Cylindrical Tzitzeica Curves Implies Forced Harmonic Oscillators

Mircea Crâşmăreanu

Abstract

Elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition are obtained via the solution of the forced harmonic equation.

Mathematics Subject Classification: 53A04, 34A05.
Key words: Tzitzeica curve, cylindrical curve, forced harmonic oscillator.

Introduction

Gheorghe Ţiţeica (1873-1939), writting in French under the name of Georges Tzitzeica, was a student of Gaston Darboux and thus a member of the second generation of classical differential geometers, after Gauss and Riemann.

There are several *Tzitzeica* notions:
(1) surfaces ([1], [32])
(2) curves ([19])
(3) hypersurfaces ([7], [23])
(4) submanifolds ([6], [8], [13], [14], [21], [27])
(5) algebras ([20], [22])
(6) connections ([9], [10], [11], [16], [18])
(7) equations ([2], [4], [5], [12], [25], [26]).

Tzitzeica has introduced a class of surfaces, nowadays called *Tzitzeica surfaces*, in 1907 ([29]) and a class of curves, called *Tzitzeica curves*, in 1911 ([30]). The relation between these objects is the following: for a Tzitzeica surface with negative Gaussian curvature, the asimptotic lines are Tzitzeica curves. Since their appearence, these notions are a permanent subject of research, suitable for fruitful generalizations.

(Elliptic) cylindrical curves are studied by Gheorghe Vrânceanu in connection with Levi-Civita and Fenchel theorems from surfaces theory in [33].

We consider two types of cylindrical curves: an usual one, which we prefer to call *elliptic*, and another one, called *hyperbolic*, according to the type of circular functions(cosine and sine) used. So, our results are divided in two sections.

In this paper we are interested in Tzitzeica cylindrical curves, more precisely we ask in what conditions a cylindrical curve is a Tzitzeica one, namely the function $t \to \dfrac{\tau(t)}{d^2(t)}$ is constant, where $d(t)$ is the distance from origin to the osculating plane of curve. The Tzitzeica condition yields a third-order ODE which in our framework admits a direct integration. Therefore the final answer of main problem is given via a second order ODE which in the elliptic case is exactly the equation of a forced harmonic oscillator! In both cases, elliptic and hyperbolic, the solution depends of four real constants: one defining the Tzitzeica condition and other three obtained by integration.

1   Elliptic cylindrical curves

Let in $\mathbf{R}^3$ a curve C given in vectorial form $C : \bar{r} = \bar{r}(t)$. This curve is called *elliptic cylindrical* if has the expression

$$(1.1) \qquad \bar{r}(t) = (\cos t, \sin t, f(t))$$

for some $f \in C^\infty(\mathbf{R})$.

The torsion function is

$$\tau(t) = \frac{(\bar{r}', \bar{r}'', \bar{r}''')}{\|\bar{r}' \times \bar{r}''\|^2} = \frac{f' + f'''}{1 + f'^2 + f''^2}.$$

Then the distance from origin to the osculating plane is

$$d(t) = \frac{\pm(f + f'')}{\sqrt{1 + f'^2 + f''^2}}.$$

Let us suppose that the curve is Tzitzeica with the constant $K \neq 0$, because the curve is not contained in a plane

$$\frac{\tau(t)}{d^2(t)} = K = \frac{f'(t) + f'''(t)}{(f(t) + f''(t))^2}.$$

Integration gives

$$\frac{1}{f(t) + f''(t)} = -(Kt + C)$$

with C a real constant. The last relation reads

$$(1.2) \qquad f''(t) + f(t) = \frac{-1}{Kt + C}.$$

But the ODE (1.2) is exacly of forced harmonic oscillator type,

$$\ddot{x} + x = g(t)$$

solved by the formula ([17, p. 60-61])

$$x(t) = x(0)\cos t - \dot{x}(0)\sin t + \int_0^t g(u)\sin(u - t)\, du$$

Therefore we have

Proposition 1.1. *An elliptic cylindrical curve* (1.1) *is Tzitzeica if and only if*

$$f(t) = f(0)\cos t - \dot{f}(0)\sin t - \int_0^t \frac{\sin(u-t)}{Ku+C}\,du$$

with $f(0), \dot{f}(0), K \neq 0, C$ *real constants.*

2 Hyperbolic cylindrical curves

A space curve is called *hyperbolic cylindrical* if

(2.1) $$\bar{r}(t) = (\cosh t, \sinh t, f(t)).$$

Straightforward computation gives

$$\tau(t) = \frac{f' - f'''}{1 + (f'^2 + f''^2)(\cosh^2 t + \sinh^2 t) - 4f'f''\cosh t\sinh t}.$$

Also

$$d(t) = \frac{\pm(f - f'')}{\sqrt{1 + (f'^2 + f''^2)(\cosh^2 t + \sinh^2 t) - 4f'f''\cosh t\sinh t}}.$$

For a Tzitzeica hyperbolic curve

$$\frac{\tau(t)}{d^2(t)} = K = \frac{f' - f'''}{(f - f'')^2}$$

and integration gives

$$\frac{1}{f - f''} = -(Kt + C) \Leftrightarrow f'' - f = \frac{1}{Kt + C}.$$

Using the same formula (3) from [17, p. 60] and the identity

$$e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

it results

Proposition 2.1. *A hyperbolic curve* (2.1) *is Tzitzeica if and only if:*

$$f(t) = f(0)\cosh t + \dot{f}(0)\sinh t + \int_0^t \frac{\sinh(t+u)}{Ku+C}$$

with $f(0), \dot{f}(0), K \neq 0$ *and* C *real constants.*

References

[1] Arena, F., *Sulle superficie R de Tzitzeica e Demoulin*, Atti Soc. Peloritana Sci. Fis. Mat. Natur., 14 (1968), 117-120.

[2] Boldin, A. Yu., Safin, S. S., Sharipov, R. A., *On an old article of Tzitzeica and the inverse scattering method*, J. Math. Phys., 34 (1993), no. 12, 5801-5809.

[3] Boskoff, V., Horja, P., *The characterization of some special Barbilian spaces using the Tzitzeica construction*, Stud. Cerc. Mat., 46 (1994), no. 5, 503-514.

[4] Brezhnev, Yu. V., *Darboux transformation and some multi-phase solutions of the Dodd-Bullough-Tzitzeica equation:* $U_{xt} = e^U - e^{-2U}$, Phys. Lett. A, 211 (1996), no. 2, 94-100.

[5] Conte, R., Musette, M., Grundland, A. M., *Bäcklund transformation of partial differential equations from the Painlevé-Gambier classification II: Tzitzeica equation*, J. Math. Phys., 40 (1999), no. 4, 2092-2106.

[6] Gheorghiu, Gh. Th., *Les variétés nonholonomes Tzitzeica doublement réglées*, Rend. Sem. Mat. Messina, 9 (1964-1965), 27-50.

[7] Gheorghiu, Gh. Th., *Sur les courbures totales d'une hypersurface nonholonôme d'un* V_n *et sur les hypersurfaces nonholonômes de Tzitzeica d'un* E_n *centroaffin*, Atti Accad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur., 51 (1971), 3-24.

[8] Godeaux, L., *Quadriques de Tzitzeica et congruences de Goursat*, Acad. Roy. Belg. Bull. Cl. Sci., 52 (1966), 348-352.

[9] Ianuş, S., *On the generalized Tzitzeica connections*, Stud. Cerc. Mat., 33 (1981), 439-443.

[10] Imai, T., *Infinitesimal variations of generalized Tzitzeica connections*, Tensor, 32 (1978), no. 2, 173-176.

[11] Imai, T., *Connection of Tzitzeica on submanifolds of codimension* 2 *of a complex manifold*, Tensor, 34 (1980), no. 2, 131-137.

[12] Kaptsov, O. V., Shan'ko, Yu. V., *Multiparametric solutions of the Tzitzeica equations*, Differential Equations, 35 (1999), no. 12, 1683-1692.

[13] Marcus, F., *On a result of Tzitzeica and a new asymptotic transform of minimal projective surfaces*, Czechoslovak Math. J., 30 (1980), no. 2, 213-227.

[14] Marcus, F., *On projective spheres and Fubini and Tzitzeica- Wilczynski pseudo-spheres*, Proc. Amer. Math. Soc., 93 (1985), no. 3, 512-520.

[15] Miron, R., Papuc, D., *A local existence theorem for the immersed manifolds in a space with affine connection*, C. R. A. S. Paris, 260 (1965), 2695-2697.

[16] Nicolescu, L., Pripoae, G., *Connexions de Tzitzeica* ξ- *speciales et presque* ξ-*speciales*, Bull. Math. Soc. Sci. Math. Roumanie, 32 (1988), no. 1, 51-57.

[17] Perko, L., *Differential equations and dynamical systems*, Third Edition, Texts in Applied Math., no. 7, Springer, N. Y., 2001.

[18] Popescu, P., *Sur une généralisation de la connexion de Tzitzeica*, An. Univ. Timişoara, 20 (1982), no. 1-2, 41-48.

[19] Pripoae, G., *Sur les courbes de Tzitzeica*, Rev. Roumaine Math. Pures Appl., 29 (1984), no. 7, 589-591.

[20] Pripoae, G., *Algébres de Tzitzeica*, Bull. Math. Soc. Sci. Math. Roumanie, 29 (1985), no. 4, 365-370.

[21] Pripoae, G., *Sur les sous-variétés de Tzitzeica*, Bull. Math. Soc. Sci. Math. Roumanie, 30 (1986), no. 3, 257-263.

[22] Pripoae, G., *Sur les algébres de Tzitzeica*, Bull. Math. Soc. Sci. Math. Roumanie, 31 (1987), no. 1, 49-55.

[23] Putinar, M., *Sur les courbes et hypersurfaces de Tzitzeica*, Bull. Math. Soc. Sci. Math. Roumanie, 23 (1979), no. 4, 419-426.

[24] Putinar, M., *One generalisation of the Tzitzeica's invariant*, Bull. Math. Soc. Sci. Math. Roumanie, 25 (1981), no. 1, 75-76.

[25] Schief, W. K., *The Tzitzeica equation: a Bäcklund transformation interpreted as truncated Painlevé expansion*, J. Phys. A, 29 (1996), no. 16, 5153-5155.

[26] Schief, W. K., *Self-dual Einstein spaces via a permutability theorem for the Tzitzeica equation*, Phys. Lett. A, 223 (1996), no. 1-2, 55-62.

[27] Teleman. K., *Généralisation d'un théorèm de G. Tzitzeica*, Rev. Roumaine Math. Pures Appl., 21 (1976), no. 7, 941-946.

[28] Teleman, A. M., Teleman, K., *A combined Bäcklund-Tzitzeica theorem*, An. Univ. Bucureşti Mat., 48 (1999), no. 2, 197-202.

[29] Tzitzeica, G., *Sur une nouvelle classes de surfaces*, C. R. A. S. Paris, 144 (1907), 1257-1259.

[30] Tzitzeica, G., *Sur certaines courbes gauches*, Ann. de l'Ec. Normale Sup., 28 (1911), 9-32.

[31] Udrişte, C., Şandru, O., Dumitrescu, C., Zlatescu, A., *Tzitzeica indicatrix and figuratrix*, Tsagas, Gregorios (ed.), "Proceedings of the workshop on global analysis, differential geometry and Lie algebras", Aristotle Univ. of Thesaloniki, Greece, 1994. Bucharest, Geometry Balkan Press, BSG Proc. 2, 1998, 117-124.

[32] Udrişte, C., Bîlă, N., *Symmetry group of Tzitzeica surfaces PDE*, Balkan J. Geom. Appl., 4 (1999), no. 2, 123-140.

[33] Vrânceanu, Gh., *Sur les courbes cylindriques fermées*, Rev. Roumaine Math. Pures et Appl., 21 (1976), no. 5, 601-607.

[34] Yano. K., *Generalizations of the connection of Tzitzeica*, Kodai Math. Sem. Rep.,
      21 (1969), 167-174.

University "Al. I. Cuza" of Iassy
Faculty of Mathematics
Iaşi, 6600, Romania
email: mcrasm@uaic.ro