{2}}. \tag{5.13}$$

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Since $\|v_\lambda\|_2 = \lambda_1^{\frac{1}{p-1}}\alpha$, we obtain by (5.11) and (5.13) that for $\lambda_1 \gg 1$

$$\begin{aligned} \mu(\lambda) - \tau_\lambda &\le \|w_\lambda\|_2^{p-1} + C_{14}\|w_\lambda\|_2^{\frac{p-1}{2}} \le \|v_\lambda\|_2^{p-1} + C_{14}\|v_\lambda\|_2^{\frac{p-1}{2}} \\ &\le \lambda_1\alpha^{p-1} + C_{14}\lambda_1^{\frac{1}{2}}\alpha^{\frac{p-1}{2}}; \end{aligned}$$

this implies that

$$\mu(\lambda) \le \lambda_1\alpha^{p-1} + C_{14}\lambda_1^{\frac{1}{2}}\alpha^{\frac{p-1}{2}} + \sum_{i=2}^{k}\lambda_i C_{12}^{\frac{p_i-1}{p-1}}. \tag{5.15}$$

Next, let y_λ be the unique positive solution of

$$\begin{aligned} &y''(r) + \frac{N-1}{r}y'(r) + \mu(\lambda)y(r) - y(r)^p = 0, \quad 0 < r < 1, \\ &y(r) > 0, \quad 0 \le r < 1, \\ &y'(0) = 0, y(1) = 0. \end{aligned} \tag{5.16}$$

Then by (5.4), it is clear that v_λ is a subsolution of (5.16). Furthermore, $\Phi_\lambda(r) = \mu(\lambda)^{\frac{1}{p-1}}$ is a supersolution of (5.16). Then by (5.6) we see that

$$v_\lambda(r) \le y_\lambda(r) \le \mu(\lambda)^{\frac{1}{p-1}}. \tag{5.17}$$

Then by Lemma 5.5 and (5.17) we obtain

$$\begin{aligned} \mu(\lambda) &\ge \|y_\lambda\|_2^{p-1} + C_{13}\|y_\lambda\|_2^{\frac{p-1}{2}} \ge \|v_\lambda\|_2^{p-1} + C_{13}\|v_\lambda\|_2^{\frac{p-1}{2}} \\ &\ge \lambda_1\alpha^{p-1} + C_{13}\lambda_1^{\frac{1}{2}}\alpha^{\frac{p-1}{2}}. \end{aligned} \tag{5.18}$$

Now Theorem 2.4 follows from (5.15) and (5.18). Thus the proof is complete. ∎

In order to prove Corollary 2.5, we apply the following Lemma 5.6 instead of Lemma 5.5:

Lemma 5.6. *8, Theorem Let $\eta > (n\pi)^2$. Furthermore, let $w_{n,\eta}$ be the unique solution of*

$$\begin{aligned} -w''(r) + |w(r)|^{p-1}w(r) = \eta w(r),\ \ 0 < r < 1, \\ w(r) > 0,\ \ 0 < r < \frac{1}{n}, \\ w\left(\frac{j}{n}\right) = 0 (j = 0, 1, \cdots, n). \end{aligned} \tag{5.19}$$

Then for $\eta \gg 1$, (5.13) holds.

By using Lemma 5.6 instead of Lemma 5.5, we can prove Corollary 2.5 by the same arguments as those used in the proof of Theorem 2.4. ■

6 Proof of Theorem 2.6.

In order to prove Theorem 2.6, we apply the following lemma:

Lemma 6.1. *2, Theorem 2 Let $u_0 \in C^2(\bar{B})$ be any function on B such that $u_0 > 0$ almost everywhere in $B, u_0 = 0$ on ∂B and $\|u_0\|_2 = \alpha$. Then*

$$\mu(\lambda) \le \sup_{x \in \bar{B}} \left(\frac{-\triangle u_0(x) + \sum_{i=1}^k \lambda_i f_i(u_0(x))}{u_0(x)} \right). \tag{6.1}$$

Proof of Theorem 2.6. At first, we shall prove the continuity of $\mu(\lambda)$. We fix an arbitrary $\lambda_0 = (\lambda_{0,1}, \lambda_{0,2}, \cdots, \lambda_{0,k}) \in R_+^k \setminus \{0\}$. We may assume without loss of generality that $\lambda_{0,1} > 0$. We fix u_0 which satisfies the conditions imposed in Lemma 6.1. Then for $|\lambda - \lambda_0| \le \delta \ll 1$

$$\mu(\lambda) \le \sup_{x \in \bar{B}} \left(\frac{-\triangle u_0(x)}{u_0(x)} \right) + \sum_{i=1}^k \lambda_i \sup_{x \in \bar{B}} \left(\frac{f_i(u_0(x))}{u_0(x)} \right) \le C_{15} + C_{15} \sum_{i=1}^k \lambda_i \le C_{16}. \tag{6.2}$$

We derive from (3.3) and (6.2) that for $|\lambda - \lambda_0| \le \delta \ll 1$

$$(\lambda_{0,1} - \delta) \frac{f_1(\sigma_\lambda)}{\sigma_\lambda} \le \sum_{i=1}^k \lambda_i \frac{f_i(\sigma_\lambda)}{\sigma_\lambda} \le \mu(\lambda) \le C_{16};$$

this implies that $\sigma_\lambda \le C_{17}$ for $|\lambda - \lambda_0| \le \delta \ll 1$. Multiply (1.1) by r^{N-1} we have

$$(r^{N-1} u_\lambda'(r))' + r^{N-1}(\mu(\lambda) u_\lambda(r) - \sum_{i=1}^k \lambda_i f_i(u_\lambda(r))) = 0; \tag{6.3}$$

this implies that for $0 \le r < 1$

$$r^{N-1}u'_\lambda(r) = \int_0^r s^{N-1}(\sum_{i=1}^k \lambda_i f_i(u_\lambda(s)) - \mu(\lambda)u_\lambda(s))ds. \tag{6.4}$$

Then by (6.2) and (6.4)

$$|u'_\lambda(r)| \le \frac{1}{r^{N-1}}\int_0^r s^{N-1}|\sum_{i=1}^k \lambda_i f_i(u_\lambda(s)) - \mu(\lambda)u_\lambda(s)|ds \le C_{18}r. \tag{6.5}$$

Furthermore, by (2.1), (6.2) and (6.5) we obtain that for $0 \le r < 1$

$$|u''_\lambda(r)| \le \frac{(N-1)|u'_\lambda(r)|}{r} + \mu(\lambda)\sigma_\lambda + \sum_{i=1}^k \lambda_i f_i(\sigma_\lambda) \le C_{19}. \tag{6.6}$$

Therefore, we find from (6.5) and (6.6) that we can apply Ascoli-Arzela's theorem, and we can choose a subsequence of (λ), which we write (λ) again, such that as $\lambda \to \lambda_0$

$$u_\lambda \longrightarrow u_1, u'_\lambda \longrightarrow u'_1 \tag{6.7}$$

uniformly on any compact subsets in $[0,1]$. Furthermore, by (6.2), we can choose a subsequence of $(\mu(\lambda))$, which we write $(\mu(\lambda))$ again, such that $\mu(\lambda) \to \mu_0$ as $\lambda \to \lambda_0$. Then we easily see from (1.1), (6.7) that (u_1, λ_0, μ_0) is a weak solution of (1.1) and by a standard regularity argument, $u_1 \in C^2(B)$. Furthermore, by (6.7) we obtain $\|u_1\|_2 = \alpha$. Hence, by Berestycki [2, Théorème 4] we find that $\mu_0 = \mu(\lambda)$. Now our assertion follows from a standard compactness argument.

Finally, we shall prove (2.7). Let ϕ_1 be the first eigenfunction associated with μ_1 satisfying $\|\phi_1\|_2 = \alpha$. Then by Lemma 6.1

$$\mu(\lambda) \le \sup_{x\in\bar{B}}\left(\frac{-\triangle\phi_1(x)}{\phi_1(x)}\right) + \sum_{i=1}^k \lambda_i \sup_{x\in\bar{B}} \frac{f_i(\phi_1(x))}{\phi_1(x)} = \mu_1 + C_{20}\sum_{i=1}^k \lambda_i. \tag{6.8}$$

Thus the proof is complete. ∎

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