

Total Mean Curvature and Closed Geodesics

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The purpose of this note is to give a proof of the following theorem and to give some easy applications of it and its proof to the extrinsic geometry of convex Zoll surfaces.

In what follows we will use the word *surface* to mean *smooth closed surface*, and the words *strictly convex* to mean *of positive Gaussian curvature*.

Theorem 1 *On a strictly convex surface $\Sigma \subset \mathbb{R}^3$ there exists a closed geodesic whose length is less than or equal to one half the total mean curvature of Σ .*

The proofs and applications are based on a Riemannian version of Gromov's non-squeezing theorem and classical integral geometry.

Given a convex surface $\Sigma \subset \mathbb{R}^3$ and a point q in the unit sphere S^2 we denote by $U_\Sigma(q)$ the perimeter of the orthogonal projection of Σ onto a plane perpendicular to q . We obtain a function U_Σ on the sphere which is clearly continuous, even, and positive. Let us denote the minimum value for this function by u_Σ . The analogue of the non-squeezing theorem we wish to present is the following result.

Lemma 1 *Let $\Sigma \subset \mathbb{R}^3$ be a strictly convex surface. There exists on Σ a closed geodesic whose length is less than or equal to u_Σ .*

The theorem follows from this lemma and the following integral-geometric characterization of the total mean curvature in terms of the average of the perimeter function over the sphere.

Lemma 2 *Let $\Sigma \subset \mathbb{R}^3$ be a strictly convex surface and let $H := \frac{1}{2}(\kappa_1 + \kappa_2)$ be its mean curvature function. We have that*

$$\int_{\Sigma} H d\Sigma = \frac{1}{2\pi} \int_{S^2} U_{\Sigma} d\omega .$$

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For a proof of lemma 2 we refer the reader to pages 66 and 67 of the classic book of Bonnesen and Fenchel [3].

1 The Birkhoff Invariant and Proof of Lemma 1

We begin by stating a slight variant of Birkhoff's theorem on the existence of at least one closed geodesic on a surface diffeomorphic to a sphere.

Let (S^2, g) denote the sphere provided with an arbitrary Riemannian metric. If $f : S^2 \rightarrow \mathbb{R}$ is a smooth function with only two critical points, we define

$$\beta(f) := \max\{\text{length of } f^{-1}(c) : c \in \mathbb{R}\} .$$

We now define the *Birkhoff invariant*, β , of the metric g as the infimum over all such smooth functions of the numbers $\beta(f)$. Intuitively, β is the minimum length of a closed string which may be slipped over the surface (see [2]). The Birkhoff invariant is a Riemannian invariant of (S^2, g) resembling a symplectic capacity.

Theorem [Birkhoff] *There exists a closed geodesic on (S^2, g) with length β .*

Proof of lemma 1. If (S^2, g) is isometrically embedded in \mathbb{R}^3 as a strictly convex surface Σ we may compare β with $\beta(h_q)$, where h_q is the height function of the convex surface Σ in the direction of $q \in S^2$. From this comparison it follows immediately that $\beta \leq u_\Sigma$ and hence lemma 1.

Using the Birkhoff invariant theorem 1 can be slightly sharpened to the following result:

Theorem 1' *Let Σ be a strictly convex surface in \mathbb{R}^3 . The Birkhoff invariant of Σ is less than or equal to one half the total mean curvature of Σ .*

Proof. Using lemma 2 and the fact that $U_\Sigma(q) \geq u_\Sigma$ we have that

$$\int_\Sigma H d\Sigma = \frac{1}{2\pi} \int_{S^2} U_\Sigma d\omega \geq \frac{1}{2\pi} \int_{S^2} u_\Sigma d\omega = 2u_\Sigma .$$

The theorem follows from the inequality $\beta \leq u_\Sigma$.

Remark. During the fall of 1994 Gelfand posed the following question in his seminar:

Is it possible to localize symplectic invariants in the same way that the Gauss-Bonnet theorem localizes the Euler characteristic?

Theorem 1' arose from trying to understand and answer this question. Although this result is by no means a localization of the Birkhoff invariant, it seems to be a step in the right direction. Integral geometry can be used, much in the same way it has been done here, to give an upper bound for the capacity of a convex set in \mathbb{R}^{2n} in terms of local $U(n)$ -invariants of its boundary. This result will be published elsewhere.

2 Applications to Zoll Surfaces

All proofs and original sources for the unproved statements in this section may be found in Besse's book [1].

Definition A Zoll surface is a Riemannian metric on the 2-sphere whose geodesic flow is periodic with minimal period 2π .

Here are some of the things that are known about Zoll surfaces:

- There are infinitely many Zoll surfaces which are not isometric to the unit sphere (Zoll, Darboux).
- The area of a Zoll surface is equal to 4π (Weinstein).
- Given two Zoll surfaces there exists a diffeomorphism between the manifolds of unit covectors which preserves the canonical contact forms. In particular, the geodesic flows of any two Zoll surfaces are conjugate (Weinstein).

In this section we apply the previous results to the study of the *extrinsic* geometry of Zoll surfaces. It seems that these are the first steps in this direction.

To state our first theorem we introduce some notations:

Let us agree to call an embedding $e : S^2 \rightarrow \mathbb{R}^3$ a *Zoll embedding* if the induced metric is Zoll, and let us say that a convex subset of \mathbb{R}^3 is a *cylinder* if it is the union of a family of parallel lines. By the *perimeter* of a cylinder we shall mean the perimeter of the plane convex set obtained by intersecting the cylinder with a plane perpendicular to its lines.

Theorem 2 (Non-squeezing) *There does not exist any strictly convex Zoll embedding of S^2 into a cylinder of perimeter less than 2π .*

Proof. If $\Sigma \subset \mathbb{R}^3$ is a convex surface, then it cannot be embedded in a cylinder of perimeter less than u_Σ . Now, by Birkhoff's theorem, the β invariant of a convex Zoll surface is 2π and therefore $u_\Sigma \geq 2\pi$.

Theorem 3 *The total mean curvature of a strictly convex Zoll surface Σ is greater than or equal to 4π . Moreover, equality holds if and only if Σ is the unit sphere.*

Proof. For a convex Zoll surface Σ and for every point q in the unit sphere we have that $U_\Sigma(q) \geq u_\Sigma \geq 2\pi$. Using lemma 2 this implies that the total mean curvature is greater than or equal to 4π . It also implies that if the total mean curvature is 4π , then $U_\Sigma \equiv u_\Sigma = 2\pi$. We shall now show that this will hold if and only if all the orthogonal projections of Σ are discs of radius 1.

Let us denote by $a_\Sigma(q)$ the area of the orthogonal projection of Σ onto the plane perpendicular to q . Since $U_\Sigma(q) = 2\pi$, the isoperimetric inequality implies that $a_\Sigma(q) \leq \pi$ for all q . Now Cauchy's formula in integral geometry (see pages 47-50 in [3]) states that

$$A(\Sigma) := \int_\Sigma d\Sigma = \frac{1}{\pi} \int_{S^2} a_\Sigma d\omega .$$

Since by Weinstein's theorem the area of Σ equals 4π , we have that $a_\Sigma \equiv \pi$. The equality case in the isoperimetric inequality now implies that the projections of Σ must be discs of radius 1.

The proof is concluded by invoking Fujiwara's theorem [5] which states that if the projections of a convex surface are discs, then the surface is a sphere.

The well known integral-geometric characterization of the total mean curvature in terms of the average of the breadth function over the sphere allows us to prove the following result:

Theorem 4 *The extrinsic diameter of a strictly convex Zoll surface Σ is greater than or equal to 2. Moreover, equality holds if and only if Σ is the unit sphere.*

Lemma 3 *The extrinsic diameter of a strictly convex surface Σ is greater than or equal to $\frac{1}{2\pi}$ times its total mean curvature. Moreover, equality holds if and only if Σ is a surface of constant breadth.*

Proof. Let $B_\Sigma(q)$ denote the distance between the two planes tangent to Σ and orthogonal to $q \in S^2$. The function $q \mapsto B_\Sigma(q)$ is known as the *breadth function*, and the integral-geometric representation we mentioned is as follows:

$$\int_\Sigma H d\Sigma = \frac{1}{2} \int_{S^2} B_\Sigma d\omega .$$

Noting that the diameter of Σ equals the maximum value of B_Σ , we have that

$$\int_\Sigma H d\Sigma = \frac{1}{2} \int_{S^2} B_\Sigma d\omega \leq 2\pi \operatorname{diam}(\Sigma) .$$

The equality holds if and only if $B_\Sigma \equiv \operatorname{diam}(\Sigma)$.

Theorem 4 follows immediately from lemma 3 and theorem 3.

3 The Camel Problem for Zoll Surfaces

We close the paper by posing an interesting problem in the extrinsic geometry of Zoll surfaces which mimics a well known theorem in symplectic geometry (see [4]).

Let $W(r)$ be the "holed wall" defined by

$$W(r) := \{(x, y, z) \in \mathbb{R}^3 : y = 0, x^2 + z^2 > r\} .$$

If $r < 1$, does there exist a continuous family of Zoll embeddings of S^2 in $\mathbb{R}^3 \setminus W(r)$ joining unit round spheres on either side of the wall?

The non-squeezing theorem implies that the answer is no if we restrict to convex Zoll embeddings.

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